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The Qualitative Analysis

of

Nonlinear Parameterised Systems

Part I

Nonlinear Systems Representation

and

Dynamical Systems Theory

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The Qualitative Analysis of Nonlinear Parameterised Systems

Part I - Nonlinear Systems Representation and Dynamical Systems Theory

Abstract

The analysis of parameterised nonlinear models is considered and a new approach is introduced which provides a framework for the global analysis of both continuous and discrete time nonlinear systems.

The paper is divided into two parts. Part I provides an overview of nonlinear system representations and dynamical systems theory. This forms the basis for the results in the second part. Part II introduces a numerically based analysis tool and demonstrates that this provides a flexible framework for the analysis of a diverse range of nonlinear model types. It is shown, by extending the numerical algorithm, that the approach provides a global perspective to the results that is difficult to obtain using analytical methods alone. Simple examples are used to illustrate how the method detects typical bifurcation phenomena. A nonlinear feedback system is analysed in order to show how the new approach provides both qualitative information and a global perspective over a defined region of the systems parameter space. The method proves to be a powerful tool when used to probe the nonlinear characteristic of a system.

Introduction

Nonlinear models arise in so many scientific and engineering disciplines that it would be impossible to list even a fraction of the applications. From an engineering viewpoint, the prime reason for wishing to model a system is to understand its behaviour. Such behaviour can be surprisingly rich and varied. The nonlinear nature of a system may show itself through performance degradation, limit cycle behaviour, harmonic distortion, hysteresis, bifurcation and chaos.

On the other hand, linear models have one very attractive property, they are easy to handle. One possible way to deal with a nonlinear system is to linearise about some known operating point. Once this is done, a host of classical linear techniques may be applied. Powerful theories abound both in the time domain and frequency domain. Unfortunately in doing this the very behaviour we wish to replicate may be lost.

A nonlinear system will not obey the laws of superposition and may loose stability for certain input values. A linear system is either stable or unstable and this applies equally for all inputs whatever their magnitude. However the stability of a nonlinear system is in general, intimately related to the input excitation and stability properties are by no means as easy to define as in the linear case. In particular both local and global aspects need to be distinguished.

In Section I of this paper we review the representation of a variety of nonlinear model types. In general the method of analysis for such models depends primarily on the model structure. One approach, which can be placed under the broad heading of dynamical systems theory, is particularly attractive, for two reasons. Firstly, it is applicable to a wide variety of continuous and discrete models. This is most useful when considering nonlinear sampled data systems representing continuous plant. Secondly, the theory, being essentially qualitative in nature, begins to be of use exactly were traditional linear theory breaks down, that is, when one or more eigenvalues of the linearised system become degenerate. In Section II we review the basis of this theory and the main results.

Unfortunately, the analytical based methods rely on detailed a priori knowledge of the nonlinear models solution structure, before they can be applied. In essence they require the analyst to know where the interesting behaviour lies in the product of the problems state space and parameter space, before he can analyse what type of behaviour is occurring. In practice, this means being able to characterise the solution structure of the nonlinear equations, describing the system over some region of the state space for a given range of parameters.

In Part II of this paper we get around this problem by adopting a particularly attractive numerically based method, the *cell map* algorithm. This algorithm is first extended to allow the analysis of the type of parameterised nonlinear problems discussed in Part I. This simple approach eliminates the problems mentioned above and is applicable to a diverse range of model structures. In addition it provides a global aspect to the analysis which is very difficult to achieve using previous analytical or numerical approaches.

A number of simple examples are used to illustrate how typical bifurcation phenomena can be detected, both in continuous and discrete systems. As a by-product of the analysis almost complete characterisation of the local and global stability over a predefined region is provided.

Nonlinear System Representations

1.0. Introduction

Modern methods of analysis of dynamical systems of equations have their roots in the work by Poincare, 1890, Birkhoff, 1927 and Lyapunov, 1949. The behaviour of solutions of nonlinear systems was studied extensively by Rayleigh, 1896, Duffing, 1918, and Van der Pol, 1927. The study of nonlinear systems over the last few decades has been driven to a large extent by the type of model structures adopted. Systems of ordinary differential equations, ODE's, are commonly expressed in the form

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t),t) \qquad \mathbf{x} \in \mathbf{R}^n \; ; \; \mathbf{x}(0) = \mathbf{x}_0 \tag{1.1}$$

If $u(t)$ is defined explicitly, then an alternative notation is

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t),\mathbf{u}(t),t) \qquad \mathbf{x} \in \mathbf{R}^n \; ; \; \mathbf{u} \in \mathbf{R}^m \; ; \; \mathbf{x}(0) = \mathbf{x}_0 \tag{1.2}$$

If the differential equation is to have a unique solution, $\mathbf{x}(t)$, for every initial condition $\mathbf{x}_0$ and input vector $\mathbf{u}(t)$, it is necessary to impose some constraints on $\mathbf{F}$ and $\mathbf{u}$. For a solution to exist it is sufficient that $\mathbf{F}$ be continuous in its arguments. To ensure uniqueness a Lipschizt condition must be satisfied within some region $\mathbf{M} \in \mathbf{R}^n$

$$| \mathbf{F}(\mathbf{u},t) - \mathbf{F}(\mathbf{v},t) | \leq \mathbf{L}(\mathbf{u} - \mathbf{v}) \qquad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{M} \tag{1.3}$$

where $\mathbf{L}$ is a constant and $|\mathbf{x}|$ the Euclidean norm. A more recent notation used by dynamical systems theorists is that of the *flow*. If $\Phi_t(\mathbf{x})$ is a point in $\mathbf{R}^n$ at time t, starting from $\mathbf{x}_0$ at time $t=0$, resulting from following the vector field $\mathbf{F}$, then the *map* Φ_t describes the flow Φ_t of the ODE in $\mathbf{R}^n$ such that

$$\Phi : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n \qquad \Phi(\mathbf{x}) = x \quad \Phi_s(\Phi_t(\mathbf{x})) = \Phi_{s+t}(\mathbf{x}) \tag{1.4}$$

and represents the totality of the solution generated by $\mathbf{F}$ [Hirsh and Smale, 1974, Mees, 1981]. The ideas and methods of bifurcation theory can be used to provide qualitative information on nonlinear systems of equations exhibiting typical nonlinear phenomena, limit cycles and the like. The theory also provides local stability information on the resulting bifurcated solutions. Consider a system of differential equations depending on the k dimensional parameter vector μ

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\mu) \qquad \mathbf{x} \in \mathbf{R}^n \qquad \mu \in \mathbf{R}^k$$

$$\Phi_t \; : \; \mathbf{R}^n \times \mathbf{R}^k \rightarrow \mathbf{R}^n \qquad \mathbf{f} = (\, f_1(x_1,..,x_n),..,f_n(x_1,..,x_n)\,)^T \tag{1.5}$$

The word *bifurcate* means to split into two. In its most general form bifurcation theory is the study of the splitting of the solution structure of nonlinear systems. Its main aim is to describe the qualitative changes which take place in the solution structure of a differential equation or map, depending on a distinguishing parameter μ.

The ability to select different system parameters lends itself to further possibilities. In particular the problem of parameter sensitivity and structural stability may be addressed.

The discrete equivalent of (1.5) is

$$\mathbf{x}_{k+1} = \mathbf{f}(\,\mathbf{x}_k,\mu\,) \qquad \mathbf{x} \in \mathbf{R}^n \quad \mu \in \mathbf{R}^m \tag{1.6}$$

The representations (1.5) and (1.6) can be used to represent a very wide range of systems. As a starting point the various representations which are often used in the study of nonlinear systems are briefly reviewed and then these are related back to the general parameterised problem above.

1.1. State Space Models

A large proportion of modern control theory is based upon the state variable model

$$\begin{aligned} \dot{\mathbf{x}}(t) &= f(\,\mathbf{x}(t),\mathbf{u}(t),t\,) \qquad && \mathbf{x} \in \mathbf{R}^n \;;\; \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= g(\,\mathbf{x}(t),\mathbf{u}(t),t\,) \qquad && \mathbf{u} \in \mathbf{R}^m \;;\; \mathbf{y} \in \mathbf{R}^l \end{aligned} \tag{1.7}$$

Lyapunov, 1949, considered the stability of the class of autonomous nonlinear systems

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\,t,\mathbf{x}(t)\,) \qquad && \mathbf{x} \in \mathbf{R}^n \;;\; \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{f} &: \mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}^n \qquad && \forall t \geq 0 \;;\; t \in \mathbf{R}_+ \end{aligned} \tag{1.8}$$

where $\mathbf{f}$ is sufficiently smooth to ensure a unique solution and has an equilibrium point at the origin. Lyapunov's direct or second method briefly states that the system (1.8) is asymptotically stable on some domain $\mathbf{M}$ about the origin if there exists a scalar function V which is positive definitive and whose time derivative along the trajectory of the system is negative definitive within the same domain. The great advantage of his theory lies in the fact that stability of an equilibrium point of a dynamical system may be evaluated without having to solve the differential equation (1.8). The search for Lyapunov functions has occupied a considerable amount of effort by system theorists in recent years. The focus of attention being the estimation of the size of this domain [Genesio and Vicino, 1984].

It is convenient here to introduce the concept of an *invariant set*, which is defined as any set of points in the state space, $\mathbf{R}^n$, such that each trajectory starting within it, remains there for all subsequent time. The *domain of attraction*, or **DOA**, of an asymptotically stable equilibrium point, is an open invariant set containing the equilibrium point, with the property that every trajectory starting within it approaches the equilibrium point as $t \to \infty$. An equilibrium point is said to be *globally asymptotically stable* if the **DOA** encompasses the entire state space, $\mathbf{R}^n$. The **DOA** defines a basin about the attractor in which all solutions are captured.

1.2. Functional Series Representations

Functional expansions have been applied to virtually every branch of nonlinear system theory. The Volterra series [Volterra, 1930], is characterised by the *input output* relationship

$$y(t) = \sum_{n=1}^{\infty} y_n(t) \tag{1.9}$$

where the response $y_n(t)$ due to the nth order kernel $h_n(\tau_1,..,\tau_n)$ is given by

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,..,\tau_n) \prod_{i=1}^{n} u(t-\tau_i)\, d\tau_i \tag{1.10}$$

The original application and early work was carried out by Weiner, 1958. Brilliant, 1958, George, 1959, and Barrett, 1963. Sufficient conditions for the existence of a Volterra series expansion, for a very general class of nonlinear systems, are given in Gilbert, 1977. Early application of the functional series approach are due to Narayanan, 1967, 1969, 1970, Bedrosian and Rice, 1971, Chua and Ng, 1979a, 1979b, Weiner and Spina, 1980, Marmarelis and Marmarelis, 1978, Billings, 1980, Schetzen, 1980 and Rugh, 1981.

The Fourier transform of the nth order kernel $h_n(\tau_1,...,\tau_n)$ is given by

$$H_n(w_1,..,w_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,..,\tau_n)\, e^{-w_1\tau_1 - \cdots - w_n\tau_n}\, d\tau_1 \cdots d\tau_n \tag{1.11}$$

and is called the nth order transfer function for the system. The importance of such higher order spectra was realised by Rosenblatt and Van Ness, 1965 and Brillinger, 1965, 1967a, 1967b and 1970. Recently there has been renewed interest in the estimation of higher order spectra [Billings and Tsang, 1987, and Billings, Tsang and Tomlinson, 1988].

