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# A Functional Expansion And Stability For Nonlinear Input-Output Maps

by

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## Abstract

In this paper we shall use the global bilinearization of a linear analytic system using the tensor operator approach introduced in [6]. We shall generalize the result on finite dimensional bilinear systems [13] to this class of systems. We use the idea of diagonal dominance for tensor operators to derive an exponential stability result. Few examples will be presented to illustrate the theory.

# 1 Introduction and Notation:

Since its use by Brockett [9], the Carleman linearization [12] has been successfully applied and extended by Lo [18], Brockett [10], Krener [17], Takata [19] and Banks and his co-workers [1]–[8]. It has been always seen as a promising technique to tackle general nonlinear systems.

In this paper we shall apply this technique to transform linear analytic systems into bilinear ones in the space of tensors [16]. Then we expand the results in [13] to this class of systems. We shall present a new functional expansion of the given input-output map and derive a sufficient condition for exponential stability. This condition turns out to be the intuitively appealing result that one of the tensor operators involved verifies the 'diagonal dominance' property to be precised later.

The paper is organised as follows. In section 2 we shall present the global exponential stability for free nonlinear systems. In section 3 we present the new input-output expansion from which we deduce in section 4 some stability results; these constitute the generalization of the results in [13]. Finally, in section 5, few examples will be presented to illustrate the theory.

We shall use the same notations as in [6], that is an $n$-multi-index is an $n$-multiple $\mathbf{i} = (i_1, \ldots, i_n)$ of non-negative integers; its length (or order) is given by $\mid \mathbf{i} \mid = i_1 + \ldots + i_n$. The sum of two multi-indices $\mathbf{i}$ and $\mathbf{j}$ is defined as $\mathbf{i}+\mathbf{j} = (i_1 + j_1, \ldots, \ldots, i_n + j_n)$. We say that $\mathbf{i} \leq \mathbf{j}$ if $i_k \leq j_k$ for $k = 1, \ldots, n$. When $\mathbf{i} \leq \mathbf{j}$, we define $\mathbf{j}-\mathbf{i}$ as $(j_1 - i_1, \ldots, j_n - i_n)$. We also define $\mathbf{i}! = i_1! \ldots i_n!$ and

$$\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \ldots x_n^{i_n}$$

for $\mathbf{x} = (x_1, \ldots, x_n) \in I\!R^n$. Also, $\mathbf{1}(r)$ will denote the $n$-multi-index with 1 in the $r$th place and zero elsewhere, and

$$\delta_{\mathbf{i}}^{\mathbf{j}} = \begin{cases} 1 & \text{if } \mathbf{i} = \mathbf{j}, \\ 0 & \text{if } \mathbf{i} \neq \mathbf{j}. \end{cases}$$

We write $\mathbf{i} \geq \mathbf{0}$ if and only if $i_1 \geq 0, \ldots, i_n \geq 0$.

For an analytic function $h : I\!R^n \longrightarrow I\!R$, Taylor's formula becomes

$$h(\mathbf{x}) = \sum_{\mathbf{i} \geq \mathbf{0}} \frac{\mathbf{x}^{\mathbf{i}}}{\mathbf{i}!} h^{(\mathbf{i})}(\mathbf{0}),$$

where

$$h^{(\mathbf{i})}(\mathbf{x}) = \partial^{|\mathbf{i}|} h(\mathbf{x}) / \partial x_1^{i_1} \ldots \partial x_n^{i_n},$$

Let $l^2$ denote the standard Banach space of square summable sequences and let $l_e^2$ [5] denote the Banach space of sequences $(\alpha_n)_{n \geq 0}$ such that the sequence $(\alpha_n / n!)_{n \geq 0}$ belongs to $l^2$. Define a norm on $l_e^2$ by

$$\| (\alpha_n)_{n \geq 0} \|_e = \left( \sum_{n \geq 0} \frac{\alpha_n^2}{(n!)^2} \right)^{\frac{1}{2}} \text{ for } (\alpha_n)_{n \geq 0} \in l_e^2.$$

