**Monograph:**

Banks, S.P. and Zhang, Yi (1990) On the Stability Analysis of Nonlinear Systems. Research Report. Acse Report 389 . Dept of Automatic Control and System Engineering. University of Sheffield

# On the Stability Analysis of Nonlinear Systems

by

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Feb 1990

Research Report No 389

# Abstract

The stability of nonlinear systems and nonlinear delay systems is investegated in this paper via the methods of Carleman linearization and inequality analysis. Some new criteria for stability are obtained.


Keywords: Stability, Nonlinear Systems, Delay.

# 1 Introduction

The stability of nonlinear systems has been widely studied, particularly via the Liapunov approach(LaSalle and Lefschetz [6],1961).More recently, global linearization techniques(Banks[2],1988) and the nonlinear variation of parameter formula(Alakseev[1],1961, and Brauer[5],1966) has been used in deriving new stability and limit cycle results (Banks[3],1986,and Banks[4],1988). In this paper, we shall combine the latter two methods and derive some further new stability criteria for nonlinear ordinary and delay differential equations.

The method will be based on an analysis of an infinite-dimentional differential inequality considered in section 2. In section 3 we shall apply the results of section 2 to nonlinear ordinary differential equations by using Carleman linearization. Combining the results of section 2 and the nonlinear variation of parameters formula, in section 4 shall derive stability criteria for nonlinear delay equations. Finally in section 5, some examples will be give to illustrate the theory.

## 2 Inequality Analysis

In this section,we shall establish some inequalities, which will be used in the stability analysis of nonlinear systems.

*Theorem* 2.1. Assume that $v_i(t), i \in I$ are nonnegative continuous functions defined on $[t_0, +\infty)$, which satisfy

$$\dot{v_i}\ (t) \leq -r_i(t)v_i(t) + \sum_{j\in I} a_{ij}(t)v_i(t) \tag{1}$$

where $r_i(t), a_{ij}(t)$ are nonnegative continuous functions, $I$ is a countable index set. If there exist constants $h > 0, \delta > 0$, and $d_i > 0$ such that

$$r_i(t) \leq h < 0$$

$$\frac{1}{d_i}\sum_{j\in I} d_i \frac{a_{ij}(t)}{r_i(t)} \leq \delta < 1$$

$$\sup_{i\in I}\left[\frac{v_i(t_0)}{d_i}\right] < +\infty$$

for all $t \leq t_0$ and all $i \in I$. Then, there exist constants $\lambda$ and $m \geq 1$ such that

$$\frac{v_i(t)}{d_i} \leq m \cdot \sup_{j\in I}\left[\frac{v_j(t_0)}{d_j}\right] \cdot e^{-\lambda(t-t_0)}$$

for all $t \geq t_0$ and all $i \in I$.

*Proof*: Since $\delta < 1, h > 0$, it follows that for sufficently small $\lambda > 0$, the inequality

$$1 - \frac{\lambda}{h} - \delta > 0$$

holds.

Define

$$Q_i(t) = v_i(t)e^{\lambda(t-t_0)}/d_i, t \geq t_0, i \in I$$

Then, we have

$$\dot{Q}_i(t) \leq -r_i(t)Q_i(t) + \lambda Q_i(t) + \frac{1}{d_i}\sum_{j\in I} d_j a_{ij}(t)Q_j(t) \qquad (2)$$

for all $t \geq t_0$ and all $i \in I$.