1.3. Multidimensional Systems

The representation of systems using the Volterra series leads naturally to the consideration of the multidimensional Laplace transform. Since the output in (1.9) is given by the superposition of the response due to each kernel $h_n(\tau_1,...,\tau_n)$ it is sufficient to consider only the response $y_n(t)$ due to the nth order kernel. The multidimensional Laplace transformation ${}_LF_n(s_1,..,s_n)$ of a function $f_n(t_1,t_2,..,t_n)$ is given by [George, 1959]

$$ {}_LF_n(s_1,..,s_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_n(\tau_1,..,\tau_n) \prod_{i=1}^{n} e^{-s_i t_i} \, d\tau_i \tag{1.12} $$

and the inverse transform by

$$ f_n(t_1,..,t_n) = \frac{1}{2\pi j^n} \int_{\alpha_1 - j\infty}^{\alpha_1 + j\infty} \cdots \int_{\alpha_n - j\infty}^{\alpha_n + j\infty} F_n(s_1,..,s_n) \, e^{s_1 t_1 + \cdots + s_n t_n} \, \partial s_1 \cdots \partial s_n \tag{1.13} $$

A multidimensional kernel input output expression can then be constructed using

$$ {}_LY_n(s_1,..,s_n) = {}_LH_n(s_1,..,s_n) \prod_{i=1}^{n} {}_LU_i(s_i) \tag{1.14} $$

giving

$$ y_n(t_1,..,t_n) = L^{-1}\left[{}_LY_n(s_1,..,s_n) \right] \tag{1.15} $$

The advantage of this approach arises when considering block structured systems, which consist of interconnected linear and zero memory nonlinear subsystems [George, 1959, and Schetzen, 1980]. The nth order kernel response $y_n(t)$ may be found directly by applying the inverse transformation (1.13) sequentially . Alternatively, by realising that for the output time function $y_n(t_1,t_2,..,t_n)$ only the special case

$$ g(t) = y_n(t_1,..,t_n) \Big|_{t_1 = t_2 = \cdots = t_n} \tag{1.16} $$

is of interest, then there must exist a $G(s)$ such that

$$ G(s) = L\left[g(t) \right] \tag{1.17} $$

The process of reducing ${}_LF_n(s_1,s_2,..,s_n)$ to $G(s)$ is known as *association of variables* [George, 1959, Chen and Chie, 1973], and may be carried out using the iterative formulae [Barker and Ambati, 1972]

$$ {}_LF_{n-1}(\tau_1,s_2,..,s_n) = \frac{1}{2\pi j^n} \int_{\alpha_1 - j\infty}^{\alpha_1 + j\infty} {}_LF_n(s_1,..,s_n) \, e^{s_1 \tau_1} \, \partial s_1 \tag{1.18} $$

where $s_2, s_3, .., s_n$ are taken to be constant.

This approach has been extended to the analysis of discrete systems by Jury, 1958 and to nonlinear sampled data systems by Alper, 1964, Lavi and Narayanan, 1968, Barker and Ambati, 1972, Jagen and Reddy, 1972 and Farison and Fu, 1973. Most usefully Barker and Ambati define a sequential n-stage process for obtaining the multidimensional Z-transform from ${}_LF_n(s_1,..,s_n)$

$$ {}^{L}_{Z}F^{\,n-r}_{\;r}(z_1,..,z_r,s_{r+1},..,s_n) = \frac{1}{2\pi j^n} \int_{c_r - j\infty}^{c_r + j\infty} \frac{{}^{L}_{Z}F^{\,n-r+1}_{\;r-1}(z_1,..,z_{r-1},s_r,..,s_n)}{(1 - e^{s_r T} z_r^{-1})} \, \partial s_r $$

$$ r = 1,2,..,n \qquad z_1,..,z_{r-1},s_{r+1},..,s_n \text{ constant} \tag{1.19} $$

When F (.) has no branch points this may be calculated as

$$ {}^{L}_{Z}F^{\,n-r}_{\;r}(z_1,..,z_r,s_{r+1},..,s_n) = \sum_{\text{Residues}} \frac{{}^{L}_{Z}F^{\,n-r+1}_{\;r-1}(z_1,..,z_{r-1},s_r,..,s_n)}{(1 - e^{s_r T} z_r^{-1})} \tag{1.20} $$

Furthermore, a generalisation of the well known relationship for combining a continuous system described by a rational Laplace transform in cascade with the *zero order hold*, ZOH, is given as

$$ \frac{z-1}{z} Z\left[\frac{F(s)}{s} \right] \quad \text{becomes} \quad \frac{z-1}{z} Z\left[\frac{{}_LF_n(s_1,..,s_n)}{s} \prod_{i=1}^{n} \frac{1}{s_i} \right] \tag{1.21} $$

Stability definitions for the general k-dimensional causal discrete transfer function can be found in Huang, 1972, Justice and Shanks, 1973, Anderson and Jury, 1974 and Strintzis, 1977. This essentially multidimensional filtering approach, has found wide application, see Bose 1977a, 1977b and 1982, Jury, 1980.

1.4. Bilinear Systems

Algebraic theory for the representation, realisation and analysis of nonlinear systems has developed into a flourishing area. The fundamental results were obtained by Sandberg, 1964, 1965a, 1965b, and Zames, 1963, 1964. Without exception the above call upon advanced concepts in algebra, analysis and geometry, see Brockett, 1981, Rugh, 1981, Fliess et al, 1983, Isidori, 1985. and Vidyasagar, 1986, for introductory material.

In this work bilinear systems play an important role, taking the form

$$ \dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) + \mathbf{u}(t)\,C\,\mathbf{x}(t) $$

$$ \mathbf{y}(t) = C\,\mathbf{x}(t) \tag{1.22} $$

It is possible to write down the Volterra series for a bilinear system, D'Alessandro et al, 1974. Furthermore, a set of necessary and sufficient conditions for a Volterra series to have a bilinear realisation may also be constructed.

1.5. Linear Analytic Systems

A more general class of nonlinear systems known as the *linear analytic* system, Brockett, 1972, 1976, takes the form

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t),t) + \mathbf{u}(t)\,g(\mathbf{x}(t)) + \mathbf{u}(t)$$

$$\mathbf{y}(t) = h(\mathbf{x}(t),t) \tag{1.23}$$

Differential geometric results on existence of and uniqueness of (1.24) are provided by Sussmann, 1977, and Crouch, 1981. Notice how (1.23) imposes a linearity constraint on the form of the input vector $u(.)$.

Realisation theory for these systems is provided by Fliess, 1982 and Jakubczyk, 1987. Concrete examples of bilinear systems are provided by Mohler, 1970, 1977, and Mayne and Brockett, 1973.

An algebraic approach to the input output description of nonlinear discrete time systems is provided by Normand-Cyrot and Monaco, 1984a, 1984b, Realisation theory for nonlinear discrete time systems can be found in the work by Clancy and Rugh, 1978, Sontag, 1979a and 1979b, and Schwartz and Dickinson, 1986a, 1986b.

The emphasis in this work has been on control issues such as decoupling [Claude, 1982, Isidori et al, 1981, Isidori, 1985, Nijmeijer 1982, Nijmeijer et al 1985, Grizzle, 1985a, 1985b and 1986, and D'Andrea and Levin, 1986]; disturbance decoupling, [Hirschorn, 1981, and Isidori, 1985] disturbance rejection, [Desoer and Lin, 1985, and Anartharam and Desoer, 1985]; and feedback linearisation [Isidori, 1983, Monaco and Normand-Cyrot, 1983, Isidori and Ruberti, 1984, Cheng et al, 1985, Jakubczyk, 1987, and Calvet and Arkun, 1988]. Additionally, coverage of extensions to the familiar results found in linear theory is provided by Hermann and Krener, 1977, Vidyasagar, 1986, and Crouch and Byrnes, 1986.

1.6. Discrete Dynamical Systems

Discrete systems or *recurrences* have been studied in themselves as dynamical systems by Bernussou, 1977, and Gumowski and Mira, 1980. Extensive effort has gone into the explaining the recurrent behaviour of iterates of the form

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k, \mu) \qquad \mathbf{x} \in \mathbf{R}^n \ ; \ \mu \in \mathbf{R} \tag{1.24}$$

where μ is a parameter and g(.) is a single valued smooth function. Defining

$$\mathbf{g}^k(\mathbf{x}_k, \mu) = \mathbf{g}(\cdots \mathbf{g}(\cdot) \cdots) \tag{1.25}$$

as the *iterated* recurrence, the isolated point singularities of (1.25) are given by the roots of

$$\mathbf{g}^k(\mathbf{x}, \mu) - \mathbf{x} = 0 \qquad k = 1,2,3,... \tag{1.26}$$

A set of k points forms a **P-K** cycle and is an invariant with respect to (1.24). For $k=1$ we have a fixed point of the recurrence, for $k \geq 2$ a **P-K** cycle which describes a subharmonic, frequency division or period doubling phenomena.

1.7. Nonlinear Recursive Systems

Recursive systems are discussed by Hammer, 1984a, 1984b, 1985, and 1986a. In qualitative terms, a discrete system is said to be recursive when its output sequence can be computed from its input output sequence in a recursive manner. A system is recursive if a pair of integers, n_u, $n_y \geq 0$, exist such that any output sequence $\{y_0, y_1, ...\}$ can be computed from the input sequence $\{u_0, u_1, ...\}$ using

$$y(k+n_y+1) = f(y(k),..,y(k+n_y), u(k),..,u(k+n_u)) \tag{1.27}$$

Necessary and sufficient conditions are derived for the existence of a fractional representation of the system (1.27), such that, the numerator and denominator are stable recursive systems. The concept of rationality plays an important and key role in this theory and stabilisation conditions are then derived for the construction of feedback and feed forward compensators. [Hammer, 1986b, 1987 and 1988].

1.8. Time-Series

Although generally considered in their own right, there are strong links between the model structures used in time series analysis and those studied as nonlinear dynamical systems. The representation of nonlinear systems from the time series viewpoint has lead to the adaptation of a number of interesting model structures in addition to the widely used Autoregressive AR(k), Moving Average MA(l) and the mixed ARMA(k,l) process [Akaike, 1974, and Priestley, 1978]. A general linear model of a time series $\{x(t)\}$ may be written as

$$x(t) = \sum_{n=0}^{\infty} g_n \, \varepsilon(t-n) \tag{1.28}$$

where $\{\varepsilon(t)\}$ denotes a purely random white noise process. The time index here is used to denote a discrete sampling. Generally the sequence $\{\varepsilon(t)\}$ is assumed stationary such that $E[\varepsilon(t)] = \mu$, $\mathrm{Var}[\varepsilon(t)] = \sigma^2$ and $\mathrm{Cov}[\varepsilon(t_1), \varepsilon(t_2)] = 0$ for all $t_1 \neq t_2$

The Autoregressive, AR(k), model is represented by

$$x(t) + \alpha_1 x(t-1) + \cdots + \alpha_k x(t-k) = \varepsilon(t) \tag{1.29}$$

the Moving Average, MA(l), by

$$x(t) = \varepsilon(t) + \beta_1 \varepsilon(t-1) + \cdots + \beta_l \varepsilon(t-l) \tag{1.30}$$

and the mixed Autoregressive Moving Average, ARMA(k,l), model by

$$x(t) + \alpha_1 x(t-1) + \cdots + \alpha_k x(t-k) = \varepsilon(t) + \beta_1 \varepsilon(t-1) + \cdots + \beta_l \varepsilon(t-l) \tag{1.31}$$

A state space representation for time series, the Markovian form, also exists

$$\mathbf{x}(t) = F\,\mathbf{x}(t) + G\,\varepsilon(t+1) \qquad \mathbf{x}(t) = (x_1(t),..,x_n(t-n))^T$$

$$y(t) = H\,\mathbf{x}(t) \tag{1.32}$$

F, G and H are suitably defined matrices. A number of canonical forms exist for this representation, see Priestley, 1981, Cox, 1981, and Newbold, 1981.

1.9. Nonlinear Time Series

The first systematic study of nonlinear time series models was carried out by Weiner, 1958, who considered Volterra expansions of the form

$$x(t) = \sum_{n=0}^{\infty} g_n \, \varepsilon(t-n) + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} g_{n_1 n_2} \, \varepsilon(t-n_1)\, \varepsilon(t-n_2) + \cdots \tag{1.33}$$

Considerable literature exists on the theoretical properties of such models, see for instance the review by Brillinger, 1970. However the estimation of such models is extremely unwieldily due to the excess of parameters involved [Billings, 1980].

Bilinear, BL(k,l), models are considered from the time series modelling point of view by Granger, 1978, and Granger and Anderson, 1978a, 1978b. The general Bilinear model is given by

$$x(t) = a_0 + \sum_{i=1}^{k} a_i\, x(t-i) + \sum_{i=1}^{l} b_i\, \varepsilon(t-i) + \sum_{i=1}^{l}\sum_{j=1}^{k} c_{ij}\, \varepsilon(t-i)\, x(t-j) + \varepsilon(t) \qquad (1.34)$$

It may be shown that with suitable choices of a_i, b_i and c_{ij} (1.35) may approximate any reasonable model of the form (1.34) [Brockett, 1976]. Re-writing (1.34) in state space form gives

$$\mathbf{x}(t+1) = A\,\mathbf{x}(t) + B\,\varepsilon(t+1) + \sum_{i=1}^{l} C_i\,\mathbf{x}(t)\,\varepsilon(t-i)$$

$$y(t) = H\,\mathbf{x}(t) \qquad \mathbf{x}(t) = (\,x(t),..,x(t-k)\,)^T \qquad (1.35)$$

Stationarity, invertability and parameter estimation for such models are covered in Granger and Anderson, 1978b, and Subba-Rao, 1981.