Now consider the algebraic tensor product $\mathcal{L}_n = \otimes_n l_e^2$ and let $\| \cdot \|$ be any cross norm on $\mathcal{L}_n$ [20]. For a simple tensor $\Phi$, we have

$$\Phi = (\phi_{i_1 \ldots i_n}) = (\alpha_{i_1}^1 \ldots \alpha_{i_n}^n) = \alpha^1 \otimes \ldots \otimes \alpha^n,$$

where $\alpha^k = (\alpha_{i_k}^k)_{i_k \geq 0} \in l_e^2 (k = 1, \ldots, n)$. Then

$$\| \Phi \| = \prod_{k=1}^{n} \| \alpha^k \|_e,$$

For $\Phi = (\phi_{\mathbf{i}})_{\mathbf{i} \geq \mathbf{0}}$ in the space of tensors $\mathcal{L}_n^T$ of the form $\Phi = (\mathbf{x}^{\mathbf{i}})_{\mathbf{i} \geq \mathbf{0}}$, we have,

$$\begin{aligned} \| \Phi \| &= \prod_{k=1}^{n} \| (x_k^{i_k})_{i_k \geq 0} \|_e \\ &= \prod_{k=1}^{n} \left( \sum_{i_k \geq 0} \frac{1}{(i_k!)^2} x_k^{2 i_k} \right)^{\frac{1}{2}} \\ &= \left( \sum_{\mathbf{i} \geq \mathbf{0}} \frac{1}{(\mathbf{i}!)^2} \phi_{\mathbf{i}}^2 \right)^{\frac{1}{2}} \qquad (1.1) \end{aligned}$$

Let $A$ be a tensor operator defined by $(A\Phi)_{\mathbf{q}} = \sum_{\mathbf{p} \geq \mathbf{0}} a_{\mathbf{q}}^{\mathbf{p}} \phi_{\mathbf{p}}$, $\mathbf{p}$ and $\mathbf{q}$ being multi-indices. We say that $A$ satisfies the 'diagonal dominance' property if there exists a $\rho > 0$ such that $a_{\mathbf{p}}^{\mathbf{p}} + \sum_{\mathbf{l} \neq \mathbf{p}} | a_{\mathbf{p}}^{\mathbf{l}} | < -\rho, \forall \mathbf{p} \geq \mathbf{0}$. This is a byproduct of the generalization of Gersgorin's theorem to tensor operators [4].

## 2 Global exponential stability for free nonlinear systems:

Consider the system without input given by

$$\dot{\mathbf{x}} = f(\mathbf{x}) \quad , \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.1}$$

where $\mathbf{x}_0 \in I\!R^n$ and $f : I\!R^n \longrightarrow I\!R^n$ is an analytic function such that $f(0) = 0$. We shall assume that the solution exists for all time.

Differentiating $\phi_{\mathbf{p}} = \mathbf{x}^{\mathbf{p}}$, where $\mathbf{p}$ is an $n$-multi-index, along the trajectories of (2.1), we obtain

$$\begin{aligned} \dot{\phi}_p &= \sum_{k=1}^{n} p_k \mathbf{x}^{\mathbf{p}-\mathbf{1}(\mathbf{k})} \dot{x}_k \\ &= \sum_{k=1}^{n} p_k \mathbf{x}^{\mathbf{p}-\mathbf{1}(\mathbf{k})} \sum_{|\mathbf{m}|>0} f_k^{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \\ &= \sum_{|\mathbf{m}|>0} \sum_{k=1}^{n} p_k f_k^{\mathbf{m}} \mathbf{x}^{\mathbf{p}+\mathbf{m}-\mathbf{1}(\mathbf{k})} \\ &= \sum_{|\mathbf{l}|>0} a_{\mathbf{p}}^{\mathbf{l}} \phi_{\mathbf{p}} \end{aligned} \tag{2.2}$$

