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Reconstruction Of Linear and Nonlinear Continuous Time Models

From Discrete Time Sampled-Data Systems

by

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Abstract

A new approach for identifying continuous time models from discrete time sampled-data records is presented. The proposed method involves estimating and validating a discrete time model, linear or nonlinear, based on sampled data records, evaluating the discrete time linear and nonlinear frequency response functions and then curve fitting to the frequency response data to yield a continuous time model. No numerical differentiation and integration is involved and hence higher derivatives of input and output data records are avoided. Errors which would be introduced by the numerical approximation of differentiation and integration are therefore eliminated. The orthogonal estimator which is introduced to curve fit to the complex frequency response functions provides information on the model structure and the unknown parameter values for linear and nonlinear continuous time models. The advantage of this approach is that nonlinear differential equation models which can be related to the physical behaviour of the system can be readily computed from discrete time data.

1. Introduction

System identification based on sampled-data records is well established and finds wide application in almost all branches of science and engineering. Although the estimates obtained provide good modelling of the system under investigation and reveal the system behaviours [1,2], it is generally not easy to relate the estimated parameters to the physical behaviour or to a continuous time description of the system. In continuous systems, many characteristics can be directly related to some of the system parameters. For example, hard and soft springs in a second order mechanical system will produce jump resonance and the phenomenon can easily be identified by looking at the system equation. In discrete-time forms, such relationships are less obvious. Hence reconstruction of continuous time models from sampled-data systems is desirable to enhance the interpretation of the final parameter estimates.

Traditionally, the reconstruction of linear continuous system models from discrete data is either based on the inverse z-transform or alternatively on the computation of the frequency response function followed by curve fitting. Whereas conversion from the z-domain to the s-domain is generally found to be tedious and does not easily extend to the nonlinear case fitting to the frequency response functions is easily automated and can be used for linear and nonlinear systems. The latter approach exploits the fact that the frequency response functions are invariant descriptors of the characteristics of the underlying system. Whatever type of model is used to describe the system, discrete or continuous of any form, linear or nonlinear, the frequency response functions in each case must be identical if the models are adequate descriptors of the system. Because fitting continuous time models directly often involves numerical differentiation or integration it makes sense to estimate the most concise discrete-time representation and then reconstruct the continuous time model from the frequency response functions.

In the present study, a new methodology based on the ideas proposed above is developed. The method consists of three steps: i) estimation of a discrete time model, ii) evaluation of linear and nonlinear frequency response functions and iii) continuous time model reconstruction based on the frequency response data.

Parametric estimators for linear and nonlinear systems which provide efficient procedures for identifying both the structure and the unknown parameters of linear and nonlinear systems, are well developed and can be readily applied to solve the first stage of the problem [1,2]. This initial stage can be subdivided into several procedures including testing for nonlinearities in the data prior to analysis, structure detection (which terms should be included in the model), parameter estimation, model validation and testing. Stage (ii) of the analysis entails probing the estimated discrete-time models to produce the frequency response functions [1,2]. Finally, stage (iii) consists of using a new orthogonal estimation routine for complex number systems [5] to determine both the structure and unknown parameters in a nonlinear differential equation description of the system. Because details of stage (i) and (ii) are available in the literature [1,2] the emphasis in the present paper will be on the reconstruction of continuous time models from estimated frequency response data. Case studies are included to illustrate the effectiveness of the proposed algorithm.

2. Linear Systems

Consider a linear system governed by the transfer function

$$\frac{Y(s)}{U(s)} = H_c(s) = \frac{B(s)}{A(s)} \tag{1}$$

where

$A(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + 1$

$B(s) = b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m$

n = *order of the polynomial* $A(s)$

m = *order of the polynomial* $B(s)$

$U(s)$ = *Laplace transform of the system input* $u(t)$

$Y(s)$ = *Laplace transform of the system output* $y(t)$

The frequency response of the system can be obtained by replacing s in eqn.(1) with $j\omega$ such that

$$H_c(j\omega) = \frac{B(j\omega)}{A(j\omega)} \tag{2}$$

Alternatively $H_c(j\omega)$ can be computed directly using spectral estimation methods. Assume the system described by eqn.(1) is sampled at intervals T seconds apart and the deterministic part of the best fitted discrete model obtained by parametric linear system identification for a set of collected data is given by

$$A_z(z^{-1})y(k) = z^{-d}B_z(z^{-1})u(k) \tag{3}$$

where

$A_z(z^{-1}) = 1+a_{z_1}z^{-1}+\ldots+a_{z_{n_z}}z^{-n_z}$

$B_z(z^{-1}) = b_{z_0}+b_{z_1}z^{-1}+\ldots+b_{z_{m_z}}z^{-m_z}$

d - *input time delay of the system in discrete time*

$y(k)$ - *system output at the instant kT*

$u(k)$ - *system input at the instant kT*

n_z - *order of the polynomial* $A_z(z^{-1})$

m_z - *order of the polynomial* $B_z(z^{-1})$

The pulse transfer function of this discrete model is

$$H_z(z^{-1}) = z^{-d}\frac{B_z(z^{-1})}{A_z(z^{-1})} \quad (4)$$

and the frequency response of the model can be obtained by replacing z^{-1} in the polynomials with $e^{-j\omega T}$, where ω is the angular frequency to yield

$$H_z(e^{-j\omega T}) = e^{-j\omega dT}\frac{B_z(e^{-j\omega T})}{A_z(e^{-j\omega T})} \qquad -\pi \le \omega T \le \pi \quad (5)$$