Other forms of nonlinear time series models have been introduced by various researchers, these include the Nonlinear Autoregressive, NAR(k), model [Jones, 1976, 1978].

$$x(t) = f\,(x(t-1)\,) + \varepsilon(t) \qquad (1.36)$$

Nonlinear Moving Average, NMA(k), models [Robinson, 1977], of the form

$$x(t) = \varepsilon(t) + \alpha\,\varepsilon(t-1) + \beta\,\varepsilon(t)\,\varepsilon(t-1) \qquad (1.37)$$

Threshold Autoregressive models, TAR(k), as introduced by Tong and Lim, 1980 and Tong, 1983, have been used to study naturally occurring limit cycle behaviour

$$x(t) + \alpha_1^{(i)}\,x(t-1) + \cdots + \alpha_k^{(i)}\,x(t-k) = \varepsilon_t^{(i)} \qquad i = 1,..,q \qquad (1.38)$$

where $R^{(i-1)} < x(t-d) < R^{(i)}$, $-\infty = R^{(0)} < R^{(1)} < \cdots < R^{(q-1)} < R^{(q)} = +\infty$, d is some lag $1 \le d \le k$ and $R^{(i)}$ are some given regions within $\mathbf{R}^k$.

Exponential Autoregressive, EAR(k), models [Ozaki, 1980, Haggan and Ozaki, 1981] as a class, exhibit nonlinear amplitude dependence and jump resonance [Ozaki, 1981, 1982, 1985].

$$x(t) = \sum_{i=1}^{k} \left(\alpha_i + \beta_i\, e^{-x^2(t-1)} \right) x(t-i) + \varepsilon(t) \qquad (1.39)$$

A more general form is the EARMA (k,l) model

$$x(t) = \sum_{i=1}^{k} \left(\alpha_i + \beta_i e^{-x^2(t-k_1)}\right) x(t-i) + \sum_{i=1}^{l} \left(\gamma_i + \eta_i e^{-x^2(t-m_1)}\right) \varepsilon(t-i) + \varepsilon(t) \quad (1.40)$$

The State Dependent, SDM(k,l), models developed by Priestley, 1980, provide a more general form of nonlinear model. Given the extended state vector

$$\mathbf{x}_t = (\varepsilon(t-l+1),..,\varepsilon(t), x(t-k+1),..,x(t))^T \quad (1.41)$$

the SDM (k,l) model may be written as

$$x(t) = \mu(\mathbf{x}(t-1)) + \sum_{i=1}^{k} \phi_i (\mathbf{x}(t-1))\, x(t-i) + \sum_{i=1}^{l} \psi_i (\mathbf{x}(t-1))\, \varepsilon(t-i) + \varepsilon(t) \quad (1.42)$$

It can be clearly seen that, excluding the threshold models, the previous time series representations may be taken as special cases of (1.42). Identification of the functional form and the parameters involved in (1.42) is considered by Haggan, Heravi and Priestley, 1984.

Finally, the Nonlinear Autoregressive Moving Average, NARMA(k,l), completes this generalisation by adopting the general polynomial model structure [Chen and Billings, 1989]

$$x(t) = F(x(t-1),..,x(t-k), \varepsilon(t-1),..,\varepsilon(t-l)) + \varepsilon(t) \quad (1.43)$$

where $\{x(t)\}$ is a time series, $\{\varepsilon(t)\}$ a strictly white noise process and $F(.)$ some non-linear function. The model (1.43) is about as far as one can go in terms of specifying a general finite-dimensional non-linear relationship.

1.10. Nonlinear Discrete Input Output Models

The bilinear time series model (1.34) may be rewritten as the input output model

$$y(k) = a_0 + \sum_{i=1}^{n_y} a_i y(k-i) + \sum_{i=1}^{n_u} b_i u(k-i) + \sum_{i=1}^{n_y} \sum_{j=1}^{n_u} c_{ij} y(k-i) u(k-j) \quad (1.44)$$

The noise sequence $\varepsilon(t)$ has been replaced by the observable input $u(t)$. This model structure can be used to represent a wide variety of systems. For example, the bilinear state space model can easily be written as an input output model, such that

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) + u(k)\,C\,x(k) \\ y(k) &= D\,x(k) \end{aligned} \quad (1.45)$$

becomes

$$y(k+1) = DA\left(D^T D\right)^{-1} D^T y(k) + DB\,u(k) + u(k) DC\left(D^T D\right)^{-1} D^T y(k) \quad (1.46)$$

It is known that a continuous time system represented by a Volterra series may be realised by a bilinear system [Brockett, 1976]. Moreover many continuous time processes are naturally bilinear [Mohler, 1970, 1977]. However, sampling the continuous time bilinear system does not result in a discrete time bilinear input output map. Assuming that a zero-order-hold, ZOH, device is used, that is, $u(t) = u(t_k)$, for $t_k \le t < t_{k+1}$, with a fixed sampling interval, h. Then for $t \in [t_k, t_{k+1})$, the SISO system (1.23) becomes

$$\dot{x}(t) = [A + u(k)\,C\,]\,x(t) + B\,u(k)$$

$$y(t) = D\,x(t) \qquad (1.47)$$

where $u(k)$ replaces $u(t_k)$ and $x(k)$ replaces $x(t_k)$. Letting $t \to t_{k+1}$ and substituting $h = t_{k+1} - t_k$ yields

$$x(k+1) = e^{[A + u(k)C]h} \cdot x(k) + \int_0^h e^{[A + u(k)C](h - \tau)}\,B\,d\tau \cdot u(k)$$

$$y(k+1) = D\,x(k+1) \qquad (1.48)$$

The system (1.48) obviously is no longer a bilinear system.

1.11. Nonlinear Autoregressive Moving Average Model with eXogenous Inputs

Input output descriptions that expand the current output $y(t)$ in terms of past take the form

$$y(k) = F\,(\,y(k-1),..,y(k-n_y)\,,\,u(k-1),..,u(k-n_u)\,) \qquad (1.49)$$

This model is referred to as the NARMAX due to its resemblance to the linear ARMA models discussed earlier. Realisation conditions for the NARMAX model have been derived by Billings and Leontaritis, 1981, Leontaritis and Billings, 1985a and 1985b, and Chen and Billings, 1989.

The derivation of the (1.49) is based on *zero-initial-state* response function $\mathbf{f}_{\mathbf{x}_0 \mid k}$. If the system is initially at a zero equilibrium $\mathbf{x}_0$ at time $k = 1$ the response function for an input sequence $\mathbf{u}^k = \{\mathbf{u}(k), \mathbf{u}(k-1),..,\mathbf{u}(1)\}$ of length k, is given by

$$\mathbf{y}(k) = \mathbf{f}_{\mathbf{x}_0 \mid k}(\,\mathbf{u}(k), \mathbf{u}(k-1),..,\mathbf{u}(1)\,) = \mathbf{f}_{\mathbf{x}_0}(\,\mathbf{u}^k\,) \qquad \mathbf{f}_{\mathbf{x}_0} : \mathbf{U}^t \to \mathbf{Y} \qquad (1.50)$$

The NARMAX model provides a natural representation for sampled nonlinear continuous-time systems and may also represent a wide class of discrete-time nonlinear systems. The response function $\mathbf{f}_{\mathbf{x}_0 \mid k}$ of a system is said to be a polynomial response function if for each k, $\mathbf{f}_{\mathbf{x}_0 \mid k}$ is a polynomial of finite degree in all variables.

1.12. The Output-Affine and Rational Models

It is known that a polynomial response function **f** is finitely realisable if and only if it satisfies the *rational* difference equation [Sontag, 1979b].

$$y(k) = \frac{b(y(k-1),..,y(k-r),u(k-1),..,u(k-r))}{a(y(k-1),..,y(k-r),u(k-1),..,u(k-r))} \tag{1.51}$$

where r is the order of the system and $a(.)$ and $b(.)$ are polynomials of finite degree. Sontag further showed that **f** is a finitely realisable and bounded polynomial response function, if and only if it satisfies an *output affine* difference equation

$$a_0(u(k-1),..,u(k-r))\,y(k) = \sum_{i=1}^{r} a_i(u(k-1),..,u(k-r))\,y(k-i) + a_{r+1}(u(k-1),..,u(k-r)) \tag{1.52}$$

or

$$y(k) = \sum_{i=1}^{r} \frac{a_i(u(k-1),..,u(k-r))}{a_0(u(k-1),..,u(k-r))}\,y(k-i) + \frac{a_{r+1}(u(k-1),..,u(k-r))}{a_0(u(k-1),..,u(k-r))} \tag{1.53}$$

where $a_i(.)$, $i = 0,1,..,r+1$ are polynomials of finite degree. Such a response function admits a *state affine* representation of the form

$$\mathbf{x}(k+1) = A\,(u(k))\,\mathbf{x}(k) + B\,(u(k))$$

$$y(k) = C\,(u(k))\,\mathbf{x}(k) \tag{1.54}$$

where $A(.)$, $B(.)$ and $C(.)$ are matrix and vector valued polynomials of finite degree. Fliess and Normand-Cyrot, 1982, showed that on a finite time interval and with bounded inputs a discrete-time input output system can be arbitrarily well approximated within the set of state affine systems.

As an example, consider the approximation of the sampled continuous-time bilinear system (1.48) by a state affine system. This involves the approximation of $e^{[A + u(k)C]}$ using matrix and vector valued polynomials in $u(k)$.

State affine models can be used to represent nonlinear systems, particularly systems which appear naturally affine-in-the-states. For example, the exact sampled model (1.47) of the continuous-time bilinear state space system is affine-in-the-states.

A system admits an *output affine* model (1.52-53) if and only if, it is a finite state affine system [Sontag, 1979a]. Output affine models can therefore be employed to represent nonlinear input output systems, see Billings, Korenberg and Chen, 1987, and Chen and Billings, 1988a and 1988b.

1.13. Discussion

Because there is no one unique representation for all nonlinear systems a wide variety of nonlinear model structures have evolved. Along with each specific model class comes one or more methods of analysis that has grown up in response to that particular model form. In an attempt to avoid being dependent on any one model structure we adopt an approach based upon the general parameterised system in (1.5) and (1.6). It is usually a trivial problem to recast the different model type, discussed above, into this form. In order to justify this decision we review in section 2 of this paper the powerful theories at our disposal by adopting such model structures.

Dynamical Systems Theory

2.0. Introduction

The study of smooth or differentiable dynamical systems has seen a reemergence in recent years prompted by the work of Smale, 1963, 1967, Andronov et al, 1937, 1966, 1971, 1973, Abraham, 1979, Abraham and Shaw, 1985, Hellman, 1980, and Gurel and Rossler, 1979. The work by Smale covers the construction and use of *Poincare Maps* as a tool to explain the complex behaviour of nonlinear systems. This work was probably the first to utilise diffeomorphisms or smooth invertible mappings in the study of limit cycles dynamical systems.

It was not until the work by Lorenz, 1963, in Meteorology, and by Hayashi, 1964, on electric circuits, did concrete practical examples of the use of the qualitative geometric techniques appear. Most of the work in this area focused on explaining the phenomena of chaos [Smale, 1967].

The theory is relatively complete for systems of one and two dimensions, see for instance Hirsh and Smale, 1974, Palis and Melo, 1977, and Guckenheimer and Holmes, 1983. Extensive bibliographies exist for the subject by Shiraiwa, 1981, and Zeeman, 1981. An excellent introduction to the topological ideas is to be found in the book by Abraham and Shaw, 1985. Chow and Hale, 1982 also provide a systematic presentation of this work. Sattinger, 1973, and Iooss and Joseph, 1981, treat the subject from the analytical viewpoint. Arnold, 1982, provides a taste of the extensive Soviet contribution.

The probabilistic approach to studying dynamical systems has been pursued mostly by the physics community, see for example Cornfield et al, 1982. Here the interaction between noisy and predictable deterministic behaviour is the focus of attention. In this the introduction of *universal* properties and *scaling* methods, originated by Fiegenbaum 1978, 1979, 1982 and 1983, brought a new perspective to the field. Complex behaviour of these apparently simple discrete mappings is considered by Collet and Eckmann, 1980a, 1980b, 1981, Lanford, 1982, and Rand et al, 1981, 1982.

