Explicit Representations of Lyapunov Functions for Stable Analytic Systems

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Abstract

An explicit series expansion for a Lyapunov function for a nonlinear analytic system is presented using the Lie series.

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1 Introduction

In this note we shall consider the generation of an explicit expression for a Lyapunov function of an asymptotically stable analytic system of the form

$$\dot{x} = f(x)$$

Many Lyapunov functions for special types of nonlinear system have been found (see [1],[2] for the most common ones), but as yet no general method is known for writing down a Lyapunov function for a given system. Recently a neural network approach to the problem has been applied with considerable success ([3]) but the method is still only approximate, being based on pattern recognition principles with a finite number of state space samples. A generalization of Lyapunov's equation has been given for nonlinear analytic systems ([4]).

Here we shall use the Lie series to derive an explicit solution to this problem. The Lyapunov function V will be an analytic function of x and t and we shall find a representation for the coefficients of x^i in the Taylor series expansion of V.

2 Notation

In this paper an n-tuple of non-negative integers $(i_1, \cdots, i_n)$ will be denoted by a boldface $\mathbf{i}$. Moreover, we shall use the following abbreviations:

$$\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \cdots x_n^{i_n} \text{ , for any } x \in R^n, \mathbf{i} = (i_1, \cdots, i_n)$$

$$\sum_{\mathbf{i}=0}^{\infty} = \sum_{i_1=0}^{\infty}\cdots\sum_{i_n=0}^{\infty}$$

Thus, if $f(x)$ is an analytic function of $x \in R^n$, then Taylor's theorem can be written

$$f(x) = \sum_{\mathbf{i}=0}^{\infty} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$$

for some tensor $\alpha_{\mathbf{i}} = \alpha_{(i_1,\cdots,i_n)}$.

We shall denote by 1_k the n-tuple with 1 in the k^{th} place and zero elsewhere; thus,

$$1_k = (0, 0, \cdots, \underbrace{1}_{k}, 0, \cdots, 0)$$

Finally, note that if we require a number of vector indices, say $\mathbf{i}_1, \ldots, \mathbf{i}_m$, then we shall denote the j^{th} component of the vector $\mathbf{i}_\ell$ by $(\mathbf{i}_\ell)_j$. (Thus, $\mathbf{i}_\ell$ does *not* mean the ℓ^{th} component of $\mathbf{i}$.)

3 The Lie Series and Explicit Solutions

In this section we shall determine an explicit series solution for the nonlinear analytic system

$$\dot{x} = f(x)\ ,\ x(0) = x_0 \in R^n \tag{3.1}$$

Using the notation introduced in section 2, let f be expanded in the Taylor series

$$f(x) = \left(\sum_{\mathbf{i}=0}^{\infty} a_{\mathbf{i}}^1 \mathbf{x}^{\mathbf{i}}, \cdots, \sum_{\mathbf{i}=0}^{\infty} a_{\mathbf{i}}^n \mathbf{x}^{\mathbf{i}} \right)^T \tag{3.2}$$

The solution of (3.1) (provided it exists) is given by the Lie series ([5],[6]):

$$x(t) = exp\left(tf(x)\frac{\partial}{\partial x}\right)x\Bigg|_{x=x_0} \tag{3.3}$$

$$= \sum_{k=0}^{\infty}\frac{t^k}{k!}\left(f(x)\frac{\partial}{\partial x}\right)^k x\Bigg|_{x=x_0}$$

$$= \sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\sum_{\alpha=1}^{n}\sum_{\mathbf{i}=0}^{\infty}a_{\mathbf{i}}^{\alpha}\mathbf{x}^{\mathbf{i}}\frac{\partial}{\partial x_\alpha}\right)^k x\Bigg|_{x=x_0}$$

$$= \sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{\alpha_1=1}^{n}\cdots\sum_{\alpha_k=1}^{n}\sum_{\mathbf{i}_1=0}^{\infty}\cdots\sum_{\mathbf{i}_k=0}^{\infty}$$

$$(\mathbf{i}_1+\cdots+\mathbf{i}_{k-1}-1_{\alpha_2}-1_{\alpha_3}-\cdots-1_{\alpha_{k-1}})_{\alpha_k}\cdots(\mathbf{i}_1+\mathbf{i}_2-1_{\alpha_2})_{\alpha_3}(\mathbf{i}_1)_{\alpha_2}\cdot$$

$$a_{\mathbf{i}_k}^{\alpha_k}\cdots a_{\mathbf{i}_1}^{\alpha_1}x_0^{\mathbf{i}_1+\mathbf{i}_2+\cdots+\mathbf{i}_k-1_{\alpha_2}-\cdots-1_{\alpha_k}}\frac{\partial x}{\partial x_{\alpha_1}} \tag{3.4}$$