Integrating (2), we have

$$\begin{aligned} Q_i(t) &\leq Q_i(t_0)e^{-\int_{t_0}^t r_i(\tau)d\tau} + \lambda \int_{t_0}^t Q_i(s)e^{\int_s^t r_i(\tau)d\tau} ds \\ &+ \int_{t_0}^t \frac{1}{d_i}\sum_{j\in I} d_i a_{ij}(s)Q_j(s)e^{-\int_s^t r_i(\tau)d\tau} ds \end{aligned}$$

Let

$$Q(t) = \sup_{i\in I}\left[\sup_{t_0\leq s\leq t}[Q_i(s)]\right]$$

Then we get

$$\begin{aligned} Q_i(t) &\leq Q(t_0) + \frac{\lambda}{h}Q(t) \\ &+ \int_{t_0}^t \frac{1}{d_i}\sum_{j\in I} d_i \frac{a_{ij}(s)}{r_i(s)} r_i(s)e^{-\int_s^t r_i(\tau)d\tau} ds Q(t) \\ &\leq Q(t_0) + \frac{\lambda}{h}Q(t) + \delta \int_{t_0}^t r_i(s)e^{-\int_s^t r_i(\tau)d\tau} ds Q(t) \end{aligned}$$

Noting that

$$\int_{t_0}^{t} r_i(s) e^{-\int_s^t r_i(\tau)d\tau} ds \leq 1$$

for $t \geq t_0$, We have

$$Q_i(t) \leq Q(t_0) + \left(\frac{\lambda}{h} + \delta\right) Q(t)$$

for all $t \geq t_0$ and all $i \in I$. Hence,

$$Q(t) \leq Q(t_0) + \left(\frac{\lambda}{h} + \delta\right) Q(t)$$

and

$$Q(t) \leq \frac{1}{1 - \frac{\lambda}{h} - \delta} Q(t_0).$$

Therefore, we get

$$\frac{v_i(t)}{d_i} \leq m \cdot \sup_{i \in I} \left[\frac{v_i(t_0)}{d_i}\right] e^{-\lambda(t-t_0)}$$

for all $t \geq t_0$ and all $i \in I$, where $m = 1/(1 - \frac{\lambda}{h} - \delta)$. This completes the proof.

*Theorem* 2.2. Assume that $S(t)$ is a nonnegative continuous function defined on $[t_0, +\infty)$, which satisfies

$$\dot{S}(t) \leq -r(t)S(t) + \sum_{j=1}^{\infty} b_j(t) S^j(t)$$

for $t \geq t_0$, where $r(t), b_j(t)$ are nonnegative continuous functions. If there exist constants $d_i > 0, (i = 1, 2, \ldots)$ such that

$$\frac{1}{d_i \cdot r(t)} \sum_{j=1}^{\infty} b_j(t) \cdot d_{j+i-1} \leq \delta < 1$$

$$\sup_{j \leq 1} \left[ \frac{S^j(t_0)}{d_j} \right] < +\infty$$

for all $i \geq 1$ and all $t \geq t_0$, then, there exist constants $\lambda > 0$ and $m \geq 1$ such that

$$S(t) \leq m d_1 \cdot \sup_{j \geq 1} \left[ \frac{S^j(t_0)}{d_j} \right] \cdot e^{-\lambda(t-t_0)}$$

for $t \geq t_0$.

*Proof*: Define

$$v_i(t) = S^i(t), (i = 1, 2, \ldots)$$

We have

$$\begin{aligned} \dot{v}_i(t) &= iS^{i-1}(t) \cdot \dot{S}(t) \\ &\leq -ir(t)S^i(t) + i \sum_{j=1}^{\infty} b_j(t) S^{i+j-1}(t) \\ &= -ir(t)v_i(t) + i \sum_{j=1}^{\infty} b_j(t) \cdot v_{j+i-1}(t) \end{aligned}$$

Then, the theorem 2.1 applies.

*Corollary* 2.1. If there exist a constant $k > 0$ such that

$$\frac{1}{r(t)} \sum_{j=1}^{\infty} k^{j-1} \cdot b_j(t) \leq \delta < 1$$
$$S(t_0) \leq k$$

Then, there exist $\lambda > 0, m \geq 1$ such that

$$S(t) \leq m \cdot S(t_0) \cdot e^{-\lambda(t-t_0)}$$

for $t \geq t_0$.

