Rational Expansion

for

Nonlinear Input-Output Maps

by

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Abstract

This paper introduces a **Rational Expansion for Nonlinear Input-Output Maps**. The method is new and is based on the rational expansion of functions of several complex variables. If truncated, this series reduces to a ratio of truncated Volterra series. A 'feedback form' will be presented.

1 Introduction:

The mathematical theory of Nonlinear Systems has been dominated for many years by the use of the Volterra series expansion for Nonlinear input-output maps [5], [17]. Volterra's own work on the series is summarized in his book [18] which dates from a 1925 series of lectures. In the 1940's, Weiner became interested in the idea and since then papers dealing with such expansions have appeared periodically e.g., [1],[3]–[16] to cite a few.

However, this expansion has a major drawback, the large number of terms required to obtain a 'good 'approximation. In this paper, we shall introduce a new expansion in terms of 'rational functionals '. If truncated this series reduces to a ratio of truncated Volterra series and as such it is expected to produce a 'better' approximation in a smaller number of terms.

In order to derive this representation, we shall need some results from the theory of functions of several complex variables. These preliminaries will be presented without proofs in section 2. In section 3, we derive the new rational expansion for single-input single-output systems, while in section 4 this result is extended to the multidimensional case. In section 5, we shall present a 'feedback version' , i.e., the output is written in terms of input and output.

2 Some facts from complex analysis:

In this section we shall present some facts from the theory of functions of several complex variables. All the results in this section will be presented without proofs, and are introduced for completeness, the interested reader is urged to consult reference [2] (or a similar book) for details.

Let M and N be subsets of $\mathbf{C}^m$ where $\mathbf{C}$ is the set of complex numbers, then ∂M denotes the boundary of M, $\overline{M}$ is the closure of M, M^0 is the interior of M and $M \sqsubset N$ means that $\overline{M}$ is compact and $\overline{M} \subset N^0$. We denote by $A(D)$ (resp. $A(K)$) the class of functions holomorphic in a domain D (resp. compact set K). $A^m(D)$ will denote the class of mappings $f = (f_1, \ldots, f_m)$ that are holomorphic in D.

Let D be a bounded domain in $\mathbf{C}^m$, and suppose that h is in $C(\overline{D})$ and is positive in D. Denote by $L_h^2 = L_h^2(D)$ the collection of all functions $f \in A(D)$ with norm

$$\| f \| = \| f \|_D^h = \left(\int_D | f |^2 \, h \, dv\right)^{\frac{1}{2}}$$

L_h^2 is a Hilbert space with the inner product $< f, g > = \int_D f\overline{g}h \, dv$. It is easy to see that if a polydisc $U = U(z^0, r) = \{z \in \mathbf{C}^m : | z_j - z_j^0 | < r, j = 1, \ldots, m\}$

satisfies $U \sqsubset D$ then for every $f \in L_h^2$

$$| f(z^0) | \leq c \ \| f \|_D^h$$

where the constant c depends on r and h but not on f or z^0.

Definition 1:

Let $\chi_1, \ldots, \chi_N, N \geq m$ be holomorphic functions in the domain $D \subset \mathbf{C}^m$ and $D_1, \ldots, D_N$ domains such that $D_j \subset \chi_j(D), j = 1, \ldots, N$. The set

$$\Delta = \{z : z \in D; \chi_j(z) \in D_j, j = 1, \ldots, N\}$$

is called an analytic polyhedron.

Definition 2:

A Weil polyhedron is a connected component of an analytic polyhedron for which the boundaries ∂D_j are piecewise smooth and the intersection of k of the faces $\gamma_j = \{z : z \in D_j, \chi_j \in \partial D_j, \chi_l \in D_l, j \neq l\}$ has dimension at most 2m-k.