If the discrete model described by eqn.(3) is a good approximation to the continuous system described by eqn.(1), the discrete frequency response function eqn.(5) will be closely related to the continuous frequency response function eqn.(2) such that $e^{-j\omega dT}H_z(e^{-j\omega T}) \approx H_c(j\omega)$ in the angular frequency range $-\pi \le \omega T \le \pi$. In order to reconstruct a continuous time model based on the frequency response data obtained from $H_z(e^{-j\omega T})$, a model of the form given by eqn.(2) is built such that a close approximation to the frequency response is obtained

$$H_z(e^{-j\omega T}) = \frac{\hat{B}(j\omega)}{\hat{A}(j\omega)} \qquad -\pi \le \omega T \le \pi \quad (6)$$

where

$\hat{B}(.)$ - *estimated polynomial* $B(.)$

$\hat{A}(.)$ - *estimated polynomial* $A(.)$

Multiplying eqn. (6) with $\hat{A}(j\omega)$ and rearranging gives

$$H_z(e^{-j\omega T}) = (1-\hat{A}(j\omega))\,H_z(e^{-j\omega T}) + \hat{B}(j\omega) \qquad -\pi \le \omega T \le \pi \quad (7)$$

Consider for example a discrete frequency response function which can be approximated by a first order system

$$\hat{H}_z(e^{-j\omega T}) = \frac{\hat{B}(j\omega)}{\hat{A}(j\omega)} = \frac{\hat{b}}{j\hat{a}\omega + 1} \qquad -\pi \le \omega T \le \pi \tag{8}$$

Rearranging gives

$$H_z(e^{-j\omega T}) = (1-\hat{A}(j\omega))\, H_z(e^{-j\omega T}) + \hat{B}(j\omega) = -ja\omega\, H_z(e^{-j\omega T}) + b \tag{9}$$

Curve fitting algorithms can then be applied to eqns.(7) or (9) for the estimation of the polynomials $\hat{A}(j\omega)$ and $\hat{B}(j\omega)$ to yield a continuous time model. The orthogonal least squares algorithm can easily be implemented [5] based on eqns.(7) or (9) to recover the unknown coefficients of a_j, $j=0,\dots,n-1$ and b_k, $k=0,\dots,m$. Providing the sampling interval $\frac{1}{T}$ is greater than the Nyquist sampling rate, the constructed continuous model will provide a good approximation to the original system.

3. Nonlinear Systems

The traditional description of nonlinear systems has been based on the Volterra series [6,7]

$$y(t) = \sum_{n=1}^{\infty} y_n(t) \tag{10}$$

where $y_n(t)$ is the nth order output of the system and is defined as

$$y_n(t) = \int_{-\infty}^{\infty}\dots\int h_n(\tau_1,\dots,\tau_n)\prod_{i=1}^{n} u(t-\tau_i)\,d\tau_i$$

where $h_n(\tau_1,\dots,\tau_n)$ is the nth order Volterra kernel. The nth order frequency response function is defined by taking the multiple Fourier transform of the nth order kernel [8,9].

$$H_n(j\omega_1,\dots,j\omega_n) = \int_{-\infty}^{\infty}\dots\int h_n(\tau_1,\dots,\tau_n)\, e^{-j(\omega_1\tau_1+\dots+\omega_n\tau_n)}\,d\tau_1\dots d\tau_n \tag{11}$$

For a set of sampled-data records, nonlinear frequency response data could be obtained through multidimensional Fourier analysis. Alternatively, it could be obtained analytically by applying the probing method to a fitted parametric discrete nonlinear model [1]. By curve fitting to the estimated frequency response data a continuous time nonlinear differential equation model can be reconstructed.

In order to curve-fit a continuous time nonlinear model to a set of nonlinear frequency response functions the form of the continuous time nonlinear model must be defined. Consider the nonlinear differential equation

$$
\begin{aligned}
F^{l}\left[\frac{d^{n}y(t)}{dt},\ldots,y(t),\frac{d^{m}u(t)}{dt},\ldots,u(t)\right] = \\
a_1\frac{d^{n}y(t)}{dt} + \ldots + a_n y(t) + a_{n+1}y(t) + a_{n+2}\frac{d^{m}u(t)}{dt} + \ldots + a_{n+m+1}\frac{du(t)}{dt} + a_{n+m+2}u(t) \\
+ a_{1,1}\left(\frac{d^{n}y(t)}{dt}\right)^2 + a_{1,2}\frac{d^{n}y(t)}{dt}\frac{d^{n-1}y(t)}{dt} + \ldots + a_{n+m+2,n+m+2}u^2(t) \\
\cdot \\
\cdot \\
\cdot \\
+ a_{1,1,\ldots,1}\left(\frac{d^{n}y(t)}{dt}\right)^{l} + \ldots = 0
\end{aligned}
\tag{12}
$$

where l is the degree of nonlinearity, n is the order of dynamics in the output y(t), m is the order of dynamics in the input u(t) and $F^{l}[\ldots]$ is a polynomial nonlinear function. A nonlinear system with first order dynamics and a second order nonlinearity could for example be expressed as

$$
\begin{aligned}
F^{2}\left[\frac{dy(t)}{dt},y(t),\frac{du(t)}{dt},u(t)\right] = \\
a_1\frac{dy(t)}{dt} + a_2y(t) + a_3\frac{du(t)}{dt} + a_4u(t) \\
+ a_{1,1}\left(\frac{dy(t)}{dt}\right)^2 + a_{1,2}y(t)\frac{dy(t)}{dt} + a_{1,3}\frac{dy(t)}{dt}\frac{du(t)}{dt} + a_{1,4}u(t)\frac{dy(t)}{dt} \\
+ a_{2,2}y^2(t) + a_{2,3}y(t)\frac{du(t)}{dt} + a_{2,4}y(t)u(t) \\
+ a_{3,3}\left(\frac{du(t)}{dt}\right)^2 + a_{3,4}u(t)\frac{du(t)}{dt} \\
+ a_{4,4}u^2(t) = 0
\end{aligned}
$$

To identify the coefficients, the a's in eqn.(12), nonlinear frequency response functions for such a system are required. The probing or harmonic input method is a simple and efficient way of analytically extracting symmetrical nonlinear frequency response functions from a nonlinear model.