*Proof*: Chose $d_i = k^{i-1}$ in theorem 2.2 and note that

$$S(t_0) \geq \frac{S^i(t_0)}{k^{i-1}}$$

for all $i \geq 1$. Then, the result follows.

# 3 Nonlinear Systems

We shall assume that an $n$-multi-index is an $n$-multiple $i = (i_1, \ldots, i_n)$ of nonnegative integers, $i_1 + \ldots + i_n = i, i_k \geq 0 (k = 1, \ldots, n)$. The sum of two multi-indices $i$ and $j$ is defined as $i + j = (i_1 + j_1, \ldots, i_n + j_n)$. We say that $i \geq j$ if $i_k \geq j_k$ for $k = 1, \ldots, n$. When $i \geq j$, we define $j - i$ as

$(j_1 - i_1, \ldots, j_n - i_n)$.We also define

$$x^i = x_1^{i_1} \ldots x_n^{i_n}, \qquad \| x \| = \max_{1 \le s \le n} (| x_s |)$$

for $x = (x_1, \ldots, x_n) \in R^n$. Also,$1(k)$ will denote the $n$-multi-index with 1 in the $kth$ place and zero elsewhere.

Consider the nonlinear system

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0 \tag{3}$$

where $x \in R^n$ and $f : D \times R^n \longrightarrow R^n$ is an analytic function for $t \ge t_0$, where $D = [t_0, +\infty), f(t, 0) \equiv 0$. We shall assume that the solution of (3) exists and is unique.

*Defination* 3.1. The zero solution of (3) is said to be $K$-globally exponentially stable, if there exist $\lambda > 0, \pi \ge 1$ such that $\| x_0 \| \le K$ implies that

$$\| x(t, t_0, x_0) \| \le \pi \cdot \| x_0 \| \cdot e^{-\lambda(t - t_0)}$$

for $t \ge t_0$.

Since $f(t, x(t))$ is analytic, we may write

$$f_k(t, x(t)) = \sum_{i \ge 1} f_i^k(t) \cdot x^i(t), \qquad 1 \le k \le n.$$

Define the functions

$$\phi_i(t) = x_1^{i_1}(t) \dots x_n^{i_n}(t) = x^i(t), \qquad i \geq 1$$

and differentiate $\phi_i(t)$ along the trajectories of (3). Then, we have

$$\begin{aligned}
\dot{\phi}_i(t) &= \sum_{k=1}^{n} i_k x^{(i-1(k))}(t)\dot{x}_k(t) \\
&= \sum_{k=1}^{n} i_k x^{(i-1(k))}(t) f_k(t, x(t)) \\
&= \sum_{k=1}^{n} i_k x^{(i-1(k))}(t) \sum_{j\geq 1} f_j^k(t) x^j(t) \\
&= \sum_{j\geq 1}\sum_{k=1}^{n} i_k f_{j-i+1(k)}^k(t) x^j(t)
\end{aligned}$$

where we define $f_l^k(t) \equiv 0$, if $l \leq 0$.

Hence,

$$\dot{\phi}_i(t) = a_i^i(t)\phi_i(t) + \sum_{j\neq i} a_i^j(t)\phi_j(t) \tag{4}$$

where

$$a_i^j(t) = \sum_{k=1}^{n} i_k \cdot f_{j-i+1(k)}^k(t) \tag{5}$$

*Theorem* 3.1. If there exist constants $h > 0, \delta > 0$ and $K > 0$ such that

$$a_i^i(t) \leq -h < 0$$

$$\sum_{j\neq i} K^{j-i} \cdot \frac{|\, a_i^j(t) \,|}{|\, a_i^i(t) \,|} \leq \delta < 1$$

for all $i \geq 1$ and all $t \geq t_0$. Then, the zero solution of (3) is $K$-globally exponentially stable.

