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Nonlinear Systems , the Lie Series and the Left Shift Operator : Application to Nonlinear Optimal Control

by

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Abstract

A new representation of nonlinear systems involving the Lie series is obtained and applied to nonlinear optimal control .

Keywords : Nonlinear Systems , Lie Series , Left Shift Operator , Optimal Control .

1 Introduction

A great deal of attention has been given to the global linearization of the non-linear system

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 \tag{1.1}$$

where f is a real analytic function , by the method of Carleman linearization (see [1] and the references contained therein).

In this paper we shall approach the problem in a different way by using the Lie series. This will give rise to an (exact) infinite-dimensional linear realization of the form

$$\dot{\Phi} = A\Phi, \quad \Phi(0) = \Phi_0 \tag{1.2}$$

for some 'objects' Φ, A to be specified later. In fact, A will turn out to be a left-shift operator, independent of f. (The dynamics of (1.1) will be contained entirely in Φ_0, the initial value of (1.2).)

Using a similar approach to the linear-analytic system

$$\dot{x} = f(x) + ug(x) \quad , \quad x(0) = x_0 \tag{1.3}$$

we shall obtain an infinite-dimensional bilinear realization of the form

$$\dot{\Phi} = A\Phi + uB\Phi \quad , \quad \Phi(0) = \Phi_0$$

and this will be shown to lead to an explicit solution of the linear- analytic-quadratic optimal control problem, in terms of f, g and their derivatives. This is, in general, not possible using Carleman linearization because of the complexity of A and B which are produced by this method.

In section 2 we shall give a brief introduction to Lie series, with two very simple examples to illustrate the technique. In section 3 a connection between general nonlinear systems and the left-shift operator will be established and in section 4 we shall generalize this idea to nonlinear control systems. Finally , in section 5, the method will be applied to obtain an explicit solution to the linear-analytic-quadratic optimal control problem.

2 The Lie Series

Consider the nonlinear differential equation

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 \in R^n \tag{2.1}$$

where f is real-analytic and assume that solutions exist for all $x_0 \in R^n$ and all $t \in R$. Then it is well known that the solution of the equation is given by the *Lie series*

$$x(t) = \left\{ exp\left(t \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i} \right) x \right\} \Bigg|_{x=x_0} . \tag{2.2}$$

(See [5],[2]). An elementary proof of this result will be given later. We can write (2.2) in the form

$$x(t) = exp\,(tf\partial/\partial x)\, x\ |_{x=x_0}$$

where f is regarded as a row vector and $\partial/\partial x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Thus,

$$x(t) = x_0 + tf(x) + \sum_{k=2}^{\infty} \frac{t^k}{k!} f \frac{\partial}{\partial x}\left(f\frac{\partial}{\partial x}\left(\cdots \frac{\partial f}{\partial x}\right)\cdots\right)\Bigg|_{x=x_0} \tag{2.3}$$

is the general form of the solution of (2.1).

Two examples are now presented to illustrate the solution (2.3).

(a) If the equation is linear, i.e.

$$\dot{x} = Ax \quad , \quad x(0) = x_0 \in R^n$$

then, by (2.3),

$$\begin{aligned} x(t) &= x_0 + tAx_0 + \sum_{k=2}^{\infty} \frac{t^k}{k!} Ax\frac{\partial}{\partial x}\left(Ax\frac{\partial}{\partial x}\left(\cdots \frac{\partial (Ax)}{\partial x}\right)\cdots\right)\Bigg|_{x=x_0} \\ &= x_0 + tAx_0 + \sum_{k=2}^{\infty} \frac{t^k}{k!} A^k x_0 \\ &= e^{At}x_0. \end{aligned}$$

(b) Consider the scalar equation

$$\dot{x} = x^2$$

Then, by (2.3),

$$\begin{aligned} x(t) &= x_0 + tx_0^2 + \sum_{k=2}^{\infty} \frac{t^k}{k!} x^2\frac{\partial}{\partial x}\left(x^2\frac{\partial}{\partial x}\left(\cdots \frac{\partial x^2}{\partial x}\right)\cdots\right)\Bigg|_{x=x_0} \\ &= x_0 + tx_0^2 + \sum_{k=2}^{\infty} t^k x_0^{k+1} \\ &= \frac{x_0}{1 - tx_0}. \end{aligned}$$

These examples show that the Lie series (2.2) is merely the Taylor expansion (with respect to t) of the solution of the differential equation (2.1).