Let $\gamma_{i_1,\ldots,i_m} = \bigcap_{k=1}^m \gamma_{i_k}$, we give these m-dimensional edges the natural orientation determined by the order of the faces $\gamma_{i_1}, \ldots, \gamma_{i_m}$.

Definition 3:

The distinguished boundary (or skeleton) of the polyhedron Δ, denoted σ, is the union of all m-dimensional edges $\gamma_{i_1,\ldots,i_m}$.

Definition 4:

A bounded domain D is said to be strictly pseudoconvex if there exist a neighborhood $U \supset \overline{D}$ and a function $\rho \in C^{(2)}(U)$ such that $D = \{z : z \in U, \rho(z) < 0\}$, where grad $\rho \neq 0$ on ∂D and the function ρ is strictly plurisubharmonic in U, i.e.,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k > 0$$

for $z \in U$ and all $w \in \mathbf{C}^n, w \neq 0$.

Theorem 1: (Oka and Hefer)

Suppose that D is a pseudoconvex domain in $\mathbf{C}^n$ anf $f \in A(D)$. Then for any domain $G \sqsubset D$ there exist functions $P_j(\xi, z) \in A(G \times G), j = 1, \ldots, n$, such that

$$f(\xi) - f(z) = P_1(\xi, z)(\xi_1 - z_1) + \ldots + P_n(\xi, z)(\xi_n - z_n)$$

for $\xi, z \in G$.

Let $\Omega_{i_1,\dots,i_m} = det \parallel q_{lj} \parallel$ for $l = i_1, \dots, i_m, j = 1, \dots, m$

where $q_{lj}(\xi, z) \in A^m(\overline{D} \times \overline{D})$ is obtained from theorem 1

$$\chi_l(\xi) - \chi_l(z) = \sum_{j=1}^{m} q_{lj}(\xi, z)(\xi_j - z_j)$$

Theorem 2: (Weil)

Let Δ be a Weil polyhedron and let $f \in A_C(\Delta) = A(\Delta) \cap A(\overline{\Delta})$. Then for $z \in \Delta$

$$f(z) = \frac{1}{(2\pi i)^m} \sum_{i_1 < \dots < i_m} \int_{\gamma_{i_1 \dots i_m}} \frac{f(\xi)\Omega_{i_1 \dots i_m}(\xi, z)\, d\xi}{\prod_{k=1}^{m} [\chi_{i_k}(\xi) - \chi_{i_k}(z)]}$$

This is called the Bergman-Weil integral representation.

Remark 1: This formula recovers the value of the holomorphic function f in Δ from its values on the m-dimensional distinguished boundary σ and has a kernel that is holomorphic in z, but the form of the kernel depends on the shape of the domain (unlike Cauchy integral formula for a function of a single variable).

Consider a Weil polyhedron $\Delta = \{z : z \in D, \chi_l(z) < 1, l = 1, \dots, N\}$, $\chi_l(z) \in A(D), l = 1, \dots, N$, $\Delta \sqsubset D$.

Theorem 3: (Weil)

Every function $f \in A(\Delta)$ can be expanded for $z \in \Delta$ in a series

$$f(z) = \sum_{i_1<\ldots<i_m} \sum_{k_1,\ldots,k_m=0}^{\infty} \left[\frac{1}{(2\pi i)^m} \int_{\gamma_{i_1\ldots i_m}} \frac{f(\xi)\Omega_{i_1\ldots i_m}(\xi,z)\ d\xi}{\chi_{i_1}^{k_1+1}(\xi)\ldots\chi_{i_m}^{k_m+1}(\xi)} \right] \times \chi_{i_1}^{k_1}(z)\ldots\chi_{i_m}^{k_m}(z)$$

uniformly convergent on compact subset of Δ.

Definition 5:

A polyhedron Δ is called a polynomial Weil domain if all the $\chi_l(z)$ are polynomials.

Definition 6:

A Runge domain D is a domain such that every $f \in A(D)$ can be approximated uniformly on compact subsets of D by polynomials.