*Proof*: From (4), by the method of parameter variation, we have

$$\phi_i(t) = \phi_i(t_0)e^{\int_{t_0}^{t} a_i^i(\tau)d\tau} + \sum_{j\neq i} \int_{t_0}^{t} a_i^j(s)\phi_j(s)e^{\int_s^t a_i^i(\tau)d\tau}ds$$

so that

$$| \phi_i(t) | \leq | \phi_i(t_0) | e^{\int_{t_0}^{t} a_i^i(\tau)d\tau} + \sum_{j\neq i} \int_{t_0}^{t} | a_i^j(s) | \cdot | \phi_j(s) | \cdot e^{\int_s^t a_i^i(\tau)d\tau}ds$$

Let

$$v_i(t) = | \phi_i(t_0) | e^{\int_{t_0}^{t} a_i^i(\tau)d\tau} + \sum_{j\neq i} \int_{t_0}^{t} | a_i^j(s) | \cdot | \phi_j(s) | \cdot e^{\int_s^t a_i^i(\tau)d\tau}ds$$

then, $| \phi_i(t) | \leq v_i(t)$ for all $i \geq 1$ and all $t \geq t_0$. Moreover,

$$\dot{v}_i(t) \leq a_i^i(t)v_i(t) + \sum_{j\neq i} | a_i^j(t) | \cdot v_j(t) \tag{6}$$

Choose $d_i = K^{i-1}, i \geq 1$, then, if $\| x(t_0) \| \leq K$, we have

$$\frac{v_i(t_0)}{d_i} = \frac{| \phi_i(t_0) |}{d_i} = \frac{| x_1^{i_1}(t_0) \dots x_n^{i_n}(t_0) |}{K^{i-1}} \leq \frac{\| x(t_0) \|^i}{K^{i-1}} \leq \| x(t_0) \|$$

Hence

$$\sup_{j\geq 1} \left[ \frac{v_j(t_0)}{d_j} \right] \leq \| x(t_0) \|$$

By theorem 2.1, there exist $\lambda > 0, m \geq 1$ such that

$$\frac{v_i(t)}{K^{i-1}} \leq m \cdot \sup_{j\geq 1} \left[\frac{v_j(t_0)}{d_j}\right] e^{\lambda(t-t_0)}$$

for all $t \geq t_0$ and all $i \geq 1$. Then, we have

$$| \phi_1(t) | \leq m \cdot \| x(t_0) \| e^{-\lambda(t-t_0)}$$

for all $t \geq t_0$.

Therefore, we get

$$\| x(t) \| \leq m \cdot \| x(t_0) \| e^{-\lambda(t-t_0)}$$

for all $t \geq t_0$, whenever $\| x(t_0) \| \leq K$ . This shows that the zero solution of (3) is $K$-globally exponentially stable and the proof is complete.

*Corollary* 3.1. If there exist constants $h > 0, \delta > 0$ and $K > 0$ such that

$$f^l_{1(l)}(t) \leq -h < 0$$

$$\sum_{j\neq 0} K^j \cdot \frac{| f^l_{j+1(l)}(t) |}{| f^l_{1(l)}(t) |} \leq \delta < 1$$

for all $t \geq t_0$ and $1 \leq l \leq n$, then, the zero solution of (3) is $K$-globally exponentially stable.

*Proof*: To prove corollary 3.1, we only need to check that conditions in theorem 3.1 is satisfied. Indeed, from (5), we have

$$a^i_i(t) = \sum_{l=1}^{n} i_k \cdot f^l_{1(l)}(t) = \sum_{l=1}^{n} i_l \cdot f^l_l(t) \leq -hi \leq -h < 0$$