3 Nonlinear Systems and the Left-Shift Operator

Consider again the nonlinear differential equation

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 \in R^n \quad . \tag{3.1}$$

in which f satisfies the same conditions as in section 2. In this section we propose to obtain an infinite-dimensional representation of this system in terms of the left-shift operator. In order to do this, let $g_1(x)$ be any analytic function of x (for example $g_1(x) = x$) and define, recursively,

$$g_i(x) = \frac{\partial g_{i-1}}{\partial x} f \quad , \quad i \geq 2 \quad . \tag{3.2}$$

Then

$$\begin{aligned} \frac{d g_i}{dt} &= \frac{\partial g_i}{\partial x}\frac{dx}{dt} \\ &= \frac{\partial g_i}{\partial x} f \\ &= g_{i+1} \quad . \end{aligned}$$

Hence, if we define $g = (g_1, g_2, \cdots)^T \in (\mathcal{O}(R^n))^{N^+}$, where $\mathcal{O}(R^n)$ is the ring of analytic functions on R^n, then

$$\frac{dg}{dt} = Ag \tag{3.3}$$

where A is the left-shift operator defined on $(\mathcal{O}(R^n))^{N^+}$ by

$$A(g_1, g_2, \cdots)^T = (g_2, g_3, \cdots)^T \quad .$$

We can now prove

Theorem 3.1 The solution of equation (3.1) may be written in the form

$$x = g_1^{-1}\left(P\left\{e^{At}g(x_0)\right\}\right) \tag{3.4}$$

where P is the projection operator defined by

$$P(g_1, g_2, \ldots) = g_1 \quad .$$

In particular, if $g_1(x) = x$, then

$$x = P\left\{e^{At}g(x_0)\right\} \quad . \tag{3.5}$$

Here, $g(x_0)$ is given by

$$g(x_0) = (g_1(x_0), (\partial g_1 f)(x_0), (\partial(\partial g_1 f)f)(x_0), (\partial(\partial(\partial g_1 f)f)f)(x_0), \ldots)^T \quad , \tag{3.6}$$

where we have written $\partial = \partial/\partial x$, and in particular, if $g_1(x) = x$,

$$g(x_0) = (x_0, f(x_0), ((\partial f)f)(x_0), (\partial((\partial f)f)f)(x_0), \ldots)^T \quad . \tag{3.7}$$

Proof The result follows directly from (3.3) and the definition of g, since A is a bounded operator (on any sequential Banach space) and so e^{At} is well-defined by the usual series.

□

Note that (3.5) is equivalent to (2.3) because of (3.7) and so we have proved (2.2) in a simple way. In the remainder of this paper we shall take $g_1(x) = x$ for simplicity and so equation (3.1) is equivalent to equation (3.3) with initial condition (3.7). Define the operator $N : R^n \longrightarrow \mathcal{S}$, where $\mathcal{S}$ is the space of (unrestricted) sequences with values in R^n, by

$$N_f x = (x, f(x), ((\partial f)f)(x), ((\partial(\partial f)f)f)(x), \ldots)^T \quad .$$

We define a function $|| \, . \, ||$ on $\mathcal{S}$ by

$$||s||_A = \sup_{t \in [0,1]} \left\| \sum_{i=0}^{\infty} \frac{s_i t^i}{i!} \right\| \quad ,$$

where

$$s = (s_0, s_1, s_2, \ldots)^T \quad .$$

Let

$$\mathcal{S}_A = \{ s \in \mathcal{S} : ||s||_A < \infty \} \quad .$$

Lemma 3.2 $(\mathcal{S}_A, ||.||_A)$ is a Banach space.