Let the input u(t) be a sum of n exponentials

$$u(t) = \sum_{k=1}^{n} e^{j\omega_k t} \tag{13}$$

where ω_k may be any positive or negative real number. The nth order output of the system can be expressed as

$$y_n(t) = \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} H_n(j\omega_{k_1}, \ldots, j\omega_{k_n})\, e^{j(\omega_{k_1}+\ldots+\omega_{k_n})t} \tag{14}$$

The symmetrised nth order nonlinear frequency response function $H_n(j\omega_1, \ldots, j\omega_n)$ can be obtained by equating the coefficients of $n!\,e^{j(\omega_1+\ldots+\omega_n)}$ in the system output when the input is defined as in eqn.(12) [1,8]. The procedure is recursive and is best illustrated by example.

Consider a system described by the differential equation

$$\frac{dy(t)}{dt} + a_1 y(t) + a_2 y^2(t) = u(t) \tag{15}$$

The procedure begins with

$$u(t) = e^{j\omega t}$$

From eqns.(10) and (14)

$$y(t) = H_1(j\omega)\, e^{j\omega t}$$

and therefore

$$\frac{dy(t)}{dt} = j\omega H_1(j\omega)\, e^{j\omega t}$$

Substituting into eqn.(15) gives

$$e^{j\omega t} = j\omega H_1(j\omega)\, e^{j\omega t} + a_1 H_1(j\omega)\, e^{j\omega t} + a_2 [H_1(j\omega)\, e^{j\omega t}]^2$$

and equating coefficients of $e^{j\omega t}$ on both sides yields

$$H_1(j\omega) = \frac{1}{j\omega + a_1}$$

Probing with two inputs

$$u(t) = e^{j\omega_1 t} + e^{j\omega_2 t}$$

the output from eqns.(10) and (14) becomes

$$\begin{aligned} y(t) = {} & H_1(j\omega_1)\, e^{j\omega_1 t} + H_1(j\omega_2)\, e^{j\omega_2 t} + 2!\,H_2(j\omega_1, j\omega_2)\, e^{j(\omega_1+\omega_2)t} \\ & + H_2(j\omega_1, j\omega_1)\, e^{j2\omega_1 t} + H_2(j\omega_2, j\omega_2)\, e^{j2\omega_2 t} \end{aligned} \tag{16}$$

Substituting eqn.(16), its derivative and the input into eqn.(15) and equating coefficients of $2!\,e^{j(\omega_1+\omega_2)t}$ yields

$$0 = [j(\omega_1+\omega_2) + a_1] H_2(j\omega_1, j\omega_2) + a_2 H_1(j\omega_1) H_1(j\omega_2)$$

or

$$H_2(j\omega_1,j\omega_2) = -\frac{a_2H_1(j\omega_1)H_1(j\omega_2)}{j(\omega_1+\omega_2)+a_1}$$
$$= -a_2H_1(j\omega_1)H_1(j\omega_2)H_1(j\omega_1+j\omega_2)$$

Continuing the procedure by probing with three exponentials and equating coefficients of $3!e^{j(\omega_1+\omega_2+\omega_3)t}$ on both sides of eqn.(15) yields

$$H_3(j\omega_1,j\omega_2,j\omega_3) = -\frac{2a_2}{3}[H_1(j\omega_1)H_2(j\omega_2,j\omega_3) + H_1(j\omega_2)H_2(j\omega_1,j\omega_3) + H_1(j\omega_3)H_2(j\omega_1,j\omega_2)]H_1(j\omega_1+j\omega_2+j\omega_3)$$

The procedure can be continued indefinitely to find at each step higher order nonlinear frequency response functions in terms of the lower order functions. This procedure equally applies to discrete time nonlinear models where any order of nonlinear frequency response functions can be recursively evaluated.

In general, the first order frequency response function obtained from the probing method for a nonlinear system described by eqn.(12) is given by

$$H_1(j\omega) = \frac{-\left(a_{n+2}(j\omega)^m + \ldots + a_{n+m+2}\right)}{\left(a_1(j\omega)^n + \ldots + a_n(j\omega) + a_{n+1}\right)} \tag{17}$$

Dividing top and bottom by a_{n+1} and redefining the coefficients to ensure a term 1 in the denominator gives

$$H_1(j\omega) = -\left(a_{n+2}(j\omega)^m + \ldots + a_{n+m+2}\right) - \left(a_1(j\omega)^n + \ldots + a_n(j\omega)\right)H_1(j\omega) \tag{18}$$

The second order frequency response function can now be obtained from the equation

$$\begin{aligned}
&2!\left(a_1(j\omega_1+j\omega_2)^n + \ldots + a_n(j\omega_1+j\omega_2) + 1\right)H_2(j\omega_1,j\omega_2)\\
&= -\sum_{\substack{\text{all possible } i_1,i_2\\ i_1\le n+1,\ i_2\le n+1}} a_{i_1,i_2}H_1(j\omega_1)H_1(j\omega_2)\left[(j\omega_1)^{n+1-i_1}(j\omega_2)^{n+1-i_2}+(j\omega_1)^{n+1-i_2}(j\omega_2)^{n+1-i_1}\right]\\
&-\sum_{\substack{\text{all possible } i_1,i_2\\ i_1\le n+1,\ i_2> n+1}} a_{i_1,i_2}\left[(j\omega_1)^{n+1-i_1}(j\omega_2)^{n+m+2-i_2}H_1(j\omega_1)+(j\omega_1)^{n+m+2-i_2}(j\omega_2)^{n+1-i_1}H_1(j\omega_2)\right]\\
&-\sum_{\substack{\text{all possible } i_1,i_2\\ i_1> n+1,\ i_2>n+1}} a_{i_1,i_2}\left[(j\omega_1)^{n+m+2-i_1}(j\omega_2)^{n+m+2-i_2}+(j\omega_1)^{n+m+2-i_2}(j\omega_2)^{n+m+2-i_1}\right]
\end{aligned} \tag{19}$$

