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K-Exponential Stability of Nonlinear Delay Systems

by

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Abstract

In this paper we introduce a new concept of k-exponential stability. The k-exponential stability of nonlinear delay systems is investigated via the nonlinear variation of parameters formula and nonlinear inequality analysis.

Keywords: K-Exponential Stability, Nonlinear Systems, Delay.

Classification Numbers: 34K, 93D.

1 Introduction

The stability of delay systems has been investigated by many authors. In the past years, the Liapunov approach has been developed to study delay systems(see, Hale[5] 1977, Kolmanovskii and Nosov[6] 1986 and Gopalsamy[4] 1984). However, finding Liapunov functions or Liapunov functionals is not particularly easy, and so other methods need to be developed. Lakshmikantham and Leela[7] 1966, Driver[3] 1977, Zhang Yi[8] 1988, introduced inequalities to study delay systems, many practical stability results have been obtained.

In this paper, we shall introduce a new concept of k-exponential stability. Using the nonlinear variation of parameters formula and nonlinear inequalities, we shall investigate the k-exponential stability of nonlinear delay systems. This paper will be organized as follows. In section 2, we shall give the nonlinear variation of parameters formula. In section 3, some nonlinear inequalities will be established. Combing the nonlinear variation of parameters formula and nonlinear inequalities in section 4 we shall drive k-exponential stability criteria for nonlinear delay systems. Examples will be given in section 5 to illustrate our theory.

2 Variation of Parameters Formula

Consider the systems

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t)) \tag{1}$$

and

$$\dot{y}(t) = f(t, y(t)) \tag{2}$$

Let $x(t, t_0, x_0), y(t, t_0, x_0)$ denote the solutions of (1) and (2) through (t_0, x_0), respectively. Then, the relation between $x(t, t_0, x_0)$ and $y(t, t_0, x_0)$ is known as the nonlinear variation of paprameters formula (Alekseev[1] 1961, Brauer[2] 1966) and can be expressed in the form

$$x(t, t_0, x_0) = y(t, t_0, x_0) + \int_{t_0}^{t} \Phi[t, s, x(s, t_0, x_0)] \cdot g[t, x(s, t_0, x_0)]ds$$

where

$$\Phi(t, s, x) = \frac{\partial}{\partial x} y(t, s, x)$$

is the fundamental solution of the variational system

$$\dot{Z} = \frac{\partial f(t, y(t, s, x))}{\partial y} Z.$$

From Brauer 1966, we can get the result

Lemma 2.1. Let $x_0, y_0 \in R^n$ and denote by θ the straight line between x_0 and y_0, i.e.

$$\theta(\lambda) = x_0 + \lambda(y_0 - x_0)$$

for $0 \leq \lambda \leq 1$. Then the system (2) has solutions through x_0, y_0, which satisfy

$$\| y(t, t_0, x_0) - y(t, t_0, y_0) \| \leq \max_{0 \leq \lambda \leq 1} \| \Phi(t, t_0, \theta(\lambda)) \| \cdot \| x_0 - y_0 \| .$$

3 Inequality Analysis

In this section, we shall establish some nonlinear inequalities which will be used in the stability analysis of nonlinear delay systems.

Theorem 3.1. Let $S(t)$ be a nonnegative continuous function defined on $[t_0 - \tau, +\infty)$, which satisfies

$$\dot{S}(t) \leq r(t) \left(-S(t) + \sum_{j=1}^{m} \delta_j S_t^j \right)$$

for $t \geq t_0$, where $r(t)$ is a nonnegative continuous function , $\tau > 0, b_j \geq 0$ are constants and $S_t \triangleq \sup_{t-\tau \leq \theta \leq t}[S(\theta)]$. If there exist constants $r > 0, k > 0$ such that

$$r(t) \geq r > 0$$

$$\sum_{j=1}^{m} \delta_j k^{j-1} < 1$$

then, $S_{t_0} < k$ implies that

$$S(t) \leq S_{t_0} e^{-\lambda(t-t_0)}$$

for $t \geq t_0$, where $\lambda > 0$ is a constant.