The use of reduction methods play an important role in the study of dynamical systems. They provide a method by which to reduce systematically the dimension of a problem. The *center manifold* theorem represents perhaps the easiest and most general approach to this problem, see Kelly, 1967, Marsden and MaCracken, 1976, Carr, 1981, Guckenheimer and Holmes, 1983. Other approaches include the elimination of passive coordinates, see for instance Thompson and Hunt, 1973, the Fredholm alternative, see Iooss and Joseph, 1981, and the Lyapunov-Schmidt reduction process , see Chow and Hale, 1982.

Throughout the development of dynamical systems theory as a subject, the focus of attention has often been the more interesting dynamic phenomena found in chaotic systems. In this paper we concentrate on the more simple deterministic, but just as important, topological structures that are commonly found in engineering systems. In such systems the infinitesimal structures associated with fractals and chaotic dynamics are in general of less importance.

In this section the basic theory is reviewed. This is necessary in order to set the background for what will be called the *qualitative analysis* of nonlinear systems. The motivation behind this is twofold. Firstly, as will become evident later, the theory places few restrictions on the form of the nonlinear system. Secondly, the analysis begins where traditional linear theory breaks down. That is, when one or more of the linearised system eigenvalues becomes degenerate such that the linearised system is no longer valid. The theory then focuses on describing the changes to be expected in the solution structure of the system. As part of this process local stability information is provided.

This is in no way an attempt to review all the theory and mathematical concepts used in the study of dynamical systems. Only the basic building blocks used in the theory are outlined. The approach adopted here focuses on the geometric aspects of the theory. For a more analytic discussion of local bifurcation see Iooss and Joseph, 1981, Chow and Hale, 1982, or Golubitsky and Schaeffer, 1985, Golubitsky, Stewart and Schaeffer, 1988.

2.1. Linearisation and Bifurcation in ODE's

The word *bifurcate* means to split into two. In its most general form bifurcation theory is the study of the splitting of the solution structure of nonlinear systems. Its main aim is to describe the qualitative changes which take place in the solution structure of a differential equation or map depending on a distinguishing parameter μ.

One of the fundamental results used in bifurcation theory is the implicit function theorem. Consider a system of differential equations depending on the k dimensional parameter vector μ describing the flow Φ_t of an ODE in $\mathbf{R}^n$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\mu) \qquad \mathbf{x} \in \mathbf{R}^n \qquad \mu \in \mathbf{R}^k$$

$$\Phi_t : \mathbf{R}^n \times R^k \rightarrow \mathbf{R}^n \qquad \mathbf{f} = (f_1(x_1,..,x_n),..,f_n(x_1,..,x_n))^T \tag{2.1}$$

The equilibrium solutions of (2.1) are given by the solutions to

$$\mathbf{f}(\mathbf{x},\mu) = 0 \tag{2.2}$$

The equilibrium solutions of (2.1) are equivalent to the singular points of (2.2). As the parameter μ varies the Implicit Function theorem states that the equilibria of (2.1) are described by a smooth function $\overline{\mathbf{x}}(\mu)$ if the *Jacobian* derivative of $\mathbf{f}$ evaluated at $\overline{\mathbf{x}}$ has non-zero eigenvalues.

The graph of the function $\overline{\mathbf{x}}(\mu)$ forms a *branch* of equilibria of (2.1), referred to as the *zeros, fixed points* or *stationary* solutions of $\mathbf{f}$. Enumeration of the fixed points of a general nonlinear system (2.1) is by no means a trivial task [Brindley et al, 1989].

Linearisation of (2.1) about a fixed point $\overline{\mathbf{x},\mu}$ gives

$$\xi = D_x\mathbf{f}(\overline{\mathbf{x},\mu})\cdot\xi \qquad \xi\in\mathbf{R}^n$$

$$\xi_i = x_i - \overline{x_i} \qquad |\,\xi_i\,| \ll 1 \quad i = 1,..,n \tag{2.3}$$

where the Jacobian matrix $D_x\mathbf{f}(\cdot)$, given by

$$D_x\mathbf{f}(\overline{\mathbf{x},\mu}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & ,..., & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & ,..., & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(\mathbf{x}=\overline{\mathbf{x}},\,\mu=\overline{\mu})} \tag{2.4}$$

If the eigenvalues of the Jacobian all have nonzero real part then the fixed point $\overline{\mathbf{x}}, \overline{\mathbf{u}}$ is said to be *hyperbolic* or *nondegenerate*. The asymptotic behaviour of solutions in a neighbourhood about $\overline{\mathbf{x},\mathbf{u}}$ is then determined by the linearisation (2.3). This is a consequence of Hartman's theorem and the Stable Manifold theorem [Arnold, 1973].

At a hyperbolic fixed point the Stable Manifold theorem tells us that a local *stable*, $\mathbf{W}^s_{loc}$, and *unstable*, $\mathbf{W}^u_{loc}$, manifolds exists such that [Carr, 1981]

$$\mathbf{W}^s_{loc}(\mathbf{x}) = \left\{ \overline{\mathbf{x}}\in\mathbf{U} \mid \Phi_t(\mathbf{x})\to\overline{\mathbf{x}}\ as\ t\to\infty \text{ and } \Phi_t(\mathbf{x})\in\mathbf{U},\ \forall t\geq 0 \right\} \tag{2.5}$$

$$\mathbf{W}^u_{loc}(\mathbf{x}) = \left\{ \overline{\mathbf{x}}\in\mathbf{U} \mid \Phi_t(\mathbf{x})\to\overline{\mathbf{x}}\ as\ t\to-\infty \text{ and } \Phi_t(\mathbf{x})\in\mathbf{U},\ \forall t\leq 0 \right\} \tag{2.6}$$

The stable subspace $\mathbf{E}^s$, of dimension n_s, is characterised by contracting or exponentially decaying solutions such that $\lambda_i < 0$, $i = 1,..n_s$. The unstable subspace $\mathbf{E}^u$, of dimension n_u, by expanding or exponentially growing solutions, $\lambda_i > 0$, $i = 1,..n_u$.

If any of the eigenvalues λ_i have zero real part, then the asymptotic stability of (2.1) cannot be determined by the linearisation (2.3). At a *nonhyperbolic*, or *degenerate* fixed point one or more of the linearised eigenvalues is zero. The center manifold, $\mathbf{E}^c$, of dimension n_c, is defined as the subspace spanned by the eigenvectors corresponding to the eigenvalues $\lambda_i = 0$, $i = 1,..n_c$. At such a point branches of equilibrium solutions may come together. These points are termed *bifurcation points*.

2.2. Linearisation and Bifurcation in Maps

The theory for *Maps* or *Discrete* dynamical systems parallels that for flows. Consider the mapping

$$\mathbf{x}_{n+1} = \mathbf{g}(\mathbf{x}_n, \mu) \qquad \mathbf{x} \in \mathbf{R}^n \ ; \ \mu \in \mathbf{R}^k \qquad (2.7)$$

$$\mathbf{g} : \mathbf{R}^n \times \mathbf{R}^k \rightarrow \mathbf{R}^n \qquad \mathbf{f} = (f_1(x_1,..,x_n),..., f_n(x_1,..,x_n)) \qquad (2.8)$$

The stationary solutions $\overline{\mathbf{x}}$ of (2.8) are given by

$$\mathbf{g}^k(\overline{\mathbf{x}}, \mu) - \overline{\mathbf{x}} = 0 \qquad k = 1,2,3,... \qquad (2.9)$$

However, the solution to (2.8) may prove even more troublesome than for the continuous system (2.1) [Bernussou, 1977, Gumowski and Mira, 1980].

Linearisation of (2.7) about the fixed point $\overline{\mathbf{x}, \mu}$ gives

$$\xi_{n+1} = D_x\mathbf{g}(\overline{\mathbf{x}}, \overline{\mu})^{\cdot} \xi_n \qquad \xi \in \mathbf{R}^n$$

$$\xi_i = x_i - \overline{x_i} \qquad |\xi_i| \ll 1 \ ; \ i = 1,..,n \qquad (2.10)$$

The stable subspace $\mathbf{E}^s$, of dimension n_s, is characterised by contracting solutions, $\lambda_i < 1$, $i = 1,..n_s$. The unstable subspace $\mathbf{E}^u$, of dimension n_u, by expanding solutions, $\lambda_i > 1$, $i = 1,..n_u$. The center manifold, $\mathbf{E}^c$, of dimension n_c, is the subspace spanned by the eigenvectors corresponding to the eigenvalues $|\lambda_i| = 1$, $i = 1,..n_c$ such that

$$\mathbf{W}^s_{loc}(\mathbf{x}) = \left\{ \overline{\mathbf{x}} \in \mathrm{U} \mid g^n(\mathbf{x}) \rightarrow \overline{\mathbf{x}} \ \ as \ n \rightarrow \infty \text{ and } g^n(\mathbf{x}) \in \mathrm{U}, \ \forall n \geq 0 \right\} \qquad (2.11)$$

$$\mathbf{W}^u_{loc}(\mathbf{x}) = \left\{ \overline{\mathbf{x}} \in \mathrm{U} \mid g^{-n}(\mathbf{x}) \rightarrow \overline{\mathbf{x}} \ as \ n \rightarrow \infty \text{ and } g^{-n}(\mathbf{x}) \in \mathrm{U}, \ \forall n \geq 0 \right\} \qquad (2.12)$$

For a k-periodic cycle of the form

$$\mathbf{p}_j = \mathbf{g}^j(\mathbf{p}_0) \qquad j = 0,..,k-1 \qquad \mathbf{p}_0 = \mathbf{g}^k(\mathbf{p}_0) \qquad (2.13)$$

the stability is dependent upon the linearised map

$$D_x g^k(\mathbf{p}_0) \quad \text{or} \quad D_x g^k(\mathbf{p}_j) \qquad j = 1 \text{ or } 2 \text{ or } \cdots \text{ or } k \tag{2.14}$$

2.3. Local Codimension One Bifurcation

It is possible to list the simplest bifurcations found in families of ODE's depending upon one parameter [Guckenheimer and Holmes, 1983]. These elementary bifurcations are *generic*, that is, they appear in all the typical problems. Sometimes alone, but more often as elements in a more complex overall picture, as part of a *global* bifurcation. Given the one parameter system of ODE's described by

$$\dot{x}(t) = f(x(t), \mu) \qquad x \in \mathbf{R} \; ; \; \mu \in \mathbf{R} \; ; \; f : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \tag{2.15}$$

If the equilibrium $\overline{x,\mu}$ has a simple zero eigenvalue $\lambda = 0$ and

$$f_x(\overline{x}, \overline{\mu}) = 0 \; ; \; f_\mu(\overline{x}, \overline{\mu}) \neq 0 \; ; \; f_{xx}(\overline{x}, \overline{\mu}) \neq 0 \tag{2.16}$$

then a smooth curve of equilibrium of quadratic tangency exists which is qualitatively equivalent to the generic *Saddle Node*, Fig. (1), and is described by the normal form

$$\dot{x}(t) = \mu - x^2 \tag{2.17}$$

The great importance of the Saddle Node or *Fold* bifurcation lies in its structural stability. Indeed all bifurcations in one parameter families with a single degenerate eigenvalue can be perturbed to a Saddle Node.

If the form of the ODE is constrained such that the trivial equilibrium solution $x = 0$ exists for all μ then the Saddle Node is no longer possible and

$$f(0, \mu) = 0 \qquad \forall \mu \in \mathbf{R} \tag{2.18}$$

If the equilibrium $\overline{x,\mu}$ has a simple zero eigenvalue $\lambda = 0$ and

$$f_x(\overline{x}, \overline{\mu}) = 0 \qquad f_{\mu x}(\overline{x}, \overline{\mu}) \neq 0 \qquad f_{xx}(\overline{x}, \overline{\mu}) = 0 \tag{2.19}$$

then the resulting structurally stable bifurcation is *Transcritical*, Fig. (2), and is described by

$$\dot{x}(t) = \mu x - x^2 \tag{2.20}$$

An exchange of stability occurs at the bifurcation point $\overline{x,\mu}$, the stable origin becoming unstable in the supercritical case and the nontrivial equilibrium gains stability.