and

$$
\begin{aligned}
&\sum_{j\neq i} K^{j-i}\frac{| a_i^j(t) |}{| a_i^i(t) |} \\
= \;& 1+\frac{1}{| a_i^i(t) |}\left[a_i^i(t)+\sum_{j\neq i} K^{j-i} \mid a_i^j(t) \mid\right] \\
\leq \;& 1+\frac{1}{hi}\left[\sum_{l=1}^{n} i_l \cdot f_{1(l)}^l(t)+\sum_{j\neq i} K^{j-i}\cdot\sum_{l=1}^{n} i_l\cdot \mid f_{j-i+1(l)}^l(t) \mid\right] \\
\leq \;& 1+\frac{1}{hi}\sum_{l=1}^{n} i_l\cdot \mid f_{1(l)}^l(t) \mid \left[-1+\sum_{j\neq i} K^{j-i}\cdot\frac{| f_{j-i+1(l)}^l(t) |}{| f_{1(l)}^l(t) |}\right] \\
\leq \;& 1+\frac{1}{hi}\sum_{l=1}^{n} i_l\cdot \mid f_{1(l)}^l(t) \mid \left[-1+\sum_{j\neq 0} K^{j}\cdot\frac{| f_{j+1(l)}^l(t) |}{| f_{1(l)}^l(t) |}\right] \\
\leq \;& 1-\frac{(1-\delta)}{hi}\sum_{l=1}^{n} i_l\cdot \mid f_{1(l)}^l(t) \mid \\
\leq \;& 1-\frac{(1-\delta)}{hi}\sum_{l=1}^{n} i_l\cdot h \\
= \;& \delta \\
< \;& 1
\end{aligned}
$$

This completes the proof.

Now, let $x^{[p]}, p \geq 1$, denote $N(n,p)$-dimensional vector

$$N(n,p)=\binom{n+p-1}{n}$$

of homogeneous $p$-forms in the components of $x$. The elements of the vector $x^{[p]}$ are of the form $x_1^{p_1}\ldots x_n^{p_n}$ with $p_1+\ldots p_n=p, p_i\geq 0, 1\leq i\leq n$.

Write $f(t, x(t))$ as

$$f_k(t, x(t)) = \sum_{p=1}^{\infty} \sum_{p_1+\ldots+p_n} f^k_{p_1,\ldots,p_n}(t) \cdot x_1^{p_1}(t) \ldots x_n^{p_n}(t)$$

Similarly as disscussed by Sira-Ramirez[7] (1988), we may write

$$\frac{d}{dt} x^{[p]} = \sum_{l=p}^{\infty} F_l^p(t) \cdot x^{[l]}(t) \tag{7}$$

where $F_l^p(t)$ is an $N(n,p) \times N(n,l)$ matrix ($l \geq p$) defined as

$$F_l^p(t) = \left( \sum_{k=1}^{n} i_k f^k_{l_1-p_1,\ldots,l_k-p_k+1,\ldots,l_n-p_n}(t) \right)_{\sum_{i=1}^{n} l_i = l}^{\sum_{i=1}^{n} p_i = p}$$

In the space of $R^{N(n,p)}$, we define

$$\| x^{[p]} \| = \sup_{p_1+\ldots+p_n=p} [\| x_1^{p_1} \ldots x_n^{p_n} \|]$$

where, $x \in R^n$.

Suppose that $\Phi_p(t,s)$ satisfies

$$\frac{\partial \Phi_p(t,s)}{\partial t} = F_p^p(t) \cdot \Phi_p(t,s)$$

$$\Phi_p(t,t) = I_p$$

where $I_p$ is an indentity matrix. We also suppose there exist a nonnegative continuous function $r_p(t)$ such that

$$\| \Phi_p(t,s) \| \leq e^{-\int_s^t r_p(\tau) d\tau}$$

for $t \geq s \geq t_0$.

*Theorem* 3.2. If there exist $h > 0, \delta > 0$ and $K > 0$ such that

$$r_p(t) \geq h > 0$$
$$\sum_{l=p+1}^{\infty} K^{l-p} \cdot \frac{\| F_l^p(t) \|_l}{r_p(t)} \leq \delta < 1$$

for all $p \geq 1$ and all $t \geq t_0$. Then, the zero solution of (3) is $K$-globally exponentially stable.