Proof: Since $r > 0$, $\sum_{j=1}^{m} \delta_j k^{j-1} < 1$, it follows for sufficiently small $\lambda > 0$ that

$$-1 + \frac{\lambda}{h} + \sum_{j=1}^{m} \delta_j k^{j-1} e^{\lambda\tau j} < 0 \tag{3}$$

Define

$$Q(t) = S(t) e^{\lambda(t-t_0)}, \qquad t \geq t_0 - \tau$$

Then, we have

$$\begin{aligned} \dot{Q}(t) &\leq \lambda Q(t) + r(t)\left(-Q(t) + e^{\lambda(t-t_0)} \sum_{j=1}^{m} \delta_j S_t^j\right) \\ &\leq r(t)\left(-Q(t) + \frac{\lambda}{r} Q(t) + \sum_{j=1}^{m} \delta_j e^{\lambda\tau j} Q_t^j\right) \end{aligned} \tag{4}$$

for $t \geq t_0$, where $Q_t = \sup_{t-\tau \leq \theta \leq t}[Q(\theta)]$.

Let $l \in (1, k/S_{t_0})$ be an arbitrary constant,we shall prove that

$$Q(t) < l S_{t_0} \triangleq M \tag{5}$$

for $t \geq t_0$. In fact, if (5) does not hold, then there must exist a $t_1 > t_0$ such that

$$Q(t_1) = M; \quad Q(t) < M, \quad t_0 - \tau \leq t < t_1$$

This follows from $Q(t) < M$ for $t \in [t_0 - \tau, t_0]$. Hence, $\dot{Q}(t_1) \geq 0$. However, from (4) by (3) ,we have

$$\begin{aligned}
\dot{Q}(t_1) &\leq r(t_1)\left(-M + \frac{\lambda}{r}M + \sum_{j=1}^{m} \delta_j e^{\lambda\tau j} M^j\right) \\
&\leq r(t_1)\left(-1 + \frac{\lambda}{r} + \sum_{j=1}^{m} \delta_j e^{\lambda\tau j} k^{j-1} \cdot \frac{M^{j-1}}{k^{j-1}}\right) M \\
&\leq r(t_1)\left(-1 + \frac{\lambda}{r} + \sum_{j=1}^{m} \delta_j e^{\lambda\tau j} \cdot k^{j-1}\right) M \\
&< 0
\end{aligned}$$

This yields a contradiction. Hence, the inequality (5) holds. Letting $l \longrightarrow 1$, we get

$$Q(t) \leq S_{t_0}$$

for $t \geq t_0$, that is

$$S(t) \leq S_{t_0} e^{-\lambda(t-t_0)}$$

for $t \geq t_0$. The proof is complete.

Theorem 3.2. Let $S_i(t)(i = 1, 2)$ be nonnegative continuous functions defined on $[t_0 - \tau, +\infty)$, which satisfy

$$\dot{S}_1(t) \le -rS_1(t) + \sum_{j=1}^{m}\left(b_j S_{1t}^j + c_j S_{2t}^j\right)$$
$$S_2(t) \le \sum_{j=1}^{m}\left(a_j S_{1t}^j + d_j S_{2t}^j\right)$$

for $t \ge t_0$, where $\tau > 0, r > 0, b_j \ge 0, c_j \ge 0, a_j \ge 0, d_j \ge 0$ are constants and $S_{it} \triangleq \sup_{t-\tau \le \theta \le t}[S_i(\theta)]$. If there exist constans $\alpha_1 > 0, \alpha_2 > 0, k > 0$ such that

$$\frac{1}{\alpha_1 r}\sum_{j=1}^{m}(b_j\alpha_1^j + c_j\alpha_2^j)k^{j-1} < 1 \tag{6}$$
$$\frac{1}{\alpha_2}\sum_{j=1}^{m}(a_j\alpha_1^j + d_j\alpha_2^j)k^{j-1} < 1 \tag{7}$$

then, $\frac{S_{1t_0}}{\alpha_1} + \frac{S_{2t_0}}{\alpha_2} < k$ implies that

$$S_1(t) + S_2(t) \le (\alpha_1 + \alpha_2)\left(\frac{S_{1t_0}}{\alpha_1} + \frac{S_{2t_0}}{\alpha_2}\right)e^{-\lambda(t-t_0)}$$

for $t \ge t_0$, where $\lambda > 0$ is a constant.