If the ODE represents a system with symmetry then the function f is odd and

$$f(-x\,,\mu) \;=\; -f(x\,,\mu) \qquad (2.21)$$

The system cannot satisfy the condition $f_{\mu x}(\overline{x,\mu})=0$, however if

$$f_x(\overline{x}\,,\overline{\mu}) = 0 \qquad f_\mu(\overline{x}\,,\overline{\mu}) \neq 0 \qquad f_{xxx}(\overline{x}\,,\overline{\mu}) \neq 0 \qquad (2.22)$$

then the resulting structurally stable bifurcation is the *Pitchfork*, Fig. (3), and is described by

$$\dot{x}(t) \;=\; \mu x - x^3 \qquad (2.23)$$

Here the sink at the origin looses stability and two stable symmetrically placed sinks are produced in the supercritical case.

If the ODE is such that the linearisation has a simple pair of purely imaginary eigenvalues $\lambda=\pm i\omega$ and no other eigenvalues with zero real part then the center manifold system will be such that center manifold E^c is of dimension $n_c=2$.

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{f}(\,\mathbf{x}(t)\,,\mu\,) \qquad \mathbf{x}\in\mathbf{R}^2 \qquad \mu\in\mathbf{R} \qquad (2.24)$$

As $\mathbf{f}_x(\mathbf{x},\mu)$ is invertible the implicit function theorem guarantees a smooth curve of equilibrium $\mathbf{x}(\mu)$ near $\overline{\mathbf{x},\mu}$. However the dimensions of the stable and unstable manifolds change if the eigenvalues $\lambda(\mu)$ cross the imaginary axis. Thus a qualitative change not involving equilibria must occur. If $\mathbf{f}_x\,(\overline{\mathbf{x},\mu})$ in (2.24) has a simple pair of pure imaginary eigenvalues and

$$\lambda_{1,2}(\mu) \;=\; \pm i\omega \qquad \text{Real}\left[\lambda_3,..,\lambda_n\right] \;<\; 0 \qquad \frac{\partial}{\partial\mu}\text{Real}\,[\lambda(\mu)]\;\big|_{\,\mu=\overline{\mu}} \;\neq\; 0 \qquad (2.25)$$

then a unique center manifold exists passing through $\overline{\mathbf{x},\mu}$ with normal form

$$\dot{x}(t) = (\,d\mu+a\,(x^2+y^2)\,)x \;-\; (\,w+c\mu+b\,(x^2+y^2)\,)y$$

$$\dot{y}(t) = (\,w+c\mu+b\,(x^2+y^2)\,)x \;+\; (\,d\mu+a\,(x^2+y^2)\,)y \qquad a\neq 0$$

A *Hopf* bifurcation occurs by which a surface of periodic solutions exists in the center manifold. If $a>0$ the periodic solutions are attracting, see Fig. (4).

Following on from the study of degenerate equilibrium points one should next consider the recurrent behaviour found in periodic cycles. Two main types of phenomena appear. Firstly the *Cyclic Fold* which is similar to the Saddle Node bifurcation for equilibria. In this two limit cycles coexist over a small parameter region and at the bifurcation point they collide. The Cyclic Fold is often associated with jump resonance or hysteresis.

Next there is the periodic Flip bifurcation in which a stable limit cycle losses stability while another closed orbit whose period is twice that of the original cycle is born.

The classification of bifurcating solutions based upon transversality becomes more difficult as one continues. Primarily because the behaviour sought, along with the degenerate eigenvalue combinations, become less typical, indeed less generic. This is particularly so if one attempts the classification of codimension two families, see for example Aronson et al, 1982.

2.4. Local Codimension One Bifurcations in Maps

There are a number of similarities between the local bifurcational behaviour found in maps or discrete dynamical systems and those that occur in ODE's or flows. Consider the Map

$$x_{k+1} = g(x_k, \mu) \qquad x \in \mathbf{R} \quad \mu \in \mathbf{R} \quad g: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \tag{2.26}$$

The fixed points $\overline{x(\mu)}$ of (2.26) then satisfy the relation

$$g(\overline{x(\mu)}, \mu) - \overline{x(\mu)} = 0 \tag{2.27}$$

and are hyperbolic if the linearisation of (2.23) has no eigenvalue $|\lambda| = 1$. Again a set of elementary bifurcations exist which may be classified according to a set of *transversality* conditions applied to their normal forms. The bifurcation theory for fixed points with eigenvalue $\lambda = 1$ is exactly analogous to that for equilibria with eigenvalue $\lambda = 0$

If the fixed point $\overline{x,\mu}$ of (2.26) is non-hyperbolic, $\lambda(\overline{\mu}) = 1$, and

$$g_x(\overline{x}, \mu) = \lambda(\mu) \qquad g_\mu(\overline{x}, \overline{\mu}) \neq 0 \qquad g_{xx}(\overline{x}, \overline{\mu}) \neq 0 \tag{2.28}$$

then there is a Saddle Node bifurcation in the vicinity of $\overline{x,\mu}$, Fig. (1). The generic one parameter family has a center manifold topologically equivalent to the normal form

$$x_{n+1} = x_n + \mu - x_n^2 \tag{2.29}$$

The signs of the inequalities in the above determine the exchange of stability that occurs at the bifurcation point [Whitley, 1982].

If the form of the Map is restricted such that the trivial fixed point $0,\mu$ exists for all μ then

$$g(0, \mu) = 0 \qquad \forall \mu \in \mathbf{R} \tag{2.30}$$

If the equilibrium $\overline{x,\mu}$ has a simple eigenvalue $\lambda(\overline{\mu}) = 0$ and

$$g_x(0, \overline{\mu}) = \lambda(\mu) \qquad \frac{\partial \lambda}{\partial \mu} = g_{x\mu}(0, \overline{\mu}) \neq 0 \qquad g_{xx}(0, \overline{\mu}) = 0 \tag{2.31}$$

then there is a Transcritical bifurcation as in Fig. (2), described by the normal form

$$x_{n+1} = (\mu - 1)\, x_n\, (1 - x_n) \tag{2.32}$$

If the map represents a system with symmetry then the function g is odd such that $g(-x,\mu) = -g(x,\mu)$ and additionally if $\lambda(\overline{\mu}) = 1$ and

$$g_x(\overline{x}, \overline{\mu}) = \lambda(\mu) \qquad \frac{\partial \lambda}{\partial \mu} = g_{x\mu}(\overline{x}, \overline{\mu}) \neq 0 \qquad g_{xxx}(\overline{x}, \overline{\mu}) \neq 0 \tag{2.33}$$

then a Pitchfork bifurcation is present in the vicinity of $\overline{x,\mu}$, Fig. (3), the corresponding normal form for which is given by

$$x_{n+1} = (1-\mu)\, x_n - x_n^3 \tag{2.34}$$

If the mapping possesses a fixed point $\overline{x,\mu}$ at which the eigenvalue $\lambda = -1$ then the implicit function theorem guarantees a smooth curve $\overline{x(\mu)}$ of fixed points passing through $\overline{x,\mu}$. Furthermore if the composite mapping $g^2(x,\mu)$ has an eigenvalue $\lambda = +1$ then there may be fixed points of $g^2(x,\mu)$ which do not appear in $g(x,\mu)$. Such points are period two, or **P-2**, cycles.

The transversality conditions for this Flip bifurcation are are summarised as, given $\lambda(\overline{\mu}) = -1$ and

$$g_x(x, \mu) = \lambda(\mu) \qquad \frac{\partial \lambda}{\partial \mu} = g_{x\mu}(\overline{x}, \overline{\mu}) \neq 0 \qquad g^2_{xxx}(\overline{x}, \overline{\mu}) \neq 0 \tag{2.35}$$

The Flip is shown in Fig. (5), the corresponding normal form for which is given by

$$x_{n+1} = -(1+\mu)\, x_n + x_n^3 \tag{2.36}$$

If the mapping g is a one parameter family with a fixed point $\overline{x,\mu}$ such that

$$g : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n \tag{2.37}$$

and a single eigenvalue passes through $\lambda = \pm 1$ then any of the above bifurcational phenomena may occur. Construction of the appropriate center manifold reduced order system, and checking the transversality conditions above, will then indicate which of the above occurs.

Alternatively a Hopf type bifurcation, similar to that which occurs in ODE's, may occur if a complex conjugate pair of eigenvalues exists, $\lambda = \pm\omega i$, having non-zero real parts.

If $g(\mathbf{x},\mu)$ is a smooth mapping having a fixed point $\overline{\mathbf{x},\mu}$ at which a complex pair of eigenvalues exist such that

$$|\lambda(\overline{\mu})| = 1 \qquad |\lambda^n(\overline{\mu})| \neq 1 \quad n = 1,2,3,4 \qquad \frac{\partial\, |\lambda(\overline{\mu})|}{\partial \mu} = d \neq 0 \tag{2.38}$$

then a smooth change of coordinates exists such that $g(\mathbf{x},\mu)$ may be written in polar form

$$r_{n+1} = r\,(1 + d\,(\mu - \overline{\mu})) + a\,r^2 + HOT's$$

$$\theta_{n+1} = \theta + c + b\,r^2 + HOT's \qquad\qquad a\,(\overline{\mu}) \neq 0 \tag{2.39}$$

and a 2-D surface $\sum$ exists in $\mathbf{R}^2 \times \mathbf{R}$ which is invariant for g. If the intersection of $\sum \cap (\mathbf{R}^2 \times \{\overline{\mu}\})$ forms a simple closed curve then cyclic behaviour will be evident. The stability of which is determined by the coefficient a [Lanford, 1973, or Marsden and MaCracken, 1976]. Stability formulae for the above have also been derived by Hassard and Wan, 1978, Iooss and Joseph, 1981 and Guckenheimer and Holmes, 1983.

To understand the dynamics of a family of Maps after the Hopf bifurcation see Takens, 1974, Arnold, 1977, Iooss, 1979, or Whitley, 1982.

2.5. Structural Stability

The idea of a *robust* or *structurally stable* system as one which retains its qualitative properties under small perturbations, originates in the work of Andronov and Pontryagin, 1937. Consider a map $\mathbf{f} \in \mathbf{C}^r(\mathbf{R}^n)$ and a perturbation of this map $\mathbf{g}$. Defining how close $\mathbf{g}$ is to $\mathbf{f}$ requires consideration of the functional spaces and topology [Hirsh, 1976]. Here we make use of the following [Golubitsky and Schaeffer, 1985].

If $\mathbf{f} \in \mathbf{C}^r(\mathbf{R}^n)$ and $r, k \in \mathbf{Z}^+$, $k \leq r$ and $\varepsilon > 0$ then $\mathbf{g}$ is a $\mathbf{C}^k$ *perturbation* of size ε if there is a compact set $\mathbf{K} \subset \mathbf{R}^n$ such that $\mathbf{f} = \mathbf{g}$ on the set $\mathbf{R}^n - \mathbf{K}$ and

$$\left| \frac{\partial^i}{\partial x_1^i \ldots \partial x_n^i} (\mathbf{f} - \mathbf{g}) \right| < \varepsilon \qquad \forall i_1,..,i_n \qquad i_1 + \cdots + i_n = i \leq k \tag{2.40}$$

The functions $\mathbf{f}$ and $\mathbf{g}$ may be either vector fields or mappings. The closeness of the two vector fields, or maps, is then defined in terms of a topological equivalence.

Two $\mathbf{C}^r$ maps $\mathbf{f}$ and $\mathbf{g}$ are $\mathbf{C}^k$ *conjugate* if there exists a $\mathbf{C}^k$ homeomorphism $\mathbf{h}$ such that

$$\mathbf{h} \circ \mathbf{f} = \mathbf{g} \circ \mathbf{h} \tag{2.41}$$

$\mathbf{C}^0$ equivalence is called *topological equivalence.* The definition implies that $\mathbf{h}$ maps an orbit within $\{ \mathbf{f}^n(\mathbf{x}) \}$ into an orbit of $\{ \mathbf{g}^n(\mathbf{x}) \}$. This property is defined as *orbit equivalence.*

Two $\mathbf{C}^r$ maps $\mathbf{f}$ and $\mathbf{g}$ are $\mathbf{C}^k$ *equivalent* if there exists a $\mathbf{C}^k$ diffeomorphism h which takes orbits $\Phi_t^f(\mathbf{x})$, of $\mathbf{f}$, to orbits $\Phi_t^g(\mathbf{x})$, of $\mathbf{g}$, preserving the sense of the flows.