Consider next the nonautonomous analytic system

$$\dot{x} = f(x,t)\ ,\ x(0) = x_0 \in R^n \tag{3.5}$$

By the usual trick of augmenting the state we can write this as an $(n+1)$ -dimensional autonomous system:

$$\dot{y} = F(y)\ ,\ y(0) = (x_0, 0) \in R^{n+1}$$

where

$$(y_1, \cdots, y_n)^T = x\ ,\ y_{n+1} = t.$$

Then, if F is analytic in y we have the expansion

$$F(y) = \left(\sum_{\mathbf{i}=0}^{\infty}\overline{a}_{\mathbf{i}}^{1}y^{\mathbf{i}}, \cdots, \sum_{\mathbf{i}=0}^{\infty}\overline{a}_{\mathbf{i}}^{n}y^{\mathbf{i}}, \sum_{\mathbf{i}=0}^{\infty}\overline{a}_{\mathbf{i}}^{n+1}y^{\mathbf{i}}\right)^T$$

where

$$\overline{a}^{n+1}_{(0,0,\ldots,0,1)} = 1 \ , \ \overline{a}^{n+1}_{\mathbf{i}} = 0 \text{ for all } \mathbf{i} \neq (0,0,\ldots,0,1).$$

4 Lyapunov Functions

Consider first the nonautonomous system

$$\dot{y} = F(y,t) \ , \ y \in R^n \ , \ F(0,t) = 0 \ , \ t \in R^+ \tag{4.1}$$

and suppose that this system is exponentially asymptotically stable in the region $\Gamma \subseteq R^n \times R$. Then we have the following result:

Lemma 4.1 *A Lyapunov function for (4.1) in* Γ *is given by*

$$V(t_0, y_0) = \int_{t_0}^{\infty} ||y(t;t_0,y_0)||^2 dt \ , \ (y_0,t_0) \in \Gamma \tag{4.2}$$

where $y(t;t_0,y_0)$ *is the solution of (4.1) through* (y_0,t_0).

Proof Since the solution $y(t;t_0,y_0)$ is exponentially stable by assumption, the integral in (4.2) certainly exists, and

$$V(t_0,y_0) > 0 \quad y_0 \neq 0.$$

Also, if $y(t;t_0,y_0)$ is the solution through (y_0,t_0) and $y_1 = y(t_1;t_0,y_0)$, then for $t_1 > t_0$ we have

$$\begin{aligned} V(t_0,y_0) &= \int_{t_0}^{\infty} ||y(t;t_0,y_0)||^2 dt \\ &= \int_{t_1}^{\infty} ||y(t;t_0,y_0)||^2 dt + \int_{t_0}^{t_1} ||y(t;t_0,y_0)||^2 dt \end{aligned}$$

$$
\begin{aligned}
&> \int_{t_1}^{\infty} ||y(t;t_0,y_0)||^2 dt \\
&= \int_{t_1}^{\infty} ||y(t;t_1,y_1)||^2 dt \\
&= V(t_1,y_1)
\end{aligned}
$$

and so V decreases along trjectories. □

Next consider the autonomous system

$$\dot{x} = f(x)\ ,\ f(0) = 0 \tag{4.3}$$

which is assumed to be asymptotically stable in the connected region $\Omega \subseteq R^n$, with $0 \in \Omega$. In general, however, this system will not be exponentially stable in Ω. Thus we cannot define, in general, a Lyapunov function for (4.3) of the form

$$V(x_0) = \int_0^{\infty} ||x(t;x_0)||^2 dt \tag{4.4}$$

since $||x||^2$ may not be integrable. However, we can put

$$y(t) = e^{-\alpha t} x(t) \tag{4.5}$$

for some $\alpha > 0$ and consider the nonautonomous system

$$\dot{y} = e^{-\alpha t} f(e^{\alpha t} y) - \alpha y. \tag{4.6}$$

This system is exponentially asymptotically stable on the region

$$\Gamma = \{(y,t) :\ y \in e^{-\alpha t}\Omega\}.$$

Let $V(t_0, y_0)$ be a Lyapunov function for (4.6) defined by lemma 3.1. Then we have