Proof: From (6) and (7), it follows that for sufficiently small $\lambda > 0$, the inequalities

$$-r + \lambda + \frac{1}{\alpha_1}\sum_{j=1}^{m} e^{\lambda\tau j}(b_j\alpha_1^j + c_j\alpha_2^j)k^{j-1} < 0 \tag{8}$$

and

$$\frac{1}{\alpha_2}\sum_{j=1}^{m} e^{\lambda\tau j}(b_j\alpha_1^j + c_j\alpha_2^j)k^{j-1} < 1 \tag{9}$$

hold.

Defining

$$Q_i(t) = S_i(t)e^{\lambda(t-t_0)}/\alpha_i, \quad t \geq t_0 - \tau, \quad (i = 1, 2),$$

we have for $t \geq t_0$ that

$$\dot{Q}_1(t) \leq -rQ_1(t) + \lambda Q_1(t) + \frac{1}{\alpha_1}\sum_{j=1}^{m} e^{\lambda\tau j}(b_j\alpha_1^j Q_{1t}^j + c_j\alpha_2^j Q_{2t}^j) \qquad (10)$$

and

$$Q_2(t) \leq \frac{1}{\alpha_2}\sum_{j=1}^{m} e^{\lambda\tau j}(a_j\alpha_1^j Q_{1t}^j + d_j\alpha_2^j Q_{2t}^j) \qquad (11)$$

Let $l \in \left(1, \frac{k}{S_{1t_0}/\alpha_1 + S_{2t_0}/\alpha_2}\right)$ be an arbitrary constant; we assert that

$$Q_i(t) < l\left(\frac{S_{1t_0}}{\alpha_1} + \frac{S_{2t_0}}{\alpha_2}\right) \triangleq M \qquad (12)$$

for $t \geq t_0$ and $i = 1, 2$. Suppose (12) does not hold, then there exist a $t_1 > t_0$ and some i such that

$$Q_i(t_1) = M; \quad Q_i(t) < M, \quad t_0 - \tau \leq t < t_1;$$

$$Q_j(t) \leq M, j \neq i, t_0 - \tau \leq t \leq t_1.$$

Case 1: $i = 1$.

Then, we must have $\dot{Q}_1(t_1) \geq 0$. However, frome (10) by (8), we have

$$\dot{Q}_1(t_1) \;\; \leq \;\; -rM + \lambda M + \frac{1}{\alpha_1}\sum_{j=1}^{m} e^{\lambda\tau j}(b_j\alpha_1^j + c_j\alpha_2^j)M^j$$

$$
\begin{aligned}
&\leq \left[-r + \lambda + \frac{1}{\alpha_1} \sum_{j=1}^{m} e^{\lambda \tau j} (b_j \alpha_1^j + c_j \alpha_2^j) k^{j-1} \right] M \\
&< 0
\end{aligned}
$$

This is a contradiction.

Case 2: $i = 2$.

That is $Q_2(t_1) = M$. From (11) by (9), we have

$$
\begin{aligned}
Q_2(t_1) &\leq \frac{1}{\alpha_2} \sum_{j=1}^{m} e^{\lambda \tau j} (a_j \alpha_1^j + d_j \alpha_2^j) M^j \\
&\leq \frac{1}{\alpha_2} \sum_{j=1}^{m} e^{\lambda \tau j} (a_j \alpha_1^j + d_j \alpha_2^j) k^{j-1} \cdot M \\
&< M
\end{aligned}
$$

This is also a contradiction.

Hence, our assertion is correct. Let $l \longrightarrow 1$ in (12), then we have

$$
Q_i(t) \leq \left(\frac{S_{1t_0}}{\alpha_1} + \frac{S_{2t_0}}{\alpha_2} \right)
$$

for $t \geq t_0$ and $i = 1, 2$. Thus,

$$
S_1(t) + S_2(t) \leq (\alpha_1 + \alpha_2) \left(\frac{S_{1t_0}}{\alpha_1} + \frac{S_{2t_0}}{\alpha_2} \right) e^{-\lambda(t - t_0)}
$$

for all $t \geq t_0$ and the proof is complete.