If **h** also preserves the parameterisation with time then we have a *conjugate* equivalence.

$$\mathbf{h}\,(\Phi^{f}_{t_1}(\mathbf{x})\,) \;=\; \Phi^{g}_{t_2}(\,\mathbf{h}(\mathbf{x})\,) \tag{2.42}$$

A map $\mathbf{f} \in \mathbf{C}^r(\mathbf{R}^n)$, or a $\mathbf{C}^r$ vector field is *structurally stable* if there is an $\varepsilon > 0$ such that all $\mathbf{C}^1$, ε perturbations of **f** are topologically equivalent to **f**. Hence vector field or map which possesses nonhyperbolic fixed points cannot be structurally stable. Bifurcation of equilibria usually produce changes in the topological type of a flow or mapping. A value $\overline{\mu}$ for which the flow or mapping is not structurally stable is a *bifurcation* value of μ. If the linearised Jacobian has a degenerate eigenvalue a small perturbation may remove the fixed point. Purely imaginary eigenvalues may perturb to yield a hyperbolic sink, saddle or source. Similar effects are produced for degenerate periodic orbits or cycles.

Structural stability thus requires all fixed points and cycles to be hyperbolic, such that any sufficiently *close* system has the same qualitative behaviour. Generic properties of dynamical systems are discussed further in Peixoto, 1962, and Hirsh and Smale, 1974. In this work we do not concern ourselves with measuring this *closeness* but content ourselves with observing qualitative similar behaviour experimentally.

2.6. Classification of Bifurcation

Bifurcation of equilibria generally produces a qualitative change in the topological type of a flow or map. Attempts to construct a systematic bifurcation theory based upon the above lead however to complications as the fine detail of the function must be considered. A number of distinct lines of attack have been developed.

The simplest approach is that based upon studying the *degeneracy* of eigenvalues. The classification of bifurcation behaviour has been formulated based around the qualitative features that persist under perturbation. A number of normal forms are then constructed to represent this behaviour along with the different arrangements of degenerate eigenvalues.

A second approach is that based upon the *Singularity Theory*. The use of singularity methods [Golubitsky and Schaeffer, 1985] provides a comprehensive approach to the classification for problems up to *codimension* three. Where the codimension of a system is defined as the smallest dimension of parameter space which contains all the qualitative bifurcation behaviour in a persistent manner. The classification again comprises of a set of normal forms, Table (2.1), along with the corresponding unperturbed bifurcation diagrams, Fig. (6), for each normal form

$$h\,(x\,,\mu\,) \;=\; 0 \qquad \mathbf{x} \in \mathbf{R} \qquad \mu \in \mathbf{R} \tag{2.43}$$

Normal Form		Codim	Nomenclature
1.	$\varepsilon x^2 + \delta\lambda$	0	Limit Point
2.	$\varepsilon x^2 - \delta\lambda$	1	Simple Bifurcation
3.	$\varepsilon(x^2 + \delta\lambda^2)$	1	Isola Center
4.	$\varepsilon x^3 + \delta\lambda$	1	Hysteresis
5.	$\varepsilon x^2 + \delta\lambda^3$	2	Asymmetric Cusp
6.	$\varepsilon x^3 + \delta\lambda x$	2	Pitchfork
7.	$\varepsilon x^4 + \delta\lambda$	2	Quartic Fold
8.	$\varepsilon x^2 + \delta\lambda^4$	3	
9.	$\varepsilon x^3 + \delta\lambda^2$	3	Winged Cusp
10.	$\varepsilon x^4 + \delta\lambda x$	3	
11.	$\varepsilon x^5 + \delta\lambda$	3	
N.B $\varepsilon = \pm 1$ and $\delta = \pm 1$			

Table 2.1

Normal Forms for Singularities of codimension ≤ 3

Golubitsky and Schaeffer make extensive use of the following definition.

Define- an *Unfolding* of a bifurcation as a family which contains that bifurcation in a *persistent* way.

The key to the above definition is the term persistent, which can be taken to imply qualitatively similar dynamics. A classification in terms of the type or number of degenerate eigenvalues is not sufficient however, as it does not take into account the natural unfolding found in the families of bifurcation problems.

For each normal forms $h\,(x,\mu)$ a *universal unfolding* $H(x,\mu,\alpha)$ exists which characterises all possible perturbations introduced by the parameter vector α up to codimension $k \leq 3$, see Table (2.2).

$$H\,(x\,,\mu\,,\alpha) \;=\; 0 \qquad x \in \mathbf{R}\;;\;\mu \in \mathbf{R}\;;\;\alpha \in \mathbf{R}^k \tag{2.44}$$

Fig. (7) illustrates the family of perturbations to be expected within the unfolding for two of the more common families known as the *hysteresis* and *pitchfork* varieties. The attraction of this theory lies in the fact that beyond the perturbations listed no further qualitative behaviour is to be expected. Note how each universal unfolding reduces to the corresponding normal form when $\alpha = 0$.

	Universal Unfolding	Codim	Nomenclature
1.	$\varepsilon x^2 + \delta\lambda$	0	Limit Point
2.	$\varepsilon(x^2 - \delta\lambda^2 + \alpha)$	1	Simple Bifurcation
3.	$\varepsilon(x^2 + \delta\lambda^2 + \alpha)$	1	Isola Center
4.	$\varepsilon x^3 + \delta\lambda + \alpha x$	1	Hysteresis
5.	$\varepsilon x^2 + \delta\lambda^3 + \alpha_1 + \alpha_2\lambda$	2	Asymmetric Cusp
6.	$\varepsilon x^3 + \delta\lambda x + \alpha_1 + \alpha_2 x^2$	2	Pitchfork
7.	$\varepsilon x^4 + \delta\lambda + \alpha_1 x + \alpha_2 x^2$	2	Quartic Fold
8.	$\varepsilon x^2 + \delta\lambda^4 + \alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2$	3	
9.	$\varepsilon x^3 + \delta\lambda^2 + \alpha_1 + \alpha_2 x + \alpha_3\lambda x$	3	Winged Cusp
10.	$\varepsilon x^4 + \delta\lambda x + \alpha_1 + \alpha_2\lambda + \alpha_3 x^2$	3	
11.	$\varepsilon x^5 + \delta\lambda + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$	3	
	N.B $\varepsilon = \pm 1$ and $\delta = \pm 1$		

Table 2.2

Universal Unfoldings for Elementary Bifurcations of Codimension ≤ 3

The mechanism for this classification of behaviour is split into two distinct steps. First the so called *recognition* problem for the normal form $h(x,\mu)$ is stated as follows. Given

$$g(x,\mu) = 0 \qquad x \in \mathbf{R} \qquad \mu \in \mathbf{R} \tag{2.45}$$

obtained from a reduced order ODE or Map, what conditions are required for g to be equivalent to one of the normal forms, h, listed in Table (2.1), such that

$$g(x,\mu) \sim h(x,\mu) \tag{2.46}$$

As part of the second stage a k parameter unfolding of $g(x,\mu)$ is constructed, that is

$$G(x,\mu,\alpha) = 0 \qquad x \in \mathbf{R} \; ; \; \mu \in \mathbf{R} \; ; \; \alpha \in \mathbf{R}^k \tag{2.47}$$

such that

$$G(x,\mu,0) = g(x,\mu) \qquad \alpha_1 = \cdots = \alpha_k = 0 \tag{2.48}$$

The unfolding G is universal if it contains no redundancy in parameters, moreover the addition of any higher order terms to the function G will not add to the qualitative behaviour to be expected.

2.7. Reduction Methods in Bifurcation Theory

Virtually all rigorous methods in dynamical systems theory are limited to finite dimensional systems. In an attempt to reduce the dimension of the state space of the problems being considered, analysts have in general relied upon the invariant manifold theory. The use of some reduction method is implicit in all the classifications discussed above.

The *Center Manifold* theorem provides a means of systematically reducing the dimension of the state space involved in a bifurcation problem [Kelly, 1967, Carr, 1981]. Attention is then focused on the behaviour of the system in the reduced center manifold that contains all the essential behaviour of the system in the vicinity of the equilibrium point about which it is constructed.

The center manifold is an invariant manifold tangent to the center eigenspace $\mathbf{E}_c$ on which the interesting asymptotic behaviour of the system lies. The theory is useful because it allows the elimination of algebraically complexity due to none essential behaviour. For a rigorous statement of the theorem see Guckenheimer and Holmes, 1983.

Assuming now that the unstable manifold $\mathbf{E}_u$ is empty and that the linear part of the system has been transformed to block diagonal form with a known equilibrium at the origin, then the system (2.1) may be written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{B}\,\mathbf{x} + f(\mathbf{x},\mathbf{y}) \qquad && (\mathbf{x},\mathbf{y}) \in \mathbf{R}^{n_c} \times \mathbf{R}^{n_s} \;;\; \mathbf{B} : n_c \times n_c \;;\; \mathbf{C} : n_s \times n_s \\ \dot{\mathbf{y}}(t) &= \mathbf{C}\,\mathbf{y} + g(\mathbf{x},\mathbf{y}) \qquad && \mathbf{E}_c \in \mathbf{R}^{n_c} \;;\; \mathbf{E}_s \in \mathbf{R}^{n_s} \end{aligned} \tag{2.49}$$

Here the eigenvalues of **B** have zero real part and those of **C** have all negative real part and the functions f, g, f_x, g_x, f_y and g_y all vanish at the origin. The center manifold $\mathbf{W}^c_{loc}$ is a tangent to the center subspace $\mathbf{E}^c$, that is the $\mathbf{y}=\mathbf{0}$ subspace in (2.49). It may thus be represented by the local graph

$$\mathbf{W}^c_{loc} = \{ (\mathbf{x},\mathbf{y}) \mid \mathbf{y} = \mathbf{h}(\mathbf{x}) \} \qquad h : \mathbf{R}^{n_c} \rightarrow \mathbf{R}^{n_s} \;;\; \mathbf{U} \subset \mathbf{R}^{n_c} \tag{2.50}$$

where **U** is some neighbourhood of the fixed point at the origin. Projection of the vector field on

$\mathbf{y}=\mathbf{h}(\mathbf{x})$ onto $\mathbf{E}^c$ gives an approximation to the motion on $\mathbf{W}^c_{loc}$. The construction of the center manifold system involves the approximation of $\mathbf{h}(\mathbf{x})$ using

$$\mathbf{h}(x) \;=\; a\,x^2 \;+\; bx^3 \;+\; \cdots \;+\; HOT's \tag{2.51}$$

in

$$\dot{\mathbf{y}}(t) \;=\; D_x\,\mathbf{h}(\mathbf{x})\,\dot{\mathbf{x}} \;=\; D_x\,\mathbf{h}(\mathbf{x})\,[\,\mathbf{B}\,\mathbf{x} + f\,(\mathbf{x},\mathbf{h}(\mathbf{x}))\,] \;=\; [\,\mathbf{C}\,\mathbf{h}(\mathbf{x}) + g\,(\mathbf{x},\mathbf{h}(\mathbf{x}))\,]$$

such that the reduced order system becomes

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{B}\,\mathbf{x} + f\,(\,\mathbf{x},\mathbf{h}(\mathbf{x})\,) \tag{2.52}$$

The important point to note here is that the invariance properties of the center manifold guarantee that any small solutions bifurcating from the origin (0,0,0) must lie in the center manifold. Thus one can then follow the local evolution, or behaviour, of bifurcating families in the suspended family of center manifolds.

In much the same way as in ODE's, a center manifold system may be constructed for a discrete dynamical system or Map given

$$\mathbf{x}_{n+1} \;=\; \mathbf{B}\,\mathbf{x}_n \;+ f\,(\mathbf{x}_n\,,\,\mathbf{y}_n) \qquad (\mathbf{x},\mathbf{y}) \in \mathbf{R}^{n_c} \times \mathbf{R}^{n_s}\,;\, \mathbf{B} : n_c \times n_c\,;\, \mathbf{C} : n_s \times n_s$$

$$\mathbf{y}_{n+1} \;=\; \mathbf{C}\,\mathbf{y}_n \;+ g\,(\mathbf{x}_n\,,\,\mathbf{y}_n) \qquad \mathbf{E}_c \in \mathbf{R}^{n_c}\;;\; \mathbf{E}_s \in \mathbf{R}^{n_s} \tag{2.53}$$

Where the eigenvalues of $\mathbf{B}$ are on the unit circle, $|\,\lambda_i\,| = 1,\ i = 1,..,n_c$, and those of $\mathbf{C}$ are within the unit circle, $|\,\lambda_i\,| \leq 1,\ i = 1,..,n_s$. Again approximating $\mathbf{h}(\mathbf{x})$ with the Taylor series (2.51) gives the reduced order discrete center manifold system

$$\mathbf{x}_{n+1} \;=\; \mathbf{B}\,\mathbf{x}_n \;+ f\,(\mathbf{x}_n,\mathbf{h}(\mathbf{x}_n)) \tag{2.54}$$

Parameterised families may be reduced in a similar fashion. Subsequent application of either of the classification schemes discussed earlier completes the analysis.