Corollary 1:

Every polynomial Weil domain is a Runge domain.

Remark 2: The series in theorem 3 is not in general unique.

Definition 7:

Let M be a compact connected set. M is said to be approximated from the outside by a sequence of domains $D_p, p = 1, 2, \ldots$ if $D_{p+1} \sqsubset D_p$ and $M = \bigcap_p D_p$.

Let $D_p, p = 1, 2, \ldots$ be a sequence of domains approximating M from the outside such that each D_p has a one-sheeted envelope of holomorphy $\mathcal{K}(D_p)$; then we say that the compact set M has a one-sheeted envelope of holomorphy, and define $\mathcal{K}(M) = \bigcap_p \mathcal{K}(D_p)$. We have

Proposition 1:

$\mathcal{K}(M)$ is compact and does not depend on the choice of the sequence D_p, and each function holomorphic on M is holomorphic also on $\mathcal{K}(M)$.

Proposition 2:

If every function holomorphic on M is also holomorphic on a set E, then $E \subset \mathcal{K}(M)$.

Proposition 3:

On $\mathcal{K}(M)$ each function in $A(M)$ takes only the values it takes on M.

Definition 8:

A domain $D \subset \mathbf{C}^m$ is said to be linearly convex if for each $\xi \in \partial D$ there is an analytic plane of complex dimension (n-1) passing through ξ and not intersecting D.

Remark 3: Every convex domain in $\mathbf{C}^m$ is also linearly convex. A topological product $D = D_1 \times \ldots \times D_m$ of domains in $\mathbf{C}^1$ is a linearly convex domain.

Definition 9:

A compact set M is said to be linearly convex if there exists a sequence of linearly convex domains approximating it from the outside.

Theorem 4: (Aizenberg)

Suppose that the compact set M has a one-sheeted envelope of holomorphy $\mathcal{K}(M)$. Then every $f \in A(M)$ can be represented in some neighborhood of M (depending on f) by a uniformly convergent series

$$f(z) = \sum_{p=1}^{\infty} \frac{A_p}{\prod_{j=1}^{m}(a_{pj1}z_1 + \ldots + a_{pjm}z_m + b_{pj})}$$

where $\sum_{k=1}^{m} \mid a_{pjk} \mid^2 = 1, j = 1, \ldots, m; p = 1, 2, \ldots, \sum_{p=1}^{\infty} \mid A_p \mid < \infty$ if and only if the envelope of holomorphy $\mathcal{K}(M)$ is a linearly convex compact set.

Theorem 5: (Znamenskii)

A domain (compact set) is strongly linearly convex if and only if it intersects every complex line in a connected and simply connected set.

Theorem 6: (Trutnev)

Every $f \in A(M)$ can be represented in a neighborhood of M by a uniformly convergent series

$$f(z) = \sum_{p=1}^{\infty} \frac{A_p}{(a_{p1}z_1 + \ldots + a_{pm}z_m + b_m)^m} \tag{2.1}$$

where $\sum_{k=1}^{m} \mid a_{pk} \mid^2 = 1, \sum_{p=1}^{\infty} \mid A_p \mid < \infty$ if and only if the envelope of holomorphy $\mathcal{K}(M)$ is a strongly linearly convex compact set.

Corollary 2: (Aizenberg)

For every $f \in A(M)$ to be representable in a neighborhood of M by a uniformly convergent series (2.1) it is necessary that $\mathcal{K}(M)$ be a linearly convex compact set, and sufficient that $\mathcal{K}(M)$ be approximated from the outside by regular linearly convex domains.