4 Stability Theorems

Consider the nonlinear delay system described as follows

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + g(t, x(t), x(t-\tau)) \\ x(t) = \varphi(t), \quad t_0 - \tau \le t \le t_0 \end{cases} \tag{13}$$

where $f : I \times R^n \longrightarrow R^n, I = [t_0, +\infty)$, is continuously differentiable in the region $I \times R^n, f(t,0) \equiv 0, \quad g : I \times R^n \times R^n \longrightarrow R^n$ is a continuous function with $g(t,0,0) \equiv 0$ and $\varphi(t)$ is continuous on $[t_0 - \tau, t_0]$. The delay $\tau(t)$ is a nonnegative continuous function, which satisfies $0 \le \tau(t) \le \tau$, where τ is a constant. Define $\| \varphi \| = \sup_{t_0 - \tau \le t \le t_0} [\| \varphi(t) \|]$.

Definition 4.1. The zero solution of (13) is said to be k-exponentially stable if there exist $\lambda > 0, \pi \ge 1$ such that $\| \varphi \| < k$ implies that

$$\| x(t, t_0, \varphi) \| \le \pi \cdot \| \varphi \| \cdot e^{-\lambda(t - t_0)}$$

for $t \ge t_0$.

Let $\Phi(t, s, x)$ be the solution of

$$\frac{\partial \Phi(t,s,x)}{\partial t} = \left[\frac{\partial f(t,x)}{\partial x} \right] \Phi(t,s,x)$$

$$\Phi(t,t,x) \equiv I$$

where I is the identity matrix.

Theorem 4.1. If the system (13) satisfies the conditions

(i). $\| \Phi(t,s,x) \| \leq he^{-\int_s^t r(\rho)d\rho}$, for all $(t,s,x) \in I \times I \times R^n$, where $h \geq 1, r(t) \geq r > 0, r, h$ are constants.

(ii).

$$\| g(t,x(t),x(t-\tau(t))) \| \leq \sum_{j=1}^{m} b_j(t) \cdot \left[\sup_{t-\tau \leq \theta \leq t} \| x(\theta) \| \right]^j$$

where $b_j(t)$ is a nonnegative continuous function.

(iii).

$$\frac{b_j(t)}{r(t)} \leq \delta_j, \quad t \geq t_0$$

where δ_j is a constant.

(iv).There exist a constant $k > 0$ such that

$$h \sum_{j=1}^{m} \delta_j \cdot (hk)^{j-1} < 1.$$

Then, the zero solution of (13) is k- exponentially stable.

Proof: By the nonlinear variation of parameters formula, we have

$$x(t) = \begin{cases} y(t) + \int_{t_0}^{t} \Phi(t,s,x(s)) \cdot g(s,x(s),x(s-\tau(s)))ds & t \geq t_0 \\ \varphi(t) & t_0 - \tau \leq t \leq t_0 \end{cases}$$

By lemma 2.1 and conditions (i) and (ii), it follows that

$$\| x(t) \| \leq \| y(t) \|$$

$$
\begin{aligned}
&+ \int_{t_0}^{t} \| \Phi(t,s,x(s)) \| \cdot \| g(s,x(s),x(s-\tau(s))) \| \, ds \\
&\leq h \| \varphi \| e^{-\int_{t_0}^{t} r(\rho)d\rho} \\
&+ h \sum_{j=1}^{m} \int_{t_0}^{t} e^{-\int_s^t r(\rho)d\rho} \cdot b_j(s) \cdot [\sup_{s-\tau\leq\theta\leq s} \| x(\theta) \|]^j ds.
\end{aligned}
$$