The third order frequency response function is given by

$$
\begin{aligned}
&3!\{a_1(j\omega_1+j\omega_2+j\omega_3)^n+\ldots+a_n(j\omega_1+j\omega_2+j\omega_3)+1\}H_3(j\omega_1,j\omega_2,j\omega_3) \\
&= -\sum_{\substack{\text{all possible}\\ i_1,i_2,i_3\le n+1}} a_{i_1,i_2,i_3} \sum_{\substack{\text{all possible}\\ k_1,k_2,k_3=1,2,3\\ k_1\ne k_2\ne k_3}} (j\omega_{k_1})^{n+1-i_1}(j\omega_{k_2})^{n+1-i_2}(j\omega_{k_3})^{n+1-i_3}\times \\
&\qquad\qquad H_1(j\omega_{k_1})H_1(j\omega_{k_2})H_1(j\omega_{k_3}) \\
&-\sum_{\substack{\text{all possible}\\ i_1,i_2\le n+1,\ i_3>n+1}} a_{i_1,i_2,i_3} \sum_{\substack{\text{all possible}\\ k_1,k_2,k_3=1,2,3\\ k_1\ne k_2\ne k_3}} (j\omega_{k_1})^{n+1-i_1}(j\omega_{k_2})^{n+1-i_2}(j\omega_{k_3})^{n+m+2-i_3}\times \\
&\qquad\qquad H_1(j\omega_{k_1})H_1(j\omega_{k_2}) \\
&-\sum_{\substack{\text{all possible}\\ i_1\le n+1,\ i_2,i_3>n+1}} a_{i_1,i_2,i_3} \sum_{\substack{\text{all possible}\\ k_1,k_2,k_3=1,2,3\\ k_1\ne k_2\ne k_3}} (j\omega_{k_1})^{n+1-i_1}(j\omega_{k_2})^{n+m+2-i_2}(j\omega_{k_3})^{n+m+2-i_3}H_1(j\omega_{k_1}) \\
&-\sum_{\substack{\text{all possible}\\ i_1,i_2,i_3>n+1}} a_{i_1,i_2,i_3} \sum_{\substack{\text{all possible}\\ k_1,k_2,k_3=1,2,3\\ k_1\ne k_2\ne k_3}} (j\omega_{k_1})^{n+m+2-i_1}(j\omega_{k_2})^{n+m+2-i_2}(j\omega_{k_3})^{n+m+2-i_3} \\
&-\sum_{\substack{\text{all possible}\\ i_1,i_2\le n+1}} 2!a_{i_1,i_2} \sum_{\substack{\text{all possible}\\ k_1,k_2,k_3=1,2,3\\ k_1\ne k_2\ne k_3}} (j\omega_{k_1}+j\omega_{k_2})^{n+1-i_1}(j\omega_{k_3})^{n+1-i_2}\times \\
&\qquad\qquad H_2(j\omega_{k_1},j\omega_{k_2})H_1(j\omega_{k_3}) \\
&-\sum_{\substack{\text{all possible}\\ i_1\le n+1,\ i_2>n+1}} 2!a_{i_1,i_2} \sum_{\substack{\text{all possible}\\ k_1,k_2,k_3=1,2,3\\ k_1\ne k_2\ne k_3}} (j\omega_{k_1}+j\omega_{k_2})^{n+1-i_1}(j\omega_{k_3})^{n+m+2-i_2}H_2(j\omega_{k_1},j\omega_{k_2})
\end{aligned}
\tag{20}
$$

Similarly the k'th order frequency response function can be formulated in terms of the lower order frequency response functions. To estimate the coefficients, the a's in eqn.(12), the first, second, third and higher order frequency response functions of eqns.(18), (19) and (20) are replaced by their corresponding frequency response estimates. The orthogonal least squares algorithm for complex number systems [5] can then be applied directly to identify the unknown coefficients in the continuous time model of eqn.(12).

4. Orthogonal least squares algorithm for complex number systems

Consider a complex number equation related by

$$
Z(j\omega) = \sum_{i=1}^{N}\theta_i p_i(j\omega) + \zeta(j\omega) \tag{21}
$$

where M is the number of parameters in the equation, θ_i, i=1,...,M are unknown real parameters associated with the complex variable $p_i(j\omega)$, i=1,...,M. $Z(j\omega)$ is the possibly complex dependent variable and $\zeta(j\omega)$ is some modelling error.

The conventional orthogonal least squares estimation algorithm [1,2] can be extended to accommodate complex number systems [5] by transforming eqn.(21) into the auxiliary equation

$$Z(j\omega) = \sum_{i=1}^{M} g_i w_i(j\omega) + \zeta(j\omega) \tag{22}$$

Defining

$$\begin{aligned} w_1(j\omega) &= p_1(j\omega) \\ w_i(j\omega) &= p_i(j\omega) - \sum_{k=1}^{i-1} \alpha_{ki} w_k(j\omega) \quad , \quad k < i \end{aligned} \tag{23}$$

where

$$\alpha_{ki} = \frac{\overline{w_k^*(j\omega)\, p_i(j\omega)}}{\overline{w_k^*(j\omega)\, w_k(j\omega)}}$$

such that the g_i, i=1,..,M are constant coefficients and the $w_i(j\omega)$, i=1,...,M are constructed to be orthogonal over the complex conjugate of the data records. The superscript star * denotes complex conjugate and the overbar ─── denotes time average. Estimates of g_i are given by

$$\hat{g}_i = \frac{\overline{Z(j\omega)\, w_i^*(j\omega)}}{\overline{w_i(j\omega)\, w_i^*(j\omega)}} \tag{24}$$

Once the parameters g_i, i=1,...,M have been estimated using eqn.(24), the original system parameters θ_i, i=1,...,M can be recovered according to the formula

$$\begin{aligned} \theta_M &= \hat{g}_M \\ \theta_k &= \hat{g}_k - \sum_{i=k+1}^{M} \alpha_{ki} \theta_i \quad , \quad k=M-1,\ldots,1 \end{aligned} \tag{25}$$

The structure of the system or which terms to include in the model can easily be determined by using the error reduction ratio test

$$\epsilon RR_i = \frac{\hat{g}_i \hat{g}_i^* \,\overline{w_i(j\omega)\, w_i^*(j\omega)}}{\overline{Z(j\omega)\, Z^*(j\omega)}} \times 100 \quad , \quad i=1,\ldots,M \tag{26}$$

which gives the percentage contribution each term makes to the output variance. Combining the orthogonal estimator and ϵRR test into a forward regression procedure gives a powerful estimation algorithm for complex number systems. Full details are given elsewhere [5].