Proof Only completeness presents any problems. Let $s^{(i)}$ be a Cauchy sequence in $\mathcal{S}_A$. Then for any $\epsilon > 0$ there exists m such that

$$||s^i - s^j|| = \sup_{t \in [0,1]} \left\| \sum_{k=0}^{\infty} \left(\frac{(s_k^{(i)} - s_k^{(i)}) t^k}{k!} \right) \right\| < \epsilon \qquad (3.8)$$

for $i, j \geq m$. We prove that $s_k^{(i)}$ is a Cauchy sequence for each k. For $k = 0$ this follows from (3.8) by taking $t = 0$. Assume that $s_k^{(i)}$ is a Cauchy sequence for $k \leq \ell$. Since the power series in (3.8) converges for $t \in [0,1]$ we have

$$\left\| \sum_{k=\ell+1}^{\infty} \frac{(s_k^{(i)} - s_k^{(i)})}{k!} t^k \right\| < \frac{\epsilon}{4}$$

for all i, j and for small enough $t > 0$, say $t \leq \tau$. Also, by assumption

$$\left\| \sum_{k=0}^{\ell-1} \frac{(s_k^{(i)} - s_k^{(i)})}{k!} t^k \right\| < \frac{\epsilon}{4} \quad , \quad t \in [0, \tau] \; .$$

for all $i, j \geq$ some m (depending on ℓ). Hence, by (3.8),

$$\left\| \frac{(s_\ell^{(i)} - s_\ell^{(i)})}{\ell!} t^\ell \right\| < \frac{\epsilon}{2}$$

for $t \in [0, \tau]$ and $i, j \geq m$, and so

$$\left\| s_\ell^{(i)} - s_\ell^{(j)} \right\| < \frac{\epsilon . \ell!}{\tau^\ell . 2} \quad .$$

The result now follows easily.

□

Lemma 3.3 N_f maps R^n into $\mathcal{S}_A$ for all f for which a solution of (3.1) exists on $[0, 1]$, for all $x_0 \in R^n$.

Proof This follows directly from the definition of $\|.\|_A$ and (3.4),(3.7). □

Remark Clearly, $N_f : R^n \longrightarrow \mathcal{R}(N_f) \subseteq \mathcal{S}_A$ is invertible and $N_f^{-1} = P$ where P is the projection defined above. Of course, $N_f N_f^{-1} \neq I$.

Hence, the solution of (3.1) is given by

$$x(t) = N_f^{-1} e^{At} N_f(x_0) \quad . \tag{3.9}$$

Now consider a linear-analytic control system of the form

$$\dot{x} = f(x) + uh(x) \tag{3.10}$$

(with a scalar control - the general vector control can be dealt with similarly). As before, put $g_1(x) = x$ and define inductively

$$\begin{aligned} g_i &= \frac{\partial g_{i/2}(x)}{\partial x} . f(x) \quad if\ i\ is\ even \\ g_i &= \frac{\partial g_{(i-1)/2}(x)}{\partial x} . h(x) \quad if\ i\ is\ odd \end{aligned}$$

Thus,

$$\begin{aligned}\dot{g}_i(x) &= \frac{\partial g_i(x)}{\partial x}\dot{x} \\ &= \frac{\partial g_i(x)}{\partial x}f(x) + u\frac{\partial g_i(x)}{\partial x}h(x) \\ &= g_{2i}(x) + ug_{2i+1}(x) \qquad (3.11)\end{aligned}$$

The next result follows directly from (3.11):

Theorem 3.4 The linear analytic system (3.10) can be written in the form

$$\frac{dg}{dt} = Ag + uBg \quad , \quad g(0) = g_0 \qquad (3.12)$$

where

$$g = (x, g_2, g_3, \ldots)$$

and $A = (a_{ij})$ and $B = (b_{ij})$ are infinite-dimensional matrices defined by

$$\begin{aligned} a_{ij} &= \delta_{2i,j} \\ b_{ij} &= \delta_{2i+1,j} \end{aligned}$$

□

Remark g_0 can be expanded in the form

$$g_0 = (x_0, f(x_0), h(x_0), ((\partial f)f)(x_0), ((\partial f)h)(x_0), ((\partial h)f)(x_0), ((\partial h)h)(x_0), \ldots) .$$