Let

$$
S(t) = \| \varphi \| e^{-\int_{t_0}^{t} r(\rho)d\rho} + \sum_{j=1}^{m} \int_{t_0}^{t} e^{-\int_s^t r(\rho)d\rho} \cdot b_j(s) \cdot [\sup_{s-\tau\leq\theta\leq s} \| x(\theta) \|]^j ds
$$

for $t \geq t_0$ and $S(t) = \| \varphi \|$ for $t_0 - \tau \leq t \leq t_0$. then, $\| x(t) \| \leq hS(t)$ for $t \geq t_0 - \tau$ and

$$
\begin{aligned}
\dot{S}(t) &\leq -r(t)S(t) + \sum_{j=1}^{m} b_j(t) \cdot h^j \cdot S_t^j \\
&\leq r(t) \left(-S(t) + \sum_{j=1}^{m} \delta_j \cdot h^j \cdot S_t^j \right)
\end{aligned}
$$

for $t \geq t_0$. Using theorem 3.1, we get that $S_{t_0} < k$ implies that

$$
S(t) \leq S_{t_0} \cdot e^{-\lambda(t-t_0)}
$$

for $t \geq t_0$, where $\lambda > 0$ is a constant. That is $\| \varphi \| < k$ implies that

$$
\| x(t) \| \leq h \| \varphi \| \cdot e^{-\lambda(t-t_0)}
$$

for $t \geq t_0$. Hence, the zero solution of (13) is k-exponentially stable. The proof is complete.

Now, consider the nonlinear neutral delay system

$$\begin{cases} \dot{x}(t) = f[t, x(t), x(t-\tau(t)), \dot{x}(t-\tau(t))] \\ x(t) = \varphi(t), \dot{x}(t) = \dot{\varphi}(t), t_0 - \tau \leq t \leq t_0 \end{cases} \tag{14}$$

where $f \in C[R \times R^n \times R^n \times R^n, R^n], f(t,0,0,0) \equiv 0, \partial f(t,x,0,0)/\partial x$ and $\partial f(t,x,x,0)/\partial x$ exist and are continuous, $\tau(t)$ is a nonnegative continuous function, $0 \leq \tau(t) \leq \tau$, τ is a constant and $\varphi(t), \dot{\varphi}(t)$ are continuous initial functions.

Definition: The zero solution of (14) is said to be k-exponentially stable if there exist $\lambda > 0, \pi \geq 1$ such that

$$\| \varphi \| \triangleq \sup_{t_0-\tau \leq t \leq t_0} [\| \varphi(t) \| + \| \dot{\varphi}(t) \|] < k$$

implies that

$$\| x(t) \| + \| \dot{x}(t) \| \leq \pi e^{-\lambda(t-t_0)}$$

for $t \geq t_0$.

We shall assume that

$$\begin{aligned} & \| f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3) \| \\ \leq \; & \sum_{j=1}^{m} \left(a_j \| x_1 - y_1 \|^j + b_j \| x_2 - y_2 \|^j + c_j \| x_3 - y_3 \|^j \right) \end{aligned} \tag{15}$$

where a_j, b_j, c_j are constants.

Suppose that $\Phi_1(t,s,x)$ is the solution of

$$\frac{\partial \Phi_1(t,s,x)}{\partial t} = \left[\frac{\partial f(t,x,0,0)}{\partial x}\right] \cdot \Phi_1(t,s,x)$$
$$\Phi_1(t,t,x) \equiv I$$

Theorem 4.2. If the system (14) satisfies the conditions

(i). $\| \Phi_1(t,s,x) \| \leq h_1 e^{-r_1(t-s)}$, for all $(t,s,x) \in I \times I \times R^n$, where $h_1 \geq 1, r_1 > 0$ are constants.

(ii). There exist constants $\alpha_1 > 0, \alpha_2 > 0, k > 0$ such that

$$\frac{1}{\alpha_1 r_1} \sum_{j=1}^{m} \left[b_j \cdot (h_1\alpha_1)^j + c_j \cdot \alpha_2^j\right] k^{j-1} < 1$$
$$\frac{1}{\alpha_2} \sum_{j=1}^{m} \left[(a_j + b_j) \cdot (h_1\alpha_1)^j + c_j \cdot \alpha_2^j\right] k^{j-1} < 1$$

Then, the zero solution of (14) is $\frac{\alpha_1\alpha_2 k}{\alpha_1+\alpha_2}$-exponentially stable.