5. Examples

Consider a nonlinear circuit called system S_1 described by the equation

$$0.2\frac{dy(t)}{dt} + y(t) + 0.16y^2(t) = u(t) \tag{27}$$

The system was simulated on a Vidac 336 analogue computer with a zero mean Gaussian noise of bandwidth of 5Hz as the input to the system. 500 pairs of input-output data collected by sampling at 31.25Hz were used for the identification of the system. The best discrete NARMAX model, estimated after structure detection, parameter estimation, model validation and testing was given by [1]

$$\begin{aligned} y(k) = 0.1758y(k-1) + 0.0623u(k) + 0.1616u(k-1) \\ - 0.03839y^2(k-1) + 0.569y(k-2) + 0.03143u(k-2) \end{aligned} \tag{28}$$

An inspection on eqn.(28) and its first order frequency response function, Fig.1, illustrates that the system is of first order dynamics with a second order nonlinearity. The model eqn.(28) can now be probed [1] and 400 equally spaced frequency response data in the range -5Hz to 5Hz were generated for the reconstruction of the linear part of the original system. The error reduction ratios for each candidate term and the corresponding orthogonal parameters for the first three iterations are shown in Table 1. Candidate terms that were selected for inclusion in the final model at each iteration based on the criterion of maximum error reduction ratio are all underlined in Table 1. At the end of the second iteration, the sum of the error reduction ratio for the selected candidate terms is 99.992% of the total output. Error reduction ratios for the remainder of the candidate terms after the second iteration become insignificant compared to the first two selected terms. It is therefore reasonable to argue that there are only two significant terms in the model and the associated system parameters estimated using eqn.(25) are shown in Table 2. The estimated parameters are comparable to the original system parameters eqn.(27).

For the estimation of the second order nonlinearities, 800 frequency response data were generated using $\hat{H}_2(j\omega_1, j\omega_2)$ obtained by probing the estimated discrete time model eqn.(28) [1]. Table 3 shows the first two iterations of the estimation. With the inclusion of $y^2(t)$ which contributes 99.923% to the total output, the rest of the candidate terms at the next iteration are all insignificant compared to this term. Table 4 shows the estimated system parameter for the variable $y^2(t)$. Finally, Combining the results of Tables 2 and 4 gives the estimated nonlinear differential equation

$$y(t) = -0.1931\frac{dy(t)}{dt} + 0.9965u(t) - 0.1514y^2(t) \tag{29}$$

which compares well to the original system S_1 eqn.(27).

Notice that a distinct advantage of the algorithm is that each degree of nonlinearity can be independently reconstructed thus simplifying the procedure. Furthermore because the frequency response data is generated by probing the estimated discrete model any number of noise free samples over any desired frequency range can be provided. Experimentation on the system may of course produce data corrupted by additive and/or multiplicative coloured noise. This is accommodated by fitting a noise model during the discrete model estimation to ensure the parameters associated with the process model are unbiased [1]. The noise model is discarded prior to probing [1] for the $H_i[.]$'s and hence it is assumed that the data used for the continuous model reconstruction are noise free.

The next two examples are the nonlinear circuits, NL1 and NL2 analysed in [10] representing different single-degree-of-freedom systems. Full details of the data capture, sampling rates etc. are given in [10].

Structure detection, parameter estimation and model validation gave the optimised discrete nonlinear models for system NL1 as (see Table 4(b) in [10])

$$\begin{aligned} y(k) = {} & 1.3605y(k-1) - 0.93179y(k-2) + 0.11451u(k-1) - 0.0062076u(k-2) \\ & + 0.17897 - 0.0089813y^2(k-1) - 0.010568y(k-1)u(k-2) \\ & - 0.013498y(k-1)y(k-2) + 0.0075731y^2(k-1) + 0.0092032y(k-2)u(k-1) \\ & + 0.0014783u^2(k-2) - 0.0016733y(k-1)u(k-1) \end{aligned} \tag{30}$$

and for system NL2 (see Table 8(b) in [10]) as

$$\begin{aligned} y(k) = {} & 1.6021y(k-1) - 0.94726y(k-2) + 0.061490u(k-1) + 0.13597 \\ & - 0.013829y^2(k-1) - 0.0025225y^3(k-1) \end{aligned} \tag{31}$$

respectively.

For system NL1, an inspection of the identified NARMAX model eqn.(30) and the first order frequency response function Fig. 2 shows that the system exhibits second order dynamics with a second order nonlinearity. 400 equally spaced frequency response data were used for the identification of the linear part of the system. The selected terms together with their error reduction ratios at different iterations are shown in Table 5. Table 6 shows the estimated system parameters for the selected variables. With $\frac{d^2y(t)}{dt^2}$, $\frac{dy(t)}{dt}$ and u(t) included in the final model, the sum of the error reduction ratio is 99.973%.

For the estimation of the second order nonlinearities, 800 frequency response data were used. Table 7 shows the selected six candidate terms together with the associated error reduction ratios and Table 8 shows the estimated system parameters for the selected variables. With the six candidate terms included in the final model, the sum of the error reduction ratio is 99.939%. Combining the results from Tables 6 and 8 produces the nonlinear differential equation

$$\begin{aligned} y(t) = & -1.10402E\text{-}6\frac{d^2y(t)}{dt^2} - 9.40408E\text{-}5\frac{dy(t)}{dt} + 0.200629u(t) \\ & + 0.31075 - 0.020025y^2(t) - 2.99024E\text{-}5u(t)\frac{dy(t)}{dt} \\ & - 0.013396y(t)u(t) + 6.70568E\text{-}5y(t)\frac{dy(t)}{dt} \\ & + 8.47344E\text{-}9\frac{dy(t)}{dt}\frac{du(t)}{dt} + 0.0038862u^2(t) \end{aligned} \qquad (32)$$

for the system NL1. The constant term in eqn.(32) was obtained by equating the steady-state output of eqns.(30) and (32) with the input set to zero on the assumption that the mean in eqn.(30) arose from a d.c. shift on the measured output. Equation (32) is composed of a second order system with damping ratio 0.004475 and undamped natural frequency 951.7 rad/s. The nonlinear part is dominated by the $y^2(t)$ term since this contributes more than 60% of the total output of second order frequency response function.