2.8. Discussion

The methods outlined in this section provide us with a framework and a terminology to explain the behaviour of nonlinear dynamical systems. It is this framework that we make use of in Part II of this paper.

At best, for certain classes of nonlinear problems, the current methods provide a rigorous approach to the enumeration, and classification, of the global characteristics of a parameterised system and its unfolding under perturbation. Unfortunately the more broad based results the theory provides the more restrictive the form of model that may be considered.

Singularity methods appear at first to be very attractive, however, the theory of universal unfolding has only been developed fully for systems of order $n = 1$ up to codimension $k = 3$. Inevitably dependence on reduction methods beyond this introduces problems related to the construction and validity of the reduced order systems itself. Not least of all, the need for extensive a priori knowledge of the solution structure of the system, in order to approximate the center manifold. Indeed the location of the degenerate singularities must be known precisely.

In reality of course detailed information of this type, over a range of parameters, will rarely be known. In addition, reduction methods, when employed, are only locally valid. These problems are magnified in discrete systems where a profusion of periodic behaviour is all to common.

It is important however to see how the behaviour of a model changes if the equations that make up that representation change in some manner. If only because such models are seldom known accurately. The augmentation of system models with one or more parameters allows the classification of such change. Mathematicians place the problem within the framework of structural stability and bifurcation theory which we have reviewed briefly above. The analytical basis of bifurcation theory provides a conceptual basis within which the behaviour of systems dependent upon several parameters can be considered. A number of tools have been described, most notably the invariant manifold theory and singularity theory, which facilitate detailed qualitative understanding of many problems. These theories although undoubtedly powerful do reveal practical limitations. This explosion in qualitative knowledge has stimulated demand for quantitative detail as well. All the analytical methods require precise a priori knowledge of the solution structure of the nonlinear equations before any classification can take place.

In an attempt to circumvent these limitations we adopt a numerical approach to the analysis of the general parameterised system, (2.1) and (2.7), which will be introduced in Part II of this paper.

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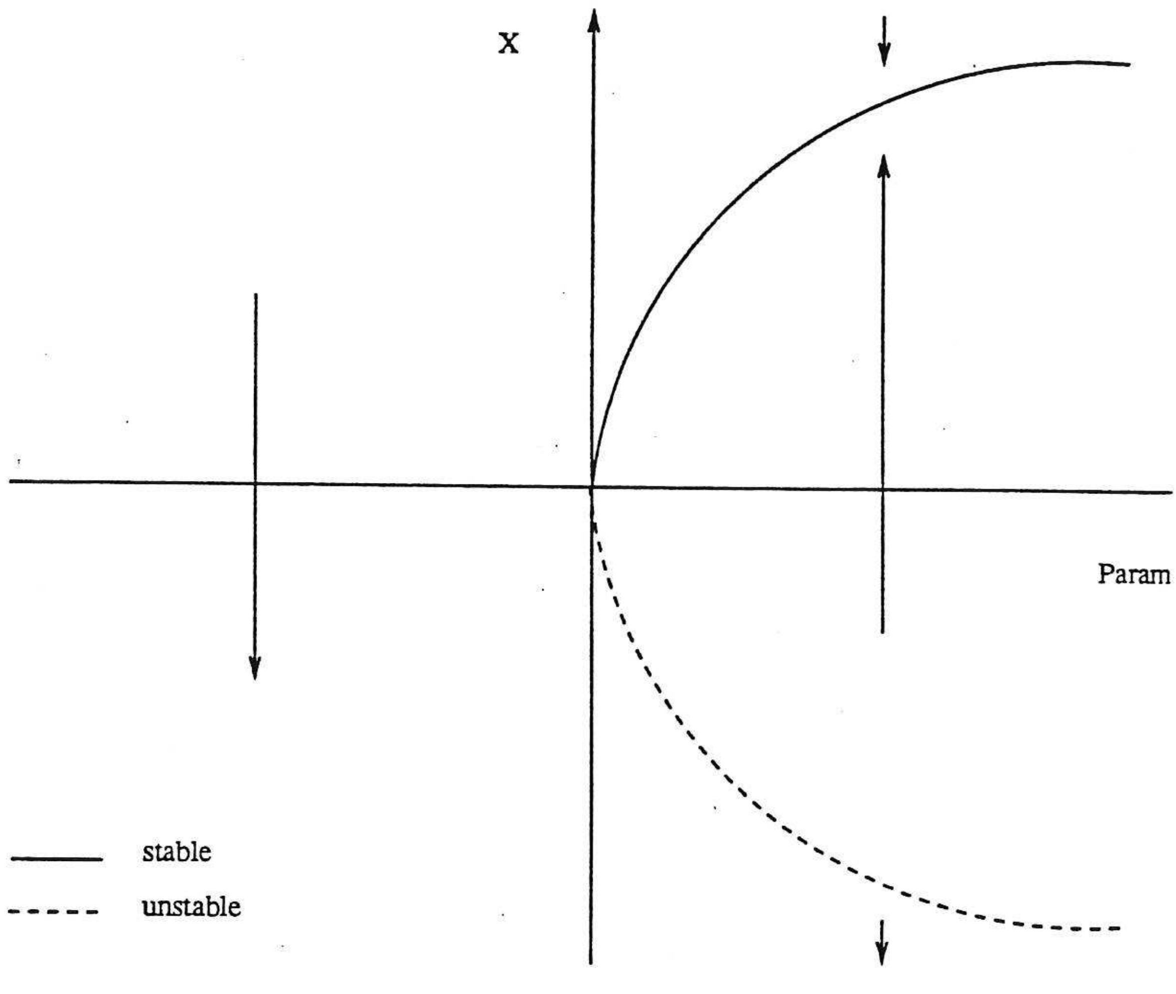