where

$$a_{\mathbf{p}}^{\mathbf{l}} = \sum_{k=1}^{n} p_k f_k^{\mathbf{l}-\mathbf{p}+\mathbf{1}(\mathbf{k})} \tag{2.3}$$

or in compact form

$$\dot{\Phi} = A\Phi \quad , \quad \Phi(0) = \Phi^0 \tag{2.4}$$

so

$$\dot{\phi}_{\mathbf{p}} = a_{\mathbf{p}}^{\mathbf{p}} \phi_{\mathbf{p}} + \sum_{\mathbf{l} \neq \mathbf{p}} a_{\mathbf{p}}^{\mathbf{l}} \phi_{\mathbf{l}} \tag{2.5}$$

hence

$$\frac{d}{dt}[e^{-a_{\mathbf{p}}^{\mathbf{p}} t} \phi_{\mathbf{p}}] = e^{-a_{\mathbf{p}}^{\mathbf{p}} t} \sum_{\mathbf{l} \neq \mathbf{p}} a_{\mathbf{p}}^{\mathbf{l}} \phi_{\mathbf{l}} \tag{2.6}$$

and

$$e^{-a_{\mathbf{p}}^{\mathbf{p}} t} \phi_{\mathbf{p}}(t) = \phi_{\mathbf{p}}(0) + \int_0^t e^{-a_{\mathbf{p}}^{\mathbf{p}} \tau} \sum_{\mathbf{l} \neq \mathbf{p}} a_{\mathbf{p}}^{\mathbf{l}} \phi_{\mathbf{l}}(\tau) d\tau \tag{2.7}$$

which can be rewritten as

$$e^{-a_{\mathbf{p}}^{\mathbf{p}}t}\phi_{\mathbf{p}}(t) = \phi_{\mathbf{p}}(0) + \int_0^t e^{-a_{\mathbf{p}}^{\mathbf{p}}\tau}\phi_{\mathbf{p}}(\tau)\sum_{\mathbf{l}\neq\mathbf{p}} a_{\mathbf{l}}^{\mathbf{p}}\phi_{\mathbf{l}-\mathbf{p}}(\tau)d\tau \qquad (2.8)$$

Suppose that $\mathbf{x}(\tau)$ is in the unit ball $\{\mathbf{x} \in I\!R^n : max_{k=1,\dots,n} \mid x_k \mid < 1\}$ for $\tau \leq t$, then,

$$\mid e^{-a_{\mathbf{p}}^{\mathbf{p}}t}\phi_{\mathbf{p}}(t) \mid \leq \mid \phi_{\mathbf{p}}(0) \mid + \int_0^t \mid e^{-a_{\mathbf{p}}^{\mathbf{p}}\tau}\phi_{\mathbf{p}}(\tau) \mid \sum_{\mathbf{l}\neq\mathbf{p}} \mid a_{\mathbf{l}}^{\mathbf{p}} \mid d\tau \qquad (2.9)$$

Using Gronwall's lemma, we conclude

$$\mid e^{-a_{\mathbf{p}}^{\mathbf{p}}t}\phi_{\mathbf{p}}(t) \mid \leq \mid \phi_{\mathbf{p}}(0) \mid e^{t\sum_{\mathbf{l}\neq\mathbf{p}}|a_{\mathbf{l}}^{\mathbf{p}}|} \qquad (2.10)$$

Thus,

$$\begin{aligned} \mid \phi_{\mathbf{p}}(t) \mid &\leq e^{(\sum_{\mathbf{l}\neq\mathbf{p}}|a_{\mathbf{l}}^{\mathbf{p}}|+a_{\mathbf{p}}^{\mathbf{p}})t} \\ &\leq e^{-\rho t} \end{aligned} \qquad (2.11)$$