For system NL2, an inspection of the identified model eqn.(31) and the first order frequency response function Fig. 3 shows that the system exhibits second order dynamics with a third order nonlinearity. 400 equally spaced frequency response data were used for the identification of the linear part of the system. The selected terms together with their error reduction ratios at different iterations are shown in Table 9. Table 10 shows the estimated system parameters for the selected variables. With $\frac{d^2y(t)}{dt^2}$, $\frac{dy(t)}{dt}$ and u(t) included in the final model, the sum of the error reduction ratio is 99.989%.

For the estimation of the second order nonlinearity, 800 frequency response data were used. Table 11 shows the selected candidate terms together with the associated error reduction ratios and Table 12 shows the estimated system parameter for the selected variable. With just a $y^2(t)$ term included in the final model, the sum of the error reduction ratio is 99.911%.

For the estimation of the third order nonlinearity, 1600 frequency response data were used. Table 13 shows the selected candidate term together with the associated error reduction ratios and Table 14 shows the estimated system parameter for the selected variable. With just $y^3(t)$ included in the final model, the sum of the error reduction ratio is 99.907%.

Combining the results of Tables 10, 12 and 14 yields the reconstructed nonlinear differential equation

$$y(t) = -1.60650E\text{-}6\frac{d^2y(t)}{dt} - 9.25612E\text{-}5\frac{dy(t)}{dt} + .18450u(t) + 0.3942 - 0.041541y^2(t) - 0.007608y^3(t) \qquad (33)$$

for system NL2. The constant term in eqn.(33) was obtained by equating the steady-state output of eqns.(31) and (33) with the input set to zero on the assumption that the mean in eqn.(31) arose from a d.c. shift on the measured output. Equation (33) is composed of a second order system with a damping ratio 0.004485 and undamped natural frequency 969 rad/s. The nonlinearity appears as a nonlinear spring type element governed by the polynomial $f(y)=y+.041541y^2+.007608y^3$.

6. Conclusion

A new method of reconstructing nonlinear differential equations from sampled data records has been introduced and illustrated using three nonlinear systems. The new algorithm is based on the application of a new orthogonal forward regression estimation for complex number systems applied to data generated by probing an identified discrete time system representation. The method exhibits several useful properties. First, higher order derivatives of the input and output are not required and therefore errors induced by numerical approximation to differentiation are reduced to a minimum. Secondly the algorithm works on short possibly noisy sampled data records and higher order nonlinear terms can easily be accommodated. Finally the physical interpretation of the system can be studied based on the identified continuous time model.

References

1. Billings S.A. and Tsang K.M. [1989]: Spectral analysis for nonlinear systems, Part I: Parametric nonlinear spectral analysis, Mechanical Systems and Signal Processing 3, 319-339.
2. Billings S.A. and Tsang K.M. [1989]: Spectral analysis for nonlinear systems, Part II: Interpretation of nonlinear frequency response functions, Mechanical Systems and Signal Processing 3, 341-359.
3. Astrom K.J. and Eykhoff P. [1971]: System identification - a survey, Automatica 7, 123-162.
4. Billings S.A. [1985]: An overview of nonlinear systems identification, 7th IFAC symp. ident. and syst. par. est., 725-729.
5. Tsang K.M. and Billings S.A. [1990]: An orthogonal estimation algorithm for complex number systems, submitted for publication.
6. Schetzen M. [1980]: The Volterra and Wiener Theories of Nonlinear Systems, Chichester, John Wiley.
7. Marmarelis P.Z. and Marmarelis V.Z. [1978]: Analysis of physiological systems- the white noise approach, New York, Plenum press.
8. Bedrosian E. and Rice S.O. [1971]: The output properties of Volterra systems (non-linear system with memory) driven by harmonic and Gaussian inputs, Proc. IEEE 59, 1688-1707.
9. Bussgang J.J., Ehrman L. and Graham J.W. [1974]: Analysis of nonlinear systems with multiple inputs, Proc. IEEE 62, 1088-1119.
10. Billings S.A., Tsang K.M. and Tomlinson G.R. [1990]: Spectral analysis for nonlinear systems, Part III: Case Study Examples, Mechanical Systems and Signal processing 4, 3-21.