*Proof*: From (7), by the method of parameter variation, we have

$$x^{[p]} = \Phi_p(t, t_0)x^{[p]}(t_0) + \sum_{l=p+1}^{\infty} \int_{t_0}^{t} \Phi_p(t, s)F_l^p(s)x^{[l]}(s)ds$$

Taking norm no both sides, we have

$$\begin{aligned} \| x^{[p]}(t) \|_p \ &\leq \ e^{-\int_{t_0}^t r_p(\tau)d\tau} \| x^{[p]}(t_0) \| \\ &+ \ \sum_{l=p+1}^{\infty} \int_{t_0}^{t} e^{-\int_s^t r_p(\tau)d\tau} \cdot \| F_l^p(s) \|_l \cdot \| x^{[l]}(s) \|_l \cdot ds \end{aligned}$$

Let

$$\begin{aligned} v_p(t) \ &= \ e^{-\int_{t_0}^t r_p(\tau)d\tau} \| x^{[p]}(t_0) \| \\ &+ \ \sum_{l=p+1}^{\infty} \int_{t_0}^{t} e^{-\int_s^t r_p(\tau)d\tau} \cdot \| F_l^p(s) \|_l \cdot \| x^{[l]}(s) \|_l \cdot ds \end{aligned}$$

Then, choose $d_p = K^{p-1}$ and note that

$$\| x^{[p]}(t_0) \| = \sup_{p_1+\ldots+p_n=p} [| x_1^{p_1}(t_0) \ldots x_n^{p_n}(t_0) |] \leq \| x(t_0) \|^p$$

As in the proof of theorem 3.1, it follows that the zero solution of (3) is $K$-globally exponentially stable. This completes the proof.

# 4 Nonlinear Delay Systems

Consider the nonlinear delay system described as follows

$$\dot{x}(t) = f(t, x(t)) + h(t, x(t), x(t-\tau))$$
$$x(t) = \varphi(t), \quad t_0 - \tau \leq t \leq t_0 \tag{8}$$

where $f : I \times R^n \longrightarrow R^n, I = [t_0, +\infty)$, is continuously differentiable in the region $I \times R^n, f(t, 0) \equiv 0, \quad h : I \times R^n \times R^n \longrightarrow R^n$ is a continuous function with $h(t, 0, 0) \equiv 0$ and $\varphi(t)$ is continuous on $[t_0 - \tau, t_0]$. We define $\| \varphi \| = \sup_{t_0 - \tau \leq t \leq t_0} [\| \varphi(t) \|]$. We shall always assume that the solution of (8) exists and is unique.

*Defination* 4.1. The zero solution of (8) is said to be $K$-globally exponentially stable if there exist $\lambda > 0, \pi \geq 1$ such that $\| \varphi \| \leq K$ implies that

$$\| x(t, t_0, \varphi(t)) \| \leq \pi \cdot \| \varphi \| \cdot e^{-\lambda(t-t_0)}$$

for $t \geq t_0$.

To analyse the stability of (8), we need to consider the system

$$\dot{y}(t) = f(t, y(t)) \tag{9}$$

Let $y(t) = y(t, t_0, \varphi(t_0)), x(t) = x(t, t_0, \varphi(t))$. Using the Alekseev nonlinear variation-of-constants formula (cf. Brauer 1966), we have the following result.

*Lemma* 4.1. The solutions of (8) and (9) are related by the formula

$$x(t) = \begin{cases} y(t) + \int_{t_0}^{t} \Phi(t, s, x(s))h(s, x(s), x(s-\tau))ds & t \leq t_0 \\ \varphi(t) & t_0 - \tau \leq t \leq t_0 \end{cases}$$

where $\Phi$ is the matrix function given by

$$\Phi(t, t_0, y_0) = \frac{\partial}{\partial y_0} y(t, t_0, y_0)$$

and it is the fundmental solution of the variational system

$$\dot{Z} = f_y[t, y(t, t_0, y_0]Z$$

We also have from Brauer(1966) the result:

*Lemma* 4.2. Let $x_0, y_0 \in R^n$ and denote by $\theta$ the straight line between $x_0$ and $y_0$,i.e.