Lemma 4.2 *If $V(t,y)$ is a Lyapunov function for (4.6) (in Γ) then $V(t,e^{-\alpha t}x)$ is a Lyapunov function for (4.3) in Ω .*

Proof This follows from

$$\begin{aligned} \dot{V}(t,e^{-\alpha t}x) &= \frac{\partial V}{\partial t}(t,e^{-\alpha t}x) + \frac{\partial V}{\partial p}(t,p)\Big|_{p=e^{-\alpha t}x} \frac{d(e^{-\alpha t}x)}{dt} \\ &= \frac{\partial V}{\partial t}(t,y) + \frac{\partial V}{\partial p}(t,p)\Big|_{p=y} \dot{y} \\ &< 0 \end{aligned}$$

□

It follows from lemmas 3.1 and 3.2 that there is no loss of generality in assuming that (4.3) is *exponentially* asymptotically stable in Ω. (Otherwise we replace (4.3) by an exponentially stable nonautonomous system and use the state augmentation trick.)

Thus, suppose that the system (3.1) is exponentially asymptotically stable. Then a Lyapunov function is given by

$$\begin{aligned} V(x) &= \int_0^\infty \left\| exp\left(tf\frac{\partial}{\partial x}\right)x\right\|^2 dt \qquad (4.7) \\ &= \int_0^\infty \sum_{k=0}^\infty \sum_{\ell=0}^\infty \frac{t^{k+\ell}}{k!\ell!} \sum_{\gamma=1}^n \sum_{\alpha_2=1}^n \cdots \sum_{\alpha_k=1}^n \sum_{\beta_2=1}^n \cdots \sum_{\beta_\ell=1}^n \sum_{\mathbf{i}_1=0}^\infty \cdots \sum_{\mathbf{i}_k=0}^\infty \sum_{\mathbf{j}_1=0}^\infty \cdots \sum_{\mathbf{j}_\ell=0}^\infty \cdot \\ &\quad C(k,\ell;\gamma,\alpha_2,\cdots,\alpha_k,\beta_2,\cdots,\beta_\ell,\mathbf{i}_1,\cdots,\mathbf{i}_k,\mathbf{j}_1,\cdots,\mathbf{j}_\ell)\cdot \\ &\quad x^{\mathbf{i}_1+\mathbf{i}_2+\cdots+\mathbf{i}_k-1_{\alpha_2}-\cdots-1_{\alpha_k}+\mathbf{j}_1+\mathbf{j}_2+\cdots+\mathbf{i}_\ell-1_{\beta_2}-\cdots-1_{\beta_\ell}} dt \end{aligned}$$

where

$$C(k,\ell;\gamma,\alpha_2,\cdots,\alpha_k,\beta_2,\cdots,\beta_\ell,\mathbf{i}_1,\cdots,\mathbf{i}_k,\mathbf{j}_1,\cdots,\mathbf{j}_\ell) =$$

$$(\mathbf{i}_1+\cdots+\mathbf{i}_{k-1}-1_{\alpha_2}-1_{\alpha_3}-\cdots-1_{\alpha_{k-1}})_{\alpha_k}\cdots(\mathbf{i}_1+\mathbf{i}_2-1_{\alpha_2})_{\alpha_3}(\mathbf{i}_1)_{\alpha_2}\times$$

$$(\mathbf{j}_1+\cdots+\mathbf{j}_{\ell-1}-1_{\beta_2}-1_{\beta_3}-\cdots-1_{\beta_{\ell-1}})_{\beta_\ell}\cdots(\mathbf{j}_1+\mathbf{j}_2-1_{\beta_2})_{\beta_3}(\mathbf{j}_1)_{\beta_2}\times$$

$$a_{\mathbf{i}_k}^{\alpha_k}\cdots a_{\mathbf{i}_1}^{\alpha_1}a_{\mathbf{j}_\ell}^{\beta_\ell}\cdots a_{\mathbf{j}_1}^{\beta_1}$$