3 Rational Expansion for Nonlinear Maps : - Scalar Case

Consider a single-input single-output system Σ defined by the causal input-output map

$$F : L^2[0,\infty;\mathbf{C}] \longrightarrow L^2[0,\infty;\mathbf{C}] \qquad (3.1)$$
$$u \longrightarrow y$$

We recall that F is called causal (strictly causal) if $u_{|[0,t]} = v_{|[0,t]}$ (respectively $u_{|[0,t)} = v_{|[0,t)}$) implies that $F(u)(t) = F(v)(t)$ for any $u, v \in L^2[0,\infty;\mathbf{C}]$ and $t \geq 0$ where $\phi_{|I}$ denotes the restriction of the function ϕ to the interval I. For a given $t \geq 0$ and $u \in L^2[0,\infty;\mathbf{C}]$ consider the truncation $u^t \in L^2[0,\infty;\mathbf{C}]$ defined by $u^t(\tau) = u(\tau)$ for $\tau \in [0,t]$ and zero elsewhere. Since F is causal we have

$$\begin{aligned} y(t) &= F(u)(t) \\ &= F(u^t)(t) \qquad (3.2) \end{aligned}$$

Let $\mathcal{E} = \{e_j, j = 1, 2, \ldots\}$ be an orthonormal basis of $L^2[0,\infty;\mathbf{C}]$. Let m be a positive integer, and approximate u^t by the finite sum $\sum_{j=1}^{m} u_j^t e_j$. Therefore,

$$y(t) = F(\sum_{j=1}^{m} u_j^t e_j)(t) \tag{3.3}$$

t being fixed, the expression in the right hand side of the previous equation can be seen as an ordinary function of m complex variables $u_1^t, \ldots, u_m^t$. Hence, we obtain

$$y = H(u_1^t, \ldots, u_m^t) \tag{3.4}$$

which can be expanded, under suitable conditions, to be precised later, in a uniformly convergent series of 'partial fractions '.

Let $\overline{u}^t = (u_1^t, \ldots, u_m^t) \in M$ and suppose that M is a compact set having a one-sheeted envelope of holomorphy $\mathcal{K}(M)$. We claim the following

Theorem 7:

Suppose that the nonlinear map F is such that $H \in A(M)$. Then F can be represented in some neighborhood of $\{u \in L^2[0,\infty;\mathbf{C}] : \overline{u}^t \in M, t \geq 0\}$ (depending on F) by a uniformly convergent series

$$y(t) = F(u)(t) = \sum_{p=1}^{\infty} \frac{A_p(t)}{[b_m(t) + \int_0^t a_m^p(t,\xi)u(\xi)\, d\xi]^m} \tag{3.5}$$

for some functions A_p, b_m and a_m^p if and only if $\mathcal{K}(M)$ is a strongly linearly convex compact set.

Proof:

Using Trutnev theorem, we conclude that H can be represented in a neighborhood of M (depending on H) by a uniformly convergent series

$$y = \sum_{p=1}^{\infty} \frac{A_p}{[b_m + a_{p1}u_1 + \ldots + a_{pm}u_m]^m} \tag{3.6}$$

where $\sum_{j=1}^{m} \mid a_{pj} \mid^2 = 1$ and $\sum_{p=1}^{\infty} \mid A_p \mid < \infty$ if and only if the envelope of holomorphy $\mathcal{K}(M)$ is a strongly linearly convex compact set.

Writing explicitely the time dependence, we obtain,

$$y(t) = \sum_{p=1}^{\infty} \frac{A_p(t)}{[b_m(t) + a_{p1}(t)u_1^t + \ldots + a_{pm}(t)u_m^t]^m} \tag{3.7}$$

But $u_j^t = < u^t, e_j >$, so

$$y(t) = \sum_{p=1}^{\infty} \frac{A_p(t)}{[b_m(t) + a_{p1}(t) < u^t, e_1 > + \ldots + a_{pm}(t) < u^t, e_m >]^m} \tag{3.8}$$

Therefore,

$$y(t) = \sum_{p=1}^{\infty} \frac{A_p(t)}{[b_m(t) + \int_0^t [a_{p1}(t)e_1^*(\xi) + \ldots + a_{pm}(t)e_m^*(\xi)]\, u(\xi)\, d\xi]^m} \tag{3.9}$$

Thus,

$$y(t) = \sum_{p=1}^{\infty} \frac{A_p(t)}{[b_m(t) + \int_0^t a_m^p(t,\xi)u(\xi)\, d\xi]^m} \tag{3.10}$$

for all $t \geq 0$ since t was fixed but arbitrary; where $a_m^p(t,\xi) = \sum_{j=1}^{m} a_{pj}(t)e_j^*(\xi)$.