$$\theta(t) = x_0 + \lambda(y_0 - x_0), 0 \leq \lambda \leq 1$$

Then the system (9) has solutions through $x_0, y_0$, which satisfy

$$\| y(t, t_0, y_0) - y(t, t_0, x_0) \| \leq \max_{0 \leq \lambda \leq 1} \| \Phi(t, t_0, \theta(\lambda) \| \cdot \| y_0 - x_0 \|$$

*Theorem* 4.1. If the system (8) satisfies the following conditions

(i).$\| \Phi(t,s,x) \| \leq m_1 e^{-r(t-s)}$ for all $(t,s,x) \in I \times I \times R^n$, where $r > 0, m \geq 1$ are constants;

(ii).$\| h(t,x(t),x(t-\tau)) \| \leq \sum_{j=1}^{\infty} (b_j \| x(t) \|^j + c_j \| x(t-\tau) \|^j)$, where $b_j \geq 0, \quad c_j \geq 0$;

(iii). There exist a constant $K > 0$ such that

$$\frac{m_1}{r} \sum_{j=1}^{\infty} (b_j + e^{r\tau} c_j) K^{j-1} < 1$$

then the zero solution of (8) is $\frac{K}{m_1+1}$-globally exponentially stable.

*Proof*: From lemma 4.2, it follows that

$$\| x(t) \| \leq \| y(t) \| + \int_{t_0}^{t} \| \Phi(t,s,x(s)) \| \cdot \| h(s,x(s),x(s-\tau)) \| ds$$

for $t \geq t_0$.

By lemma 4.2 and conditions $(i)$ and $(ii)$, we have

$$\begin{aligned} \| x(t) \| &\leq m_1 \| \varphi \| e^{-r(t-t_0)} \\ &+ m_1 \sum_{j=1}^{\infty} \int_{t_0}^{t} e^{-(t-s)} \left( b_j \| x(s) \|^j + c_j \| x(s-\tau) \|^j \right) ds \\ &\leq m_1 \| \varphi \| e^{-(t-t_0)} + m_1 \sum_{j=1}^{\infty} b_j \int_{t_0}^{t} e^{(t-s)} \| x(s) \|^j ds \\ &+ m_1 e^{r\tau} \sum_{j=1}^{\infty} c_j \int_{t_0-\tau}^{t-\tau} e^{-r(t-s)} \| x(s) \|^j ds \end{aligned}$$

$$
\begin{aligned}
&= m_1 \| \varphi \| e^{-(t-t_0)} + m_1 e^{r\tau} \sum_{j=1}^{\infty} c_j \int_{t_0-\tau}^{t_0} e^{-r(t-s)} \| x(s) \|^j ds \\
&+ m_1 \sum_{j=1}^{\infty} (b_j + e^{r\tau} c_j) \int_{t_0}^{t} e^{-r(t-s)} \| x(s) \|^j ds \\
&\le m_1 \| \varphi \| e^{-r(t-t_0)} + \frac{m_1 e^{r\tau}}{r} \sum_{j=1}^{\infty} c_j \| \varphi \|^j \cdot e^{-r(t-t_0)} \\
&+ m_1 \sum_{j=1}^{\infty} (b_j + e^{r\tau} c_j) \int_{t_0}^{t} e^{-r(t-s)} \| x(s) \|^j ds \qquad (10)
\end{aligned}
$$

for $t \ge t_0$.

When $\| \varphi \| \le K$, we have

$$
\frac{\| \varphi \|^j}{K^{j-1}} \le \| \varphi \|
$$

for $j = 1, 2, \ldots$. Then,we get

$$
\begin{aligned}
\frac{m_1 e^{r\tau}}{r} \sum_{j=1}^{\infty} c_j \| \varphi \|^j &\le \frac{m_1 e^{r\tau}}{r} \sum_{j=1}^{\infty} c_j K^{j-1} \cdot \frac{\| \varphi \|^j}{K^{j-1}} \\
&\le \frac{m_1 e^{r\tau}}{r} \sum_{j=1}^{\infty} c_j K^{j-1} \cdot \| \varphi \| \\
&\le \| \varphi \|
\end{aligned}
$$

(This follows from condition (iii)).