Fig. 1 Saddle Node Generic Codimension 1 Bifurcation

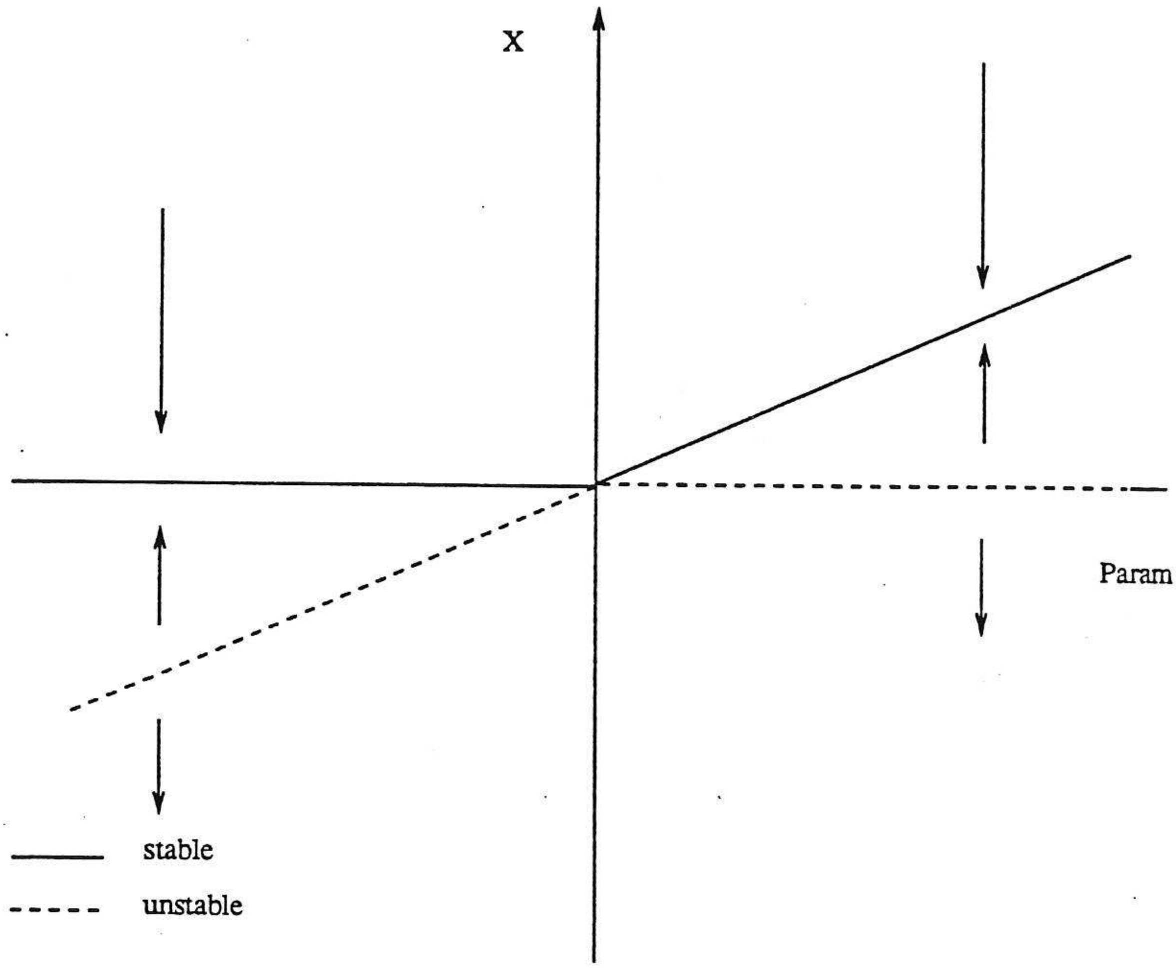

Fig. ' 2 Transcritical Generic Codimension-1 Bifurcation

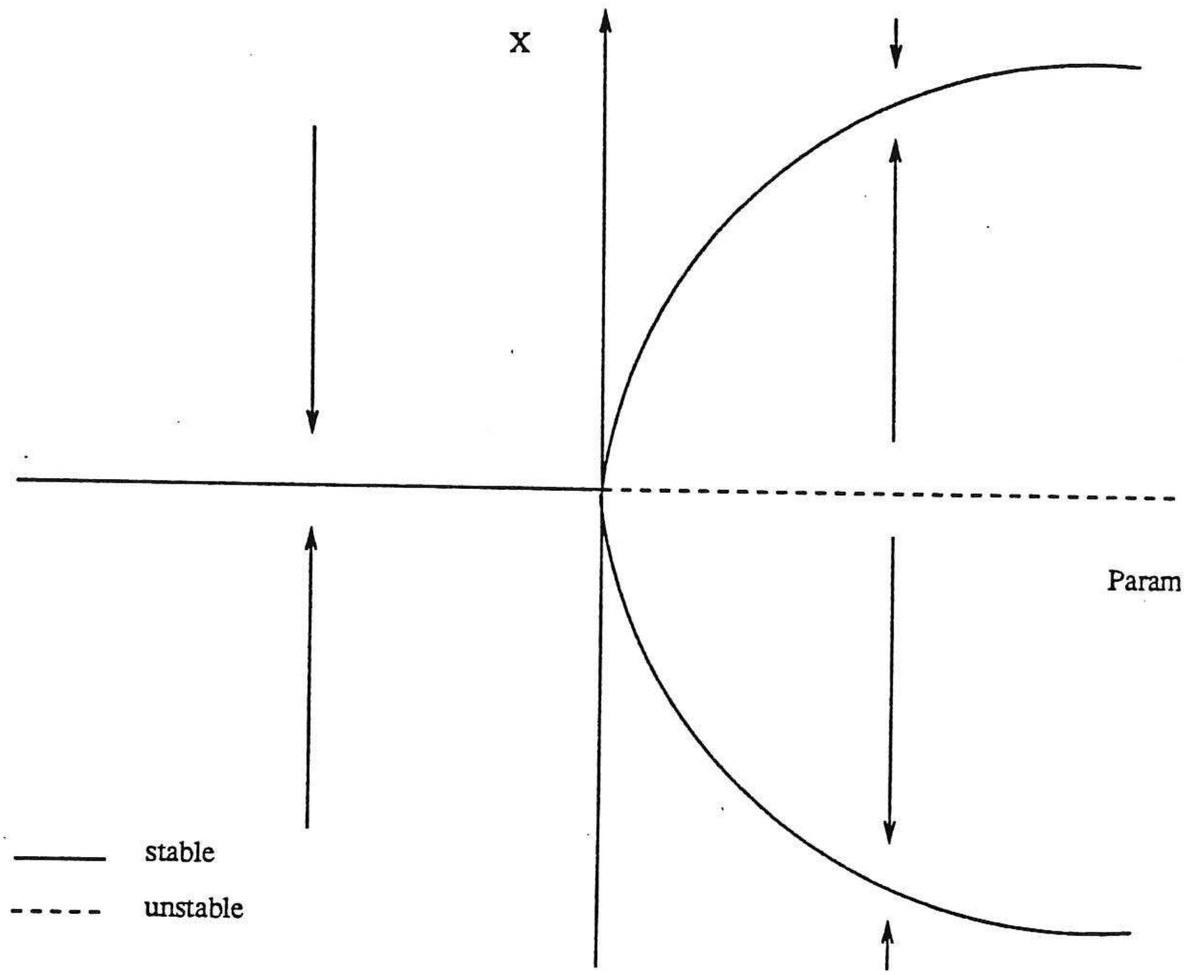

Fig. 3 Pitchfork Generic Codimension-1 Bifurcation

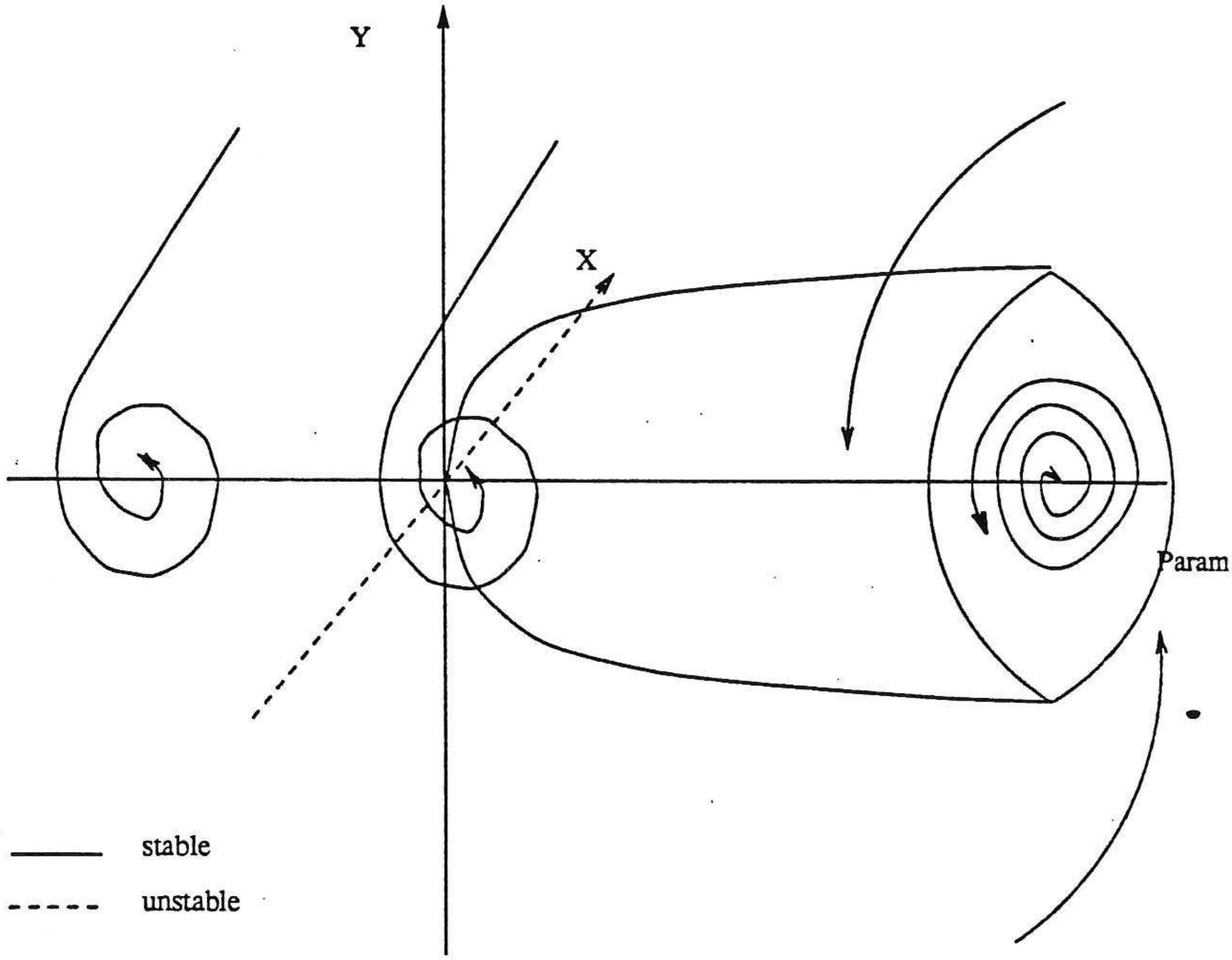

Fig. 4 Hopf Bifurcation Generic Codimension-1 Bifurcation

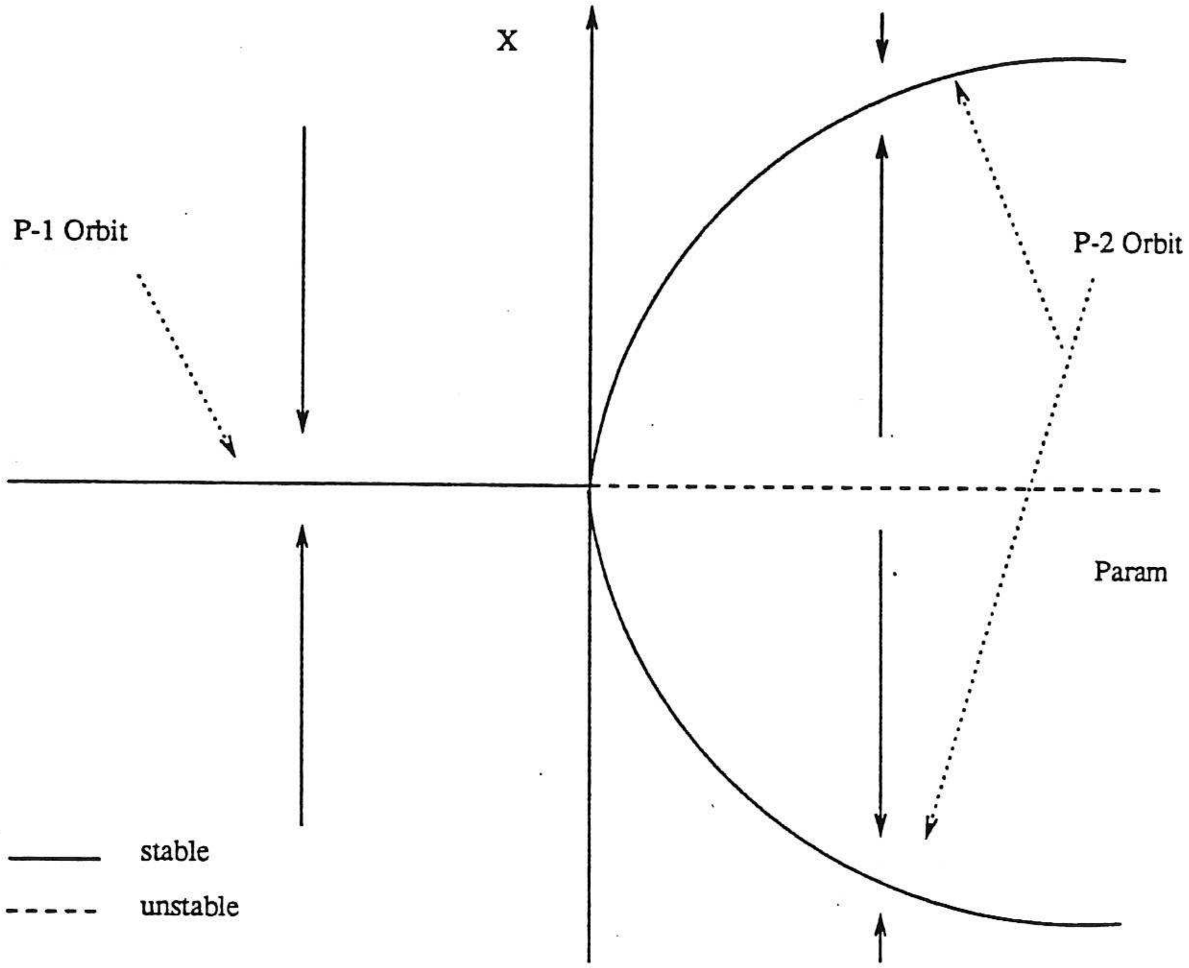

Fig. 5 Flip Period Doubling Generic Codimension-1 Bifurcation

	Normal Form
(1)	Limit Point
(2) (3)	Simple Bifurcation Isola Center
(4)	Hysteresis
(5)	Asymmetric Cusp
(6)	Pitchfork
(7)	Quartic Fold
(8)	
(9)	Winged Cusp
(10)	
(11)	

Unperturbed Bifurcation Diagram

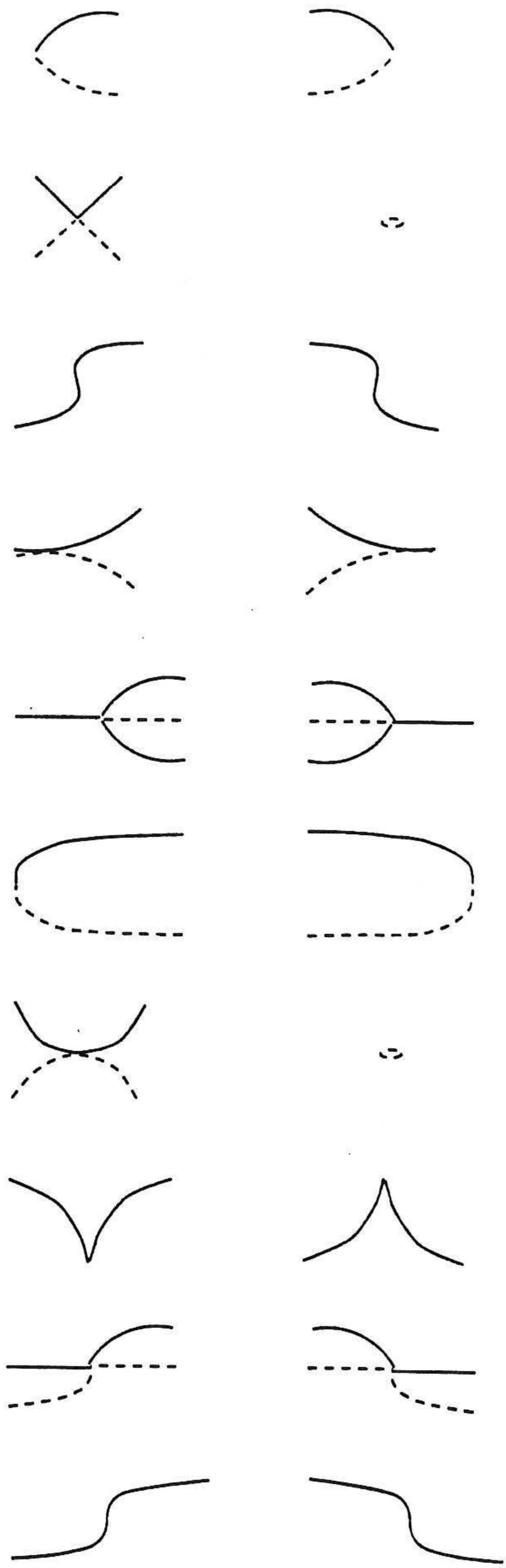

Fig. 6 Normal Form Bifurcation Diagrams - Unperturbed

Universal Unfolding | Perturbed Bifurcation Diagrams

(4) Hysteresis

Unperturbed

Perturbed

(6) Pitchfork

Unperturbed

Perturbed

Perturbed

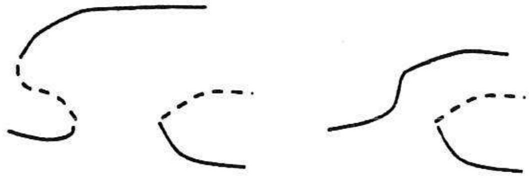

Fig. 7 Universal Unfoldings - Persistent Perturbations