Since the integrand is analytic, we may write it in the form of a series in x with coefficients which are analytic in t. Thus,

$$V(x)=\int_0^\infty\sum_{\mathbf{i}=0}^{\infty}g_{\mathbf{i}}(y)x^{\mathbf{i}}dt$$

where

$$g_{\mathbf{i}}(t)=\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{t^{k+\ell}}{k!\ell!}\sum_{\gamma=1}^{n}\sum_{\alpha_2=1}^{n}\cdots\sum_{\alpha_k=1}^{n}\sum_{\beta_2=1}^{n}\cdots\sum_{\beta_\ell=1}^{n}\sum_{\substack{\mathbf{i}_1,\cdots,\mathbf{i}_k,\mathbf{j}_1,\cdots\mathbf{j}_\ell\\ \mathbf{i}_1+\mathbf{i}_2+\cdots+\mathbf{i}_k-1_{\alpha_2}-\cdots-1_{\alpha_k}+\\ \mathbf{j}_1+\mathbf{j}_2+\cdots+\mathbf{j}_\ell-1_{\beta_2}-\cdots-1_{\beta_k}=\mathbf{i}}}$$
$$C(k,\ell;\gamma,\alpha_2,\cdots,\alpha_k,\beta_2,\cdots,\beta_\ell,\mathbf{i}_1,\cdots,\mathbf{i}_k,\mathbf{j}_1,\cdots,\mathbf{j}_\ell)\quad(4.8)$$

By the elementary properties of analytic functions we have

$$V(x)=\sum_{\mathbf{i}=0}^{\infty}v_{\mathbf{i}}x^{\mathbf{i}}\quad(4.9)$$

where

$$v_{\mathbf{i}}=\int_0^\infty g_{\mathbf{i}}(t)dt.\quad(4.10)$$

Summing up, we have proved

Theorem 4.1 *Given an exponentially stable system of the form (3.1) (with f(0)=0), a Lyapunov function is given by (4.9) where the coefficients can be calculated from (4.8) and (4.10).* □

Remark 4.1 In the case of a non-exponentially stable system we must obtain an expression similar to (4.8) by using the state augmented system and lemma 4.2.

Remark 4.2 In the linear case the Lypaunov function (4.9) is simply the usual one given by Lyapunov's equation

$$A^T P + PA = -I$$

since (4.7) becomes

$$V(x) = \int_0^\infty x^T e^{A^T t} e^{At} x dt.$$

(See also [4])

5 Conclusions

In this paper we have given an explicit expression for a Lyapunov function for a nonlinear analytic system which is exponentially stable,by using the Lie series. In the general case of not necessarily exponentially stable systems we can augment the state and reduce the problem to that of a nonautonomous exponentially stable system. The solution is given as a Taylor series expansion with an explicit characterisation of the coefficients of the monomials in the state space coordinates. These coefficients can be evaluated approximately, possibly by using a computer algebra package.

References

[1] J.LaSalle and S.Lefschetz, 'Stability by Lyapunov's Direct Method, New York:Academic Press,1961.

[2] S.P.Banks, 'Control Systems Engineering ', London:Prentice-Hall,1988.

[3] S.P.Banks and R.F.Harrison, 'Can Perceptrons Find Lyapunov Functions?-An Algorithmic Approach to Systems Stability ', Proc. 1st. IEE Int. Conf. on Artificial Neural Networks,1989.

[4] S.P.Banks, 'Generalization of the Lyapunov Equation to Non-linear Systems ',Int.J.Sys. Sci.,**19**,629-636,1988. ', Sys. Control Letts.,**19**,629-636,1988.

[5] S.P.Banks, 'Global Linear Representations of Nonlinear Systems and the Adjoint Map ', Int. J. Control,**51**,1229- 1240, 1990.

[6] P.J.Olver, 'Applications of Lie groups to differential equations ',Springer-Verlag,New York,1986.