From (10), it follows that

$$
\begin{aligned}
\| x(t) \| &\le (m_1 + 1) \| \varphi \| e^{r(t-t_0)} \\
&+ m_1 \sum_{j=1}^{\infty} (b_j + e^{r\tau} c_j) \int_{t_0}^{t} e^{-r(t-s)} \| x(s) \|^j ds
\end{aligned}
$$

for $t \geq t_0$, whenever $\| \varphi \| \leq K$.

Let

$$\begin{aligned} S(t) &= (m_1+1) \| \varphi \| e^{-r(t-t_0)} \\ &+ m_1 \sum_{j=1}^{\infty} (b_j + e^{r\tau} c_j) \int_{t_0}^{t} e^{-r(t-s)} \| x(s) \|^j \, ds \end{aligned}$$

We have $\| x(t) \| \leq S(t)$ for $t \geq t_0$, and

$$\dot{S}(t) \leq -rS(t) + m_1 \sum_{j=1}^{\infty} (b_j + e^{r\tau} c_j) S^j(t)$$

for $t \geq t_0$.

Since $S(t_0) = (m_1+1) \| \varphi \|$, then, when $\| \varphi \| \leq \frac{K}{m_1+1}$, using corollary 2.1, we get

$$S(t) \leq m(m_1+1) \| \varphi \| e^{-\lambda(t-t_0)}$$

for $t \geq t_0$, where $\lambda > 0, m \geq 1$ are constants.

Therefore, we have

$$\| x(t) \| \leq m(m_1+1) \| \varphi \| e^{-\lambda(t-t_0)}$$

for $t \geq t_0$, whenever $\| \varphi \| \leq \frac{K}{m_1+1}$. This shows that the zero solution of (8) is $\frac{K}{m_1+1}$-globally exponentially stable. The proof is complete.

# 5 Examples

*Example* 5.1. Consider the one-dimentional delay system

$$\dot{x}(t) = f(x(t)) + h(x(t-\tau)) \tag{11}$$

where $f(0) = h(0) = 0$. Assume that $f(x(t))$ and $h(x(t-\tau))$ are analytic. Then, we may write

$$f(x(t)) = \sum_{j=1}^{\infty} f_j \cdot x^j(t), \quad h(x(t-\tau)) = \sum_{j=1}^{\infty} h_j \cdot x^j(t-\tau)$$

Suppose that $f_1 < 0$, then, if there exist a constant $K > 0$ such that

$$\frac{1}{\mid f_1 \mid}\left[\mid h_1 \mid e^{|f_1|\cdot\tau} + \sum_{j=2}^{\infty}\left(\mid f_j \mid + \mid h_j \mid e^{|f_1|\cdot\tau}\right) K^{j-1}\right] < 1$$

the zero solution of (11) is $\frac{K}{2}$-globally exponentially stable.

*Example* 5.2. Consider the delay system

$$\dot{x}(t) = -x(t) + \frac{1}{40}x^2(t - \ln 2) \tag{12}$$

It is easy to check that for every $K \in (0, 10)$ the zero solution of (12) is $K$-globally exponentially stable.

*Example* 5.3. Consider the system

$$\begin{cases} \dot{x}_1(t) = -18(t+1) \cdot x_1(t) + 3t \cdot \sin^2 t \cdot x_2^2(t) - t\cos t \cdot x_1^2(t) \cdot x_2(t) \\ \dot{x}_2(t) = e^{t\sin^2 t} \cdot x_1^3(t) - 9e^t \cdot x_2(t) \end{cases} \tag{13}$$

Using corollary 3.1, we can get that for every $K \in (0, 3)$ that zero solution of (13) is $K$-globally exponentially stable.

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