Corollary 3.5 The linear analytic system (3.10) has an input-output relation in the form of the following *Volterra series*:

$$\begin{aligned} g(t) = e^{At}g_0 \quad + \quad & \sum_{k=1}^{\infty}\int_0^t\int_0^{\tau_1}\int_0^{\tau_2}\cdots\int_0^{\tau_{k-1}} e^{A(t-\tau_1)}Be^{A(\tau_1-\tau_2)}B\ldots \\ & Be^{A\tau_k}g_0u(\tau_1)\ldots u(\tau_k)d\tau_1\ldots d\tau_k \qquad (3.13)\end{aligned}$$

or

$$x(t) = P\{e^{At}g_0 \quad + \quad \sum_{k=1}^{\infty} \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} \cdots \int_0^{\tau_{k-1}} e^{A(t-\tau_1)} B e^{A(\tau_1-\tau_2)} B \ldots$$
$$B e^{A\tau_k} g_0 u(\tau_1) \ldots u(\tau_k) d\tau_1 \ldots d\tau_k\}$$

(The details of the proof of the existence of infinite-dimensional Volterra series is given in [1])

□

Lemma 3.6 If A is the infinite matrix defined by

$$A = (\delta_{2i,j})$$

then

$$e^{At} = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \delta_{2^n i,j} \right) \quad .$$

Proof This follows by induction, since

$$A^n = (\delta_{2^n i,j}) \quad . \tag{3.14}$$

□

Lemma 3.7 Let $K(t, \tau_1, \ldots, \tau_k) = e^{A(t-\tau_1)} B e^{A(\tau_1-\tau_2)} B \ldots B e^{A\tau_1}$ denote the kernel matrix in (3.13). Then

$$[K(t, \tau_1, \ldots, \tau_k)]_{ij} = \sum_{n_1, \ldots, n_{k+1}} \sum_{v_1, \ldots, v_k} \frac{(t-\tau_1)^{n_1}}{n_1!} \frac{(\tau_1-\tau_2)^{n_2}}{n_2!} \cdots \frac{(\tau_k)^{n_{k+1}}}{n_{k+1}!} \cdot$$
$$b_{2^{n_1} i, v_1} b_{2^{n_2} v_1, v_2} b_{2^{n_3} v_2, v_3} \ldots b_{2^{n_k} v_{k-1}, v_k} \delta_{2^{n_{k+1}} v_k, j} \quad .$$

Proof Denote e^{At} by $E(t) = (e_{ij}(t))$. Then

$$[K(t, \tau_1, \ldots, \tau_k)]_{ij} = \sum_{\ell_1, \ldots, \ell_{2k}} e_{i\ell_1}(t-\tau_1) b_{\ell_1 \ell_2} e_{\ell_2 \ell_3}(\tau_1-\tau_2) b_{\ell_3 \ell_4} \ldots b_{\ell_{2k-1} \ell_{2k}} e_{\ell_{2k}, j} \quad ,$$

and so

$$
\begin{aligned}
[K]_{ij} &= \sum_{\ell_1,\dots,\ell_{2k}} \sum_{n_1,\dots,n_{k+1}} \frac{(t-\tau_1)^{n_1}}{n_1!} \frac{(\tau_1-\tau_2)^{n_2}}{n_2!} \cdots \frac{(\tau_k)^{n_{k+1}}}{n_{k+1}!} . \\
&\quad \delta_{2^{n_1} i, \ell_1} b_{\ell_1 \ell_2} \delta_{2^{n_2} \ell_2, \ell_3} b_{\ell_3 \ell_4} \dots b_{\ell_{2k-1} \ell_{2k}} \delta_{2^{n_{k+1}} \ell_{2k}, j} \\
&= \sum_{\ell_2,\ell_4,\dots,\ell_{2k}} \sum_{n_1,\dots,n_{k+1}} \frac{(t-\tau_1)^{n_1}}{n_1!} \frac{(\tau_1-\tau_2)^{n_2}}{n_2!} \cdots \frac{(\tau_k)^{n_{k+1}}}{n_{k+1}!} . \\
&\quad b_{2^{n_1} i, \ell_2} b_{2^{n_2} \ell_2, \ell_4} b_{2^{n_3} \ell_4, \ell_6} \dots b_{2^{n_k} \ell_{2k-2}, \ell_{2k}} \delta_{2^{n_{k+1}} \ell_{2k}, j} \quad ,
\end{aligned}
$$