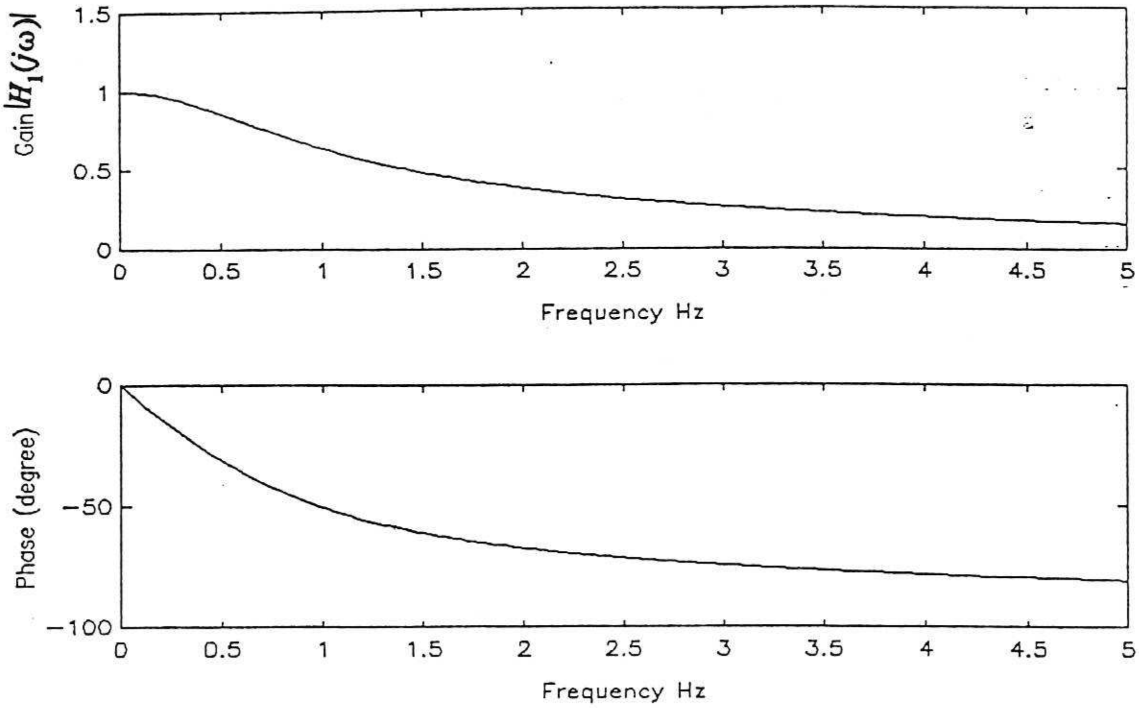

Figure 1. First order frequency response for system S_1

i	p_i	ϵRR_i	g_i
1	$-\frac{dy(t)}{dt}$	0.000	0.0000000
	$-\frac{du(t)}{dt}$	20.487	0.0119719
	$-u(t)$	23.288	-.2320950
2	$-\frac{dy(t)}{dt}$	76.704	0.1931260
	$-\frac{du(t)}{dt}$	20.487	0.0119719
3	$-\frac{du(t)}{dt}$	0.004	-.0000187

Table 1. First three iterations for the estimation of system S_1

Candidates	Parameters
$-u(t)$	-.996546
$-\frac{dy(t)}{dt}$	0.193126

Table 2. Final parameter estimates for system S_1

i	p_i	ϵRR_i	g_i
1	$-\left(\frac{dy(t)}{dt}\right)^2$	0.010	-.000050
	$-y(t)\frac{dy(t)}{dt}$	0.054	-.001157
	$-\frac{dy(t)}{dt}\frac{du(t)}{dt}$	0.005	-.000020
	$-u(t)\frac{dy(t)}{dt}$	0.039	-.000450
	$-y^2(t)$	99.923	0.151396
	$-y(t)\frac{du(t)}{dt}$	21.858	-.008864
	$-y(t)u(t)$	72.173	0.110856
	$-\left(\frac{du(t)}{dt}\right)^2$	4.998	0.000203
	$-u(t)\frac{du(t)}{dt}$	9.005	-.003724
	$-u^2(t)$	25.014	0.039428
2	$-\left(\frac{dy(t)}{dt}\right)^2$	0.039	-.000050
	$-y(t)\frac{dy(t)}{dt}$	0.098	-.001157
	$-\frac{dy(t)}{dt}\frac{du(t)}{dt}$	0.024	0.000020
	$-u(t)\frac{dy(t)}{dt}$	0.123	-.000450
	$-y(t)\frac{du(t)}{dt}$	0.113	-.000423
	$-y(t)u(t)$	0.097	-.0059817
	$-\left(\frac{du(t)}{dt}\right)^2$	0.094	0.000014
	$-u(t)\frac{du(t)}{dt}$	0.001	-.000069
	$-u^2(t)$	0.137	-.002198

Table 3. First two iterations for the estimation of system S_1

Candidate	Parameter
$-y^2(t)$	0.151396

Table 4. Second order estimate for system S_1

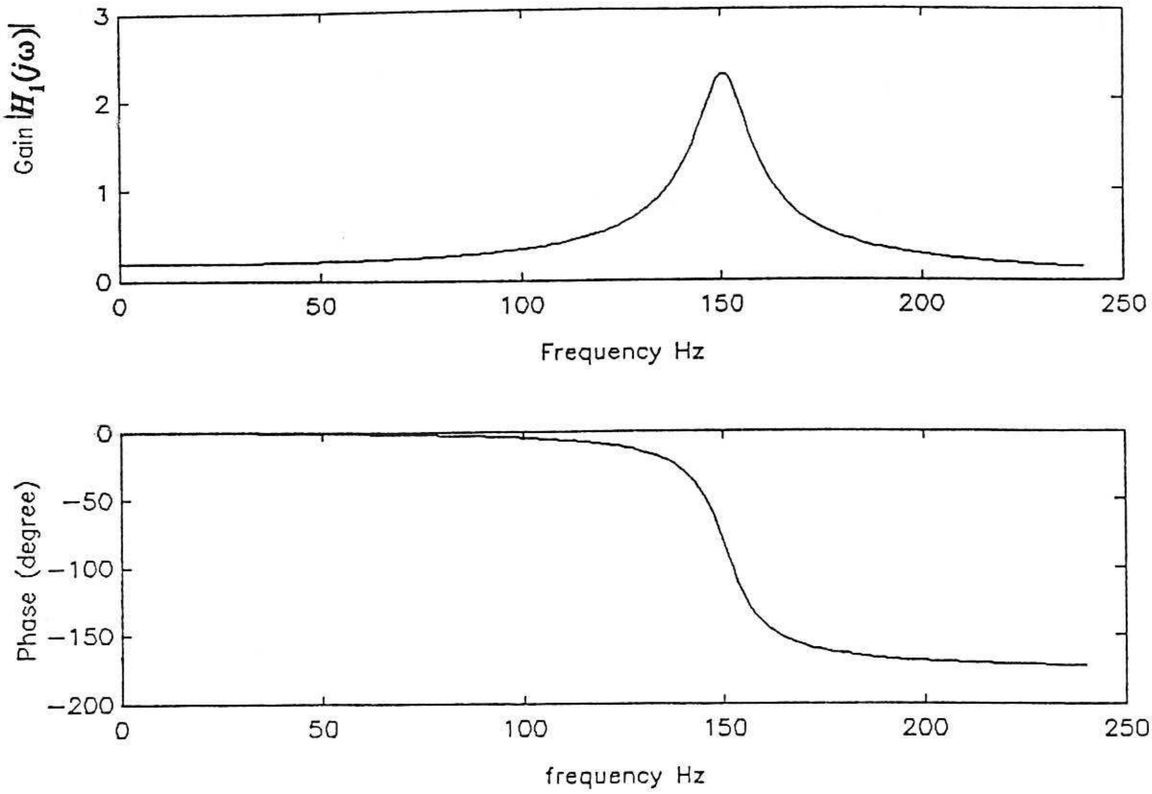

Figure 2. First order frequency response for system NL1

i	p_i	ϵRR_i	g_i
1	$-\frac{d^2y(t)}{dt^2}$	91.981	1.05691e-6
2	$-u(t)$	7.290	-0.183004
3	$-\frac{dy(t)}{dt}$	0.702	9.40408e-5