Hence, we have proved the theorem.□

Corollary 3:

Suppose the conditions of theorem 7 are fulfilled.

If u is in the ball of radius $\rho_m = \inf_{t\in[0,\infty]} \mid b_m(t) \mid$, then

$$\mid y(t) - \sum_{p=1}^{k} \frac{A_p(t)}{[b_m(t) + \int_0^t a_m^p(t,\xi)u(\xi)\ d\xi]^m} \mid \leq \frac{1}{[\rho_m - \parallel u \parallel]^m} \sum_{p=k+1}^{\infty} \mid A_p(t) \mid . \tag{3.11}$$

Proof:

Immediate and shall be ommitted.□

Remark 4: If we approximate y by the sum of the first k terms, $\hat{y}_k = \sum_{p=1}^{k} y_p$, then reducing these fractions to the same denominator and expanding all the expressions, we get

$$\hat{y}_k(t) = \frac{N_{mk}^0(t) + \sum_{l=1}^{m(k-1)} \int_0^t \ldots \int_0^t N_{mk}^l(t,\xi_1,\ldots,\xi_l)u(\xi_1)\ldots u(\xi_l)d\xi_1\ldots d\xi_l}{D_{mk}^0(t) + \sum_{l=1}^{mk} \int_0^t \ldots \int_0^t D_{mk}^l(t,\xi_1,\ldots,\xi_l)u(\xi_1)\ldots u(\xi_l)d\xi_1\ldots d\xi_l} \tag{3.12}$$

for $k \geq 1$ and $t \geq 0$, which is a ratio of truncated Volterra series.

Remark 5: Using the expansion

$$\frac{1}{(1+x)^m} = \sum_{l\geq 0} (-1)^l C_{l+m-1}^{m-1} x^l \quad , \quad \mid x \mid < 1$$

where C_{l+m-1}^{m-1} is the binomial coefficient, we could obtain a direct relationship with the Volterra series representation.

4 Rational Expansion for Nonlinear Maps : - Multidimensional Case

In this section we shall extend the idea developed in the previous section to multi-input multi-output systems. Let Σ be such a system defined by the causal operator

$$F : L^2[0,\infty;\mathbf{C}^n] \longrightarrow L^2[0,\infty;\mathbf{C}^r] \qquad (4.1)$$
$$u \longrightarrow y$$

Hence, we are considering a system with n inputs and r outputs.

For a given $t \geq 0$ and $u \in L^2[0,\infty;\mathbf{C}^n]$ consider the truncation $u^t \in L^2[0,\infty;\mathbf{C}^n]$ defined by $u^t(\tau) = u(\tau)$ for $\tau \in [0,t]$ and zero elsewhere. Since F is causal we have

$$\begin{aligned} y(t) &= F(u)(t) \\ &= F(u^t)(t) \end{aligned} \qquad (4.2)$$

Let $\mathcal{E} = \{e_j, j = 1,2,\ldots\}$ be an orthonormal basis of $L^2[0,\infty;\mathbf{C}]$. Let m be a positive integer, and approximate u^{tq} by the finite sum $\sum_{j=1}^{m} u_j^{tq} e_j$.