by lemma 3.6. Setting $v_i = \ell_{2i}$ gives the result. □

Corollary 3.8 $K(t, \tau_1, \dots, \tau_k)$ simplifies to

$$
\begin{aligned}
[K]_{ij} &= \sum_{n_1,\dots,n_{k+1}} \frac{(t-\tau_1)^{n_1}}{n_1!} \frac{(\tau_1-\tau_2)^{n_2}}{n_2!} \cdots \frac{(\tau_k)^{n_{k+1}}}{n_{k+1}!} . \\
&\quad \delta_{2^{n_{k+1}}(2^{n_k+1}(2^{n_{k-1}+1}(\dots(2^{n_1+1} i+1)\dots)+1)+1)+1, j}
\end{aligned}
$$

Proof This follows easily from lemma 3.7 since $b_{ij} = \delta_{2i+1,j}$. □

Consider next the case of a general nonlinear analytic system

$$
\dot{x} = f(x, u) \tag{3.15}
$$

again with a scalar control u. Introducing the augmented system (see [1]):

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
\dot{u} &= v
\end{aligned} \tag{3.16}
$$

(assuming differentiable controls) we can write (3.14) in the form of (3.10); i.e.

$$
\dot{y} = F(y) + vH(y) \quad , \tag{3.17}
$$

where

$$
\begin{aligned}
y &= \begin{pmatrix} x \\ u \end{pmatrix} \\
F(y) &= \begin{pmatrix} f(x,u) \\ 0 \end{pmatrix} \\
H(y) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

We can now apply theorem 3.4 directly to the system (3.16). However, if $g_3(y) = H(y)$ is defined as before then $\dot{g}_3 = 0$ and so many of the g_i's are redundant. Therefore, removing redundant g's, we define

$$
\begin{aligned}
g_1(y) &= y \\
g_2(y) &= F(y) \\
g_3(y) &= H(y) \\
g_4(y) &= \frac{\partial g_2}{\partial y}.F \\
g_5(y) &= \frac{\partial g_2}{\partial y}.H \\
\cdots & \quad \cdots \\
g_i(y) &= \frac{\partial g_{\frac{i+2}{2}}}{\partial y}.F \quad \textit{if } i \textit{ is even and } i \geq 6 \\
g_i(y) &= \frac{\partial g_{\frac{i+1}{2}}}{\partial y}.H \quad \textit{if } i \textit{ is odd and } i \geq 7
\end{aligned}
$$

Hence,

$$
\dot{g}_1(y) = g_2(y) + v g_3(y)
$$

$$
\begin{aligned}
\dot{g}_2(y) &= g_4(y) + v g_5(y) \\
\dot{g}_3(y) &= 0 \\
\cdots & \quad \cdots \\
\dot{g}_i(y) &= \frac{\partial g_i}{\partial y}(y).F + v\frac{\partial g_i}{\partial y}(y).H \\
&= g_{2i-2}(y) + v g_{2i-1}(y) \quad , \quad i \geq 4
\end{aligned}
$$

We can then write the system in the form

$$\frac{dg}{dt} = Ag + vBg$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are the infinite matrices defined by

$$
\begin{aligned}
a_{1j} &= \delta_{2j} \\
a_{2j} &= \delta_{4j} \\
a_{3j} &= 0 \\
\cdots & \quad \cdots \\
a_{ij} &= \delta_{2i-2,j}
\end{aligned}
$$

$$
\begin{aligned}
b_{1j} &= \delta_{3j} \\
b_{2j} &= \delta_{5j} \\
b_{3j} &= 0 \\
\cdots & \quad \cdots \\
b_{ij} &= \delta_{2i-1,j}
\end{aligned}
$$

for all $i \geq 4$ and all $j \geq 1$. A Volterra series can therefore be generated as in (3.13).