Proof: Rewrite (14) in the form

$$\begin{aligned} \dot{x}(t) &= f[t,x(t),0,0] \\ &+ \left(f[t,x(t),x(t-\tau(t)),\dot{x}(t-\tau(t))] - f[t,x(t),0,0]\right). \end{aligned}$$

By the nonlinear variation of parameters formula, we have

$$\begin{aligned} \| x(t) \| &\leq h_1 e^{-r_1(t-t_0)} \cdot \| \varphi \| \\ &+ h_1 \sum_{j=1}^{m} \int_{t_0}^{t} \left(b_j \| x(s-\tau(s)) \|^j + c_j \| \dot{x}(s-\tau(s)) \|^j\right) e^{-r_1(t-s)} ds \end{aligned}$$

for $t \geq t_0$.

From (14) and (15), we have

$$\| \dot{x}(t) \| \leq \sum_{j=1}^{m} \left(a_j \| x(t) \|^j + b_j \| x(t-\tau(t)) \|^j + c_j \| \dot{x}(t-\tau(t)) \|^j \right).$$

for $t \geq t_0$.

Define

$$\begin{aligned} S_1(t) &= e^{-r_1(t-t_0)} \| \varphi \| \\ &+ \sum_{j=1}^{m} \int_{t_0}^{t} \left(b_j \| x(s-\tau(s)) \|^j + c_j \| \dot{x}(s-\tau(s)) \|^j \right) e^{-r_1(t-s)} ds \end{aligned}$$

for $t \geq t_0$, $S_1(t) = \| \varphi \|$ for $t_0 - \tau \leq t \leq t_0$ and

$$S_2(t) = \begin{cases} \| \dot{x}(t) \| & t \geq t_0 \\ \| \varphi \| & t_0 - \tau \leq t \leq t_0 \end{cases}$$

Then, we have

$$\dot{S}_1(t) \leq -r_1 S_1(t) + \sum_{j=1}^{m} (b_j h_1^j \cdot S_{1t}^j + c_j \cdot S_{2t}^j)$$

$$S_2(t) \leq \sum_{j=1}^{m} [(a_j + b_j) h_1^j \cdot S_{1t}^j + c_j \cdot S_{2t}^j]$$

for $t \geq t_0$.

By the condition (ii), using theorem 3.2, it follows that $(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}) \| \varphi \| < k$ implies that

$$S_1(t) + S_2(t) \leq (\alpha_1 + \alpha_2) \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) \| \varphi \| e^{-\lambda(t-t_0)}$$

for $t \geq t_0$, where $\lambda > 0$ is a constant. That is $\| \varphi \| \leq \frac{\alpha_1 \alpha_2 k}{\alpha_1+\alpha_2}$ implies that

$$\| x(t) \| + \| \dot{x}(t) \| \leq (\alpha_1 + \alpha_2) \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) \| \varphi \| e^{-\lambda(t-t_0)}$$

for $t \geq t_0$.This shows that that zero solution of (14) is $\frac{\alpha_1 \alpha_2 k}{\alpha_1+\alpha_2}$-exponentially stable and the proof is complete.

Suppose that $\Phi_2(t, s, x)$ is the solution of

$$\frac{\partial \Phi_2(t,s,x)}{\partial t} = \left[\frac{\partial f(t,x,x,0)}{\partial x} \right] \Phi_2(t,s,x)$$

$$\Phi_2(t,t,x) \equiv I$$

where I is the identity matrix.

Theorem 4.3. If the system (14) satisfies the conditions

(i). $\| \Phi_2(t,s,x) \| \leq h_2 e^{-r_2(t-s)}$, for all $(t,s,x) \in I \times I \times R^n$, where $h_2 \geq 1, r_2 > 0$ are constants.