hence, taking into account (1.1), we obtain

$$\parallel \Phi(t) \parallel \leq e^{-\rho t} \qquad (2.12)$$

and

$$\parallel e^{At} \parallel \leq e^{-\rho t} \qquad (2.13)$$

(since $\parallel \Phi \parallel = \parallel e^{At}\Phi^0 \parallel \leq e^{-\rho t} \quad , \quad \forall \Phi^0 = (\mathbf{x}_0^{\mathbf{i}})_{\mathbf{i}\geq\mathbf{0}}$. Note that this inequality only holds on the nonlinear subspace $\mathcal{L}_n^T$ of tensors of the form $\Phi = (\mathbf{x}^{\mathbf{i}})_{\mathbf{i}\geq\mathbf{0}}$ and it does not follow that it holds for $\Phi^0$ in the closed linear span of tensors of this form [3]) provided there exists a $\rho > 0$ such that

$$a_{\mathbf{p}}^{\mathbf{p}} + \sum_{\mathbf{l}\neq\mathbf{p}} \mid a_{\mathbf{l}}^{\mathbf{p}} \mid \leq -\rho < 0 \qquad , \quad \forall \mathbf{p} \geq \mathbf{0}. \qquad (2.14)$$

Hence, we have proved the following

*Theorem 1:*

Any solution of (2.1) starting in the unit ball $\{\mathbf{x} \in I\!R^n : max_{k=1,\dots,n} \mid x_k \mid < 1\}$

will remain there and will go to zero exponentially as $t \longrightarrow \infty$ if there exists $\rho > 0$ such that

$$a_{\mathbf{p}}^{\mathbf{p}} + \sum_{\mathbf{l}\neq\mathbf{p}} \mid a_{\mathbf{l}}^{\mathbf{p}} \mid \leq -\rho < 0 \qquad , \quad \forall \mathbf{p} \geq \mathbf{0}. \tag{2.15}$$

*Remark 1:*

This diagonal dominance property known as Gersgoring's theorem has been used in [4] to prove the non-existence of limit cycles in nonlinear systems.

*Corollary 1:*

Any solution of (2.1) starting in the unit ball $\{\mathbf{x} \in \mathbb{R}^n : max_{k=1,\ldots,n} \mid x_k \mid < 1\}$ will remain there and will go to zero exponentially as $t \longrightarrow \infty$ if there exists $\rho > 0$ such that

$$f_k^{\mathbf{1(k)}} + \sum_{\mathbf{q}\geq\mathbf{0}} \mid f_k^{\mathbf{q+1(k)}} \mid \leq -\rho \tag{2.16}$$

*Proof:*

Indeed,

$$\begin{aligned} a_{\mathbf{p}}^{\mathbf{p}} + \sum_{\mathbf{l}\neq\mathbf{p}} \mid a_{\mathbf{l}}^{\mathbf{p}} \mid \quad &\leq \quad \sum_{k=1}^{n} p_k f_k^{\mathbf{1(k)}} + \sum_{\mathbf{l}\neq\mathbf{p}} \sum_{k=1}^{n} p_k \mid f_k^{\mathbf{l-p+1(k)}} \mid \\ &= \quad \sum_{k=1}^{n} p_k \left[ f_k^{\mathbf{1(k)}} + \sum_{\mathbf{l}\neq\mathbf{p}} \mid f_k^{\mathbf{l-p+1(k)}} \mid \right] \\ &= \quad \sum_{k=1}^{n} p_k \left[ f_k^{\mathbf{1(k)}} + \sum_{\mathbf{q}\neq\mathbf{0}} \mid f_k^{\mathbf{q+1(k)}} \mid \right] \end{aligned} \tag{2.17}$$

Therefore, if

$$f_k^{\mathbf{1(k)}} + \sum_{\mathbf{q}\neq\mathbf{0}} \mid f_k^{\mathbf{q+1(k)}} \mid \leq -\rho \quad , \quad k = 1, \ldots n \tag{2.18}$$

then, the previous theorem applies.