4 Application to Optimal Control

In this section we shall consider the optimal control problem

$$min\ J(u) = x^T(t_f)Fx(t_f) + \int_0^{t_f} (x^T(t)Qx(t) + u^T(t)Ru(t))dt \tag{4.1}$$

subject to the linear analytic dynamics

$$\dot{x} = f(x) + uh(x) \tag{4.2}$$

where Q, F are positive semi-definite and R is positive-definite. This has been solved before ([1]) using infinite-dimensional Taylor series representations. However, the control is difficult to evaluate in general and in the infinite-dimensional bilinear representation of (4.2) the A and B operators are tensors with large amounts of redundancy.

We now propose to solve this problem using the ideas presented above. Thus, let $g \in (\mathcal{O}(R^n))^{N^+}$ be defined as in section 3, i.e.

$$\begin{aligned} g_1(x) &= x \\ g_i(x) &= \frac{\partial g_{i/2}}{\partial x}.f(x) \quad , \quad i \ even \\ g_i(x) &= \frac{\partial g_{(i-1)/2}}{\partial x}.h(x) \quad , \quad i \ odd \end{aligned}$$

Then

$$\frac{dg}{dt} = Ag + uBg \tag{4.3}$$

where

$$A = (\delta_{2i,j}) \quad , \quad B = (\delta_{2i+1,j}) \quad .$$

Define the infinite-dimensional matrix operators $\mathcal{F}$ and $\mathcal{Q} \in \mathcal{L}((\mathcal{O}(R^n))^{N^+}, (\mathcal{O}(R^n))^{N^+})$ by

$$\begin{aligned} g^T \mathcal{F} g &= x^T F x \\ g^T \mathcal{Q} g &= x^T Q x \end{aligned}$$

for all $g \in (\mathcal{O}(R^n))^{N^+})$, where $x = g_1$. Then $\mathcal{F}, \mathcal{Q}$ are infinite-dimensional operators with matrix representations whose $(i,j)^{th}$ elements are $n \times n$ matrices with $(\mathcal{F})ij = F \quad , \quad (\mathcal{Q})_{ij} = Q$. Hence, we may write the cost functional (4.1) in the form

$$min \ \mathcal{J}(u) = g^T(t_f)\mathcal{F}g(t_f) + \int_0^{t_f} (g^T(t)\mathcal{Q}g(t) + u^T(t)Ru(t))dt \quad . \tag{4.4}$$

and so the original problem (4.1),(4.2) is equivalent to the bilinear problem (4.3),(4.4).

Infinite-dimensional bilinear-quadratic problems have been completely solved ([4] and [3]) and we may write the optimal control in the form:

$$u(t) = -\frac{1}{2}R^{-1}\sum_{i=1}^{\infty}\langle \otimes_i g, (P_i B) \otimes_i g\rangle$$

where $P_i \in \mathcal{L}(\otimes_i H)$ is given recursively by

$$P_1(t) \quad = \quad e^{\mathcal{A}_1(t_f - t)}\mathcal{F}e^{\mathcal{A}_1^T(t_f - t)} + \int_0^{t_f - t} e^{\mathcal{A}_1(t_f - t - s)}\mathcal{Q}e^{\mathcal{A}_1^T(t_f - t - s)}ds$$

$$
\begin{aligned}
P_m(t) &= -\frac{r^{-1}}{2}\sum_{\substack{i+j=m\\ i,j\geq 1}}\int_0^{t_f-t} e^{\mathcal{A}_m(t_f-t-s)}P_i(t_f-s)B\otimes P_j(t_f-s)Be^{\mathcal{A}_m^T(t_f-t-s)}ds\\
&\quad -\frac{r^{-1}}{2}\sum_{\substack{i+j=m\\ i,j\geq 1}}\int_0^{t_f-t} e^{\mathcal{A}_m(t_f-t-s)}P_i(t_f-s)B^T\otimes P_j(t_f-s)B^Te^{\mathcal{A}_m^T(t_f-t-s)}ds
\end{aligned}
\tag{4.5}
$$

and H is any Hilbert space structure on $(\mathcal{O}(R^n))^{N^+}$ (such a structure can be defined - see [1]). Here, P is a tensor operator and if C is an infinite matrix, PC is defined by

$$
(PC)^{\kappa_1\ldots\kappa_i}_{k_1\ldots k_i} = \sum_{j=1}^{i}\left(\sum_{\ell_j=1}^{\infty} P^{\kappa_1\ldots\kappa_i}_{k_1\ldots\ell_j\ldots k_i}C_{\ell_j k_j}\right)
$$