(ii).There exist constants $\alpha_1 > 0, \alpha_2 > 0, k > 0$ such that

$$\frac{1}{\alpha_1 r_2} \sum_{j=1}^{m} (b_j \tau^j + c_j) \alpha_2^j k^{j-1} < 1$$

$$\frac{1}{\alpha_2} \sum_{j=1}^{m} \left[(a_j + b_j)(h_2 \alpha_1)^j + c_j \alpha_2^j \right] k^{j-1} < 1$$

Then, the zero solution of (14) is $\frac{\alpha_1 \alpha_2 k}{\alpha_1+\alpha_2}$-exponentially stable.

Proof: Note that

$$\| x(t-\tau(t)) - x(t) \| \leq \| \int_{t-\tau(t)}^{t} \dot{x}(\theta) d\theta \|$$

$$
\begin{aligned}
&\leq \int_{t-\tau}^{t} \| \dot{x}(\theta) \| \, d\theta \\
&\leq \tau \cdot \sup_{t-\tau \leq \theta \leq t} \| \dot{x}(\theta) \|
\end{aligned}
$$

then the result follows as in the proof of theorem 4.2.

5 Examples

Example 5.1. Consider the system

$$\dot{x}(t) = -2(1+\sin^2 t)x(t) + (1+\sin^2 t)\cos x(t) \cdot x(t-\tau(t)) \tag{16}$$

where $0 \leq \tau(t) \leq \tau, \tau$ is a constant.

Taking $m = 1, r(t) = -2(1+\sin^2 t), b_1(t) = (1+\sin^2(t)), \delta_1 = \frac{1}{2}$ in theorem 4.1, it follows that for every $k \in (0, +\infty)$ the zero solution of (16) is k-exponentially stable.

Example 5.2. Consider the system

$$\dot{x}(t) = -64e^t x(t) + e^{tsin^2 t} \cdot x(t) \cdot x^2(t-\tau(t)) \tag{17}$$

where $0 \leq \tau(t) \leq \tau, \tau$ is a constant.

Taking $m = 3, r(t) = 64e^t, b_1(t) = b_2(t) = 0, b_3(t) = e^{t\sin^2 t}, \delta_1 = \delta_2 = 0, \delta_3 = \frac{1}{64}$ in theorem 4.1, we can get that for $k \in (0, 8)$ the zero solution of (17) is k-exponentially stable.

Example 5.3. Consider the system

$$\dot{x}(t) = -x(t) + \frac{1}{128}x^4(t - \tau(t)) - \frac{1}{32}\dot{x}^2(t - \tau(t)) \tag{18}$$

where $0 \leq \tau(t) \leq \tau, \tau$ is a constant.

Taking $m = 4, b_1 = b_2 = b_3 = 0, b_4 = \frac{1}{128}, c_1 = c_3 = c_4 = 0, c_2 = \frac{1}{32}, a_1 = 1, a_2 = a_3 = a_4 = 0, r_1 = 1, h_1 = 1, \alpha_1 = 1, \alpha_2 = 2$ in theorem 4.2, we can get that for $k \in (0, 8/3)$ the zero solution of (18) is k-exponentially stable.

Example 5.4. Consider the system

$$\dot{x}(t) = 2x(t) - 3x(t - \frac{1}{60}) + \frac{1}{1100}x^2(t - \frac{1}{60}) \tag{19}$$

Taking $m = 2, a_1 = 2, a_2 = 0, b_1 = 3, b_2 = 0, c_1 = 0, c_2 = \frac{1}{1100}, \tau = \frac{1}{60}, r_2 = 1, h_2 = 1, \alpha_1 = \frac{1}{4}, \alpha_2 = \frac{5}{2}$ in theorem 4.3. Then it follows that for $k \in (0, 5)$ the zero solution of (19) is k-exponentially stable.

6 Conclusions

In this paper we have studed the k-exponential stability of nonlinear delay systems via the nonlinear variation of parameters formula and nonlinear inequality analysis. By k-exponential stability, we mean that in the 'k region' the solutions decay exponentially to zero. Obviously, this will be a useful criterion in engineering. The stability theorems for nonlinear neutral delay systems are expresed in the terms of some parameters, the opitmal choice of these parameters needs further investigation.

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