Therefore,

$$y^s(t) = F^s(\sum_{j=1}^{m} u_j^{t1} e_j, \ldots, \sum_{j=1}^{m} u_j^{tn} e_j)(t) \quad s = 1, \ldots, r \tag{4.3}$$

Again, for fixed $t \geq 0$, the expression in the right hand side of the previous equation can be seen as an ordinary function of $m \times n$ complex variables $u_1^{tq}, \ldots, u_m^{tq}; q = 1, \ldots, n$. Hence, we obtain

$$y^s = H^s(u_1^{t1}, \ldots, u_m^{t1}, \ldots, u_1^{tn}, \ldots, u_m^{tn}) \tag{4.4}$$

Let $\overline{u}^t = (u_1^{t1}, \ldots, u_m^{t1}, \ldots, u_1^{tn}, \ldots, u_m^{tn}) \in M$ and suppose that M is a compact set having a one-sheeted envelope of holomorphy $\mathcal{K}(M)$. Then, using the results related to the scalar case, we obtain

Theorem 8:

Suppose that the nonlinear map F is such that $H \in A(M)$. Then F can be represented in some neighborhood of $\{u \in L^2[0, \infty; \mathbf{C}^n] : \overline{u}^t \in M, t \geq 0\}$ (depending on F) by a uniformly convergent series

$$y^s(t) = F^s(u)(t) = \sum_{p=1}^{\infty} \frac{A_p^s(t)}{[b_m^s(t) + \sum_{q=1}^{n} \int_0^t a_{mq}^{ps}(t,\xi) u^q(\xi)\, d\xi]^{mn}} \quad , \ s = 1, \ldots, r \tag{4.5}$$

for $t \geq 0$, if and only if $\mathcal{K}(M)$ is a strongly linearly convex compact set.

Corollary 4:

Suppose the conditions of theorem 8 are fulfilled.

If u is in the ball of radius $\rho_m = \min_{s=1,\ldots,r} \inf_{t\in[0,\infty]} \mid b_m^s(t) \mid$, then

$$\begin{aligned} &\mid y^s(t) - \textstyle\sum_{p=1}^{k} \frac{A_{ps}(t)}{[b_m^s(t)+\sum_{q=1}^{n} \int_0^t a_{mq}^{ps}(t,\xi)u^q(\xi)\ d\xi]^{mn}} \mid \\ &\qquad \leq \frac{1}{[\rho_m - \|u\|\ ]^{mn}} \textstyle\sum_{p=k+1}^{\infty} \mid A_{ps}(t) \mid \end{aligned} \tag{4.6}$$

for $s = 1, \ldots, r$.

5 Feedback Version:

In this section we shall derive a 'feedback version ' of the previous results. Consider the multi-input multi-output system Σ defined by the equation

$$y = F(y, u) \tag{5.1}$$

where, u and y are respectively the input and output of the system and F is the operator

$$F : L^2[0,\infty;\mathbf{C}^r] \times L^2[0,\infty;\mathbf{C}^n] \longrightarrow L^2[0,\infty;\mathbf{C}^r] \tag{5.2}$$

Defining $w = (y', u')'$ where the superscript $'$ denotes the transpose, we deduce

Corollary 5:

Provided F ($y = F(w)$) satisfies the conditions of theorem 8, we have

$$y^s(t) = \sum_{p=1}^{\infty} \frac{A_p^s(t)}{[b_m^s(t) + \sum_{q=1}^{n} \int_0^t a_{mq}^{ps}(t,\xi)u^q(\xi)\ d\xi + \sum_{q=1}^{r} \int_0^t c_{mq}^{ps}(t,\xi)y^q(\xi)\ d\xi]^{m(n+r)}} \tag{5.3}$$

for $s = 1, \ldots, r$ and $t \geq 0$.

6 Conclusion:

In this paper we have introduced a new series representation for a nonlinear input-output map. If truncated, this series reduces to a ratio of truncated Volterra series and as such, it will give a 'better' approximation in a smaller number of terms. A 'feedback form ' has been derived to reduce this number further.

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