Furthermore, $\mathcal{A}_i \in \mathcal{L}(\mathcal{L}(\otimes_i H))$ is defined by

$$
\mathcal{A}_i P_i = P_i A \quad , \quad P_i \in \mathcal{L}(\otimes_i H) \quad , \quad i \geq 1 \ .
$$

We have

$$
||\mathcal{A}_i||_{\mathcal{L}(\mathcal{L}(\otimes_i H))} \leq i||A||_{\mathcal{L}(H)} \quad ,
$$

and so $e^{\mathcal{A}_i t}$ is defined by the usual series. It can be shown ([4]) that $e^{\mathcal{A}_i t}$ is given by

$$
(e^{\mathcal{A}_i t}Q)^{\kappa_1\ldots\kappa_i}_{\ell_1\ldots\ell_i} = \sum_{k_1=1}^{\infty}\cdots\sum_{k_i=1}^{\infty} Q^{\kappa_1\ldots\kappa_i}_{k_1\ldots k_i}(e^{At})_{k_1\ell_1}(e^{At})_{k_2\ell_2}\ldots(e^{At})_{k_i\ell_i}
$$

for any Q. Now, by lemma 3.6, we have

$$
(e^{At})_{ij} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\delta_{2^n i,j} \quad ,
$$

and so

$$
\begin{aligned}
(e^{\mathcal{A}_i t}Q)^{\kappa_1\ldots\kappa_i}_{\ell_1\ldots\ell_i} &= \sum_{k_1=1}^{\infty}\cdots\sum_{k_i=1}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_i=0}^{\infty} Q^{\kappa_1\ldots\kappa_i}_{k_1\ldots k_i}\frac{t^{n_1}}{n_1!}\frac{t^{n_2}}{n_2!}\cdots\frac{t^{n_i}}{n_i!}\delta_{2^{n_1}k_1,\ell_1}\delta_{2^{n_2}k_1,\ell_2}\cdots\delta_{2^{n_i}k_1,\ell_i} \\
&= \sum_{k_1=1}^{\infty}\cdots\sum_{k_i=1}^{\infty} P^{\kappa_1\ldots\kappa_i}_{k_1\ldots k_i}\frac{t^{n_1}}{n_1!}\frac{t^{n_2}}{n_2!}\cdots\frac{t^{n_i}}{n_i!}
\end{aligned}
$$

where

$$
n_p = log_2\left(\frac{\ell_p}{k_p}\right)
$$

and the sums in the last expression are over k's for which each n_p is a natural number.

Now,

$$
\begin{aligned}
(P_i(t_f - s)B)^{\kappa_1\ldots\kappa_i}_{k_1\ldots k_i} &= \sum_{j=1}^{\ell}\left(\sum_{\ell_j=1}^{\infty} P^{\kappa_1\ldots\kappa_i}_{k_1\ldots\ell_j\ldots k_i}B_{\ell_j k_j}\right) \\
&= \sum_{j=1}^{\ell}\left(\sum_{\ell_j=1}^{\infty} P^{\kappa_1\ldots\kappa_i}_{k_1\ldots\ell_j\ldots k_i}\delta_{2\ell_j+1,k_j}\right) \\
&= \sum_{j=1}^{\ell} P^{\kappa_1\ldots\kappa_i}_{k_1\ldots\frac{(k_j-1)}{2}\ldots k_i}
\end{aligned}
$$

where the element of P in the last sum is zero if $(k_j - 1)/2$ is not a nonnegative integer. The above expressions for $e^{\mathcal{A}_i t}Q$ and $P_i B$ are now sufficient to be able to evaluate $P_m(t)$ from (4.5).

5 Conclusions

A new infinite-dimensional bilinear representation of a nonlinear control system has been given in terms of the Lie series. The simple structure for the system matrices contrasts with that obtained by using the Carleman-Taylor series representation. In this case e^{At} is easily determined and is independent of the particular nonlinear system. This has led to a very explicit form of solution for the linear-analytic-quadratic control problem.

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