Table 5. Linear estimation for system NL1

Candidates	Parameters
$-\frac{d^2y(t)}{dt^2}$	1.10402e-6
$-\frac{dy(t)}{dt}$	9.40408e-5
$-u(t)$	-0.200629

Table 6. First order parameters for system NL1

i	p_i	ϵRR_i	g_i
1	$-y^2(t)$	61.514	0.032947
2	$-u(t)\frac{dy(t)}{dt}$	20.878	1.12702e-5
3	$-y(t)u(t)$	11.850	-.019020
4	$-y(t)\frac{dy(t)}{dt}$	2.799	-4.4609e-5
5	$-\frac{dy(t)}{dt}\frac{du(t)}{dt}$	2.745	-8.3150e-9
6	$-u^2(t)$	0.153	-.003886

Table 7. Second order estimation for system NL1

Candidate	Parameter
$-y^2(t)$	0.020025
$-u(t)\frac{dy(t)}{dt}$	2.99024e-5
$-y(t)u(t)$	0.013396
$-y(t)\frac{dy(t)}{dt}$	-6.70568e-5
$-\frac{dy(t)}{dt}\frac{du(t)}{dt}$	-8.47344e-9
$-u^2(t)$	-.0038862

Table 8. Second order parameters for system NL1

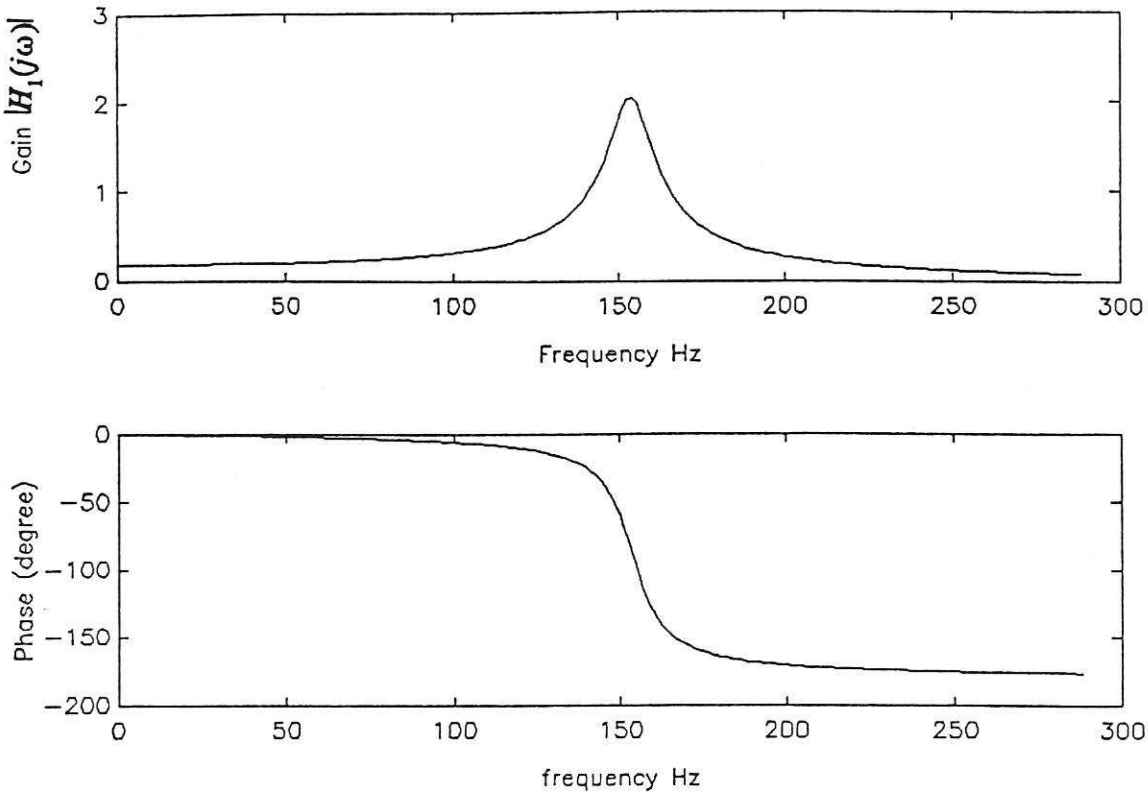

Figure 3. First order frequency response for system NL2

i	p_i	ϵRR_i	g_i
1	$-\frac{d^2y(t)}{dt^2}$	90.390	9.93406e-7
2	$-u(t)$	8.878	-0.170641
3	$-\frac{dy(t)}{dt}$	0.721	9.25612e-5

Table 9. Linear estimation for system NL2

Candidates	Parameters
$-\frac{d^2y(t)}{dt^2}$	1.0650e-6
$-\frac{dy(t)}{dt}$	9.25612e-5
$-u(t)$	-.18450

Table 10. First order parameters for system NL2

i	p_i	ϵRR_i	g_i
1	$-y^2(t)$	99.911	0.041541

Table 11. Second order estimation for system NL2

Candidate	Parameter
$-y^2(t)$	0.041541

Table 12. Second order parameters for system NL2

i	p_i	ϵRR_i	g_i
1	$-y^3(t)$	99.907	0.007608

Table 13. Third order estimation for system NL2

Candidate	Parameter
$-y^3(t)$	0.007608

Table 14. Third order parameters for system NL2