# 3 A nonlinear input-output map for linear analytic systems:

Consider the linear analytic system

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}) + \sum_{j=1}^{m} u_j g_j(\mathbf{x}) \quad , \qquad \mathbf{x}(0) = x_0 \\ y &= h(\mathbf{x}) \end{aligned} \tag{3.1}$$

where $\mathbf{x}_0 \in I\!R^n, u_j \in I\!R, j = 1, \ldots, m$ and $f, g_j (j = 1, \ldots m) : I\!R^n \longrightarrow I\!R^n$ and $h : I\!R^n \longrightarrow I\!R^r$ are analytic functions such that $f(0) = 0, g(0) = 0$. We shall assume that the solution exists for all time and (without loss of generality) that $max_{k=1,\ldots,n} \mid x_{0k} \mid \leq 1$.

Let $u_0(t) = 1$ ; $t \geq 0$ and $g_0 = f$, (3.1) becomes

$$\begin{aligned} \dot{\mathbf{x}} &= \sum_{j=0}^{m} u_j g_j(\mathbf{x}) \quad , \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ y &= h(\mathbf{x}) \end{aligned} \tag{3.2}$$

As in previous section, define $\phi_{\mathbf{p}} = \mathbf{x}^{\mathbf{p}}$ for an $n$-multi-index $p$, differentiating along the trajectories of (3.2) and replacing the $g_j$ , $(j = 0, \ldots, m)$ by their Taylor series expansions, we obtain

$$\begin{aligned} \dot{\phi_{\mathbf{p}}} &= \sum_{k=1}^{n} p_k \mathbf{x}^{\mathbf{p}-\mathbf{1}(\mathbf{k})} \dot{x}_k \\ &= \sum_{k=1}^{n} p_k \mathbf{x}^{\mathbf{p}-\mathbf{1}(\mathbf{k})} \sum_{j=0}^{m} u_j \sum_{|\mathbf{l}|\geq 0} g_{jk}^{\mathbf{l}} x^{\mathbf{l}} \\ &= \sum_{|\mathbf{l}|\geq 0} \sum_{k=1}^{n} p_k \sum_{j=0}^{m} u_j g_{jk}^{\mathbf{l}} \mathbf{x}^{\mathbf{p}+\mathbf{l}-\mathbf{1}(\mathbf{k})} \\ &= \sum_{j=0}^{m} u_j \sum_{|\mathbf{q}|\geq 0} \left[ \sum_{k=1}^{n} p_k g_{jk}^{\mathbf{q}-\mathbf{p}+\mathbf{1}(\mathbf{k})} \right] \phi_{\mathbf{q}} \end{aligned}$$

That is

$$\dot{\Phi} = \sum_{j=0}^{m} u_j N_j \Phi \qquad ; \quad \Phi(0) = \Phi^0 \tag{3.3}$$

$$y = C\Phi \tag{3.4}$$

where

$$
\begin{aligned}
\Phi &= (\phi_{\mathbf{p}})_{\mathbf{p}} \\
(N_j\Phi)_{\mathbf{p}} &= \sum_{|\mathbf{q}|\geq 0} a_j^{\mathbf{q}}(\mathbf{p})\phi_{\mathbf{q}} \\
a_j^{\mathbf{q}}(\mathbf{p}) &= \sum_{k=1}^{n} p_k g_{jk}^{\mathbf{q}-\mathbf{p}+\mathbf{1(k)}} \\
(C\Phi)_k &= \sum_{|\mathbf{q}|\geq 0} h_k^{\mathbf{q}}\phi_{\mathbf{q}}
\end{aligned}
$$

Now, for a given $k$, consider the change of variable

$$\Psi = e^{-N_k \int_0^t u_k(\tau)d\tau}\Phi \qquad (3.5)$$

We have $\Psi(0) = \Phi^0$. Differentiating $\Psi$ and taking into account (3.3) we get

$$\dot{\Psi} = e^{-N_k \int_0^t u_k(\tau)d\tau} P_k(t) e^{N_k \int_0^t u_k(\tau)d\tau}\Phi \qquad (3.6)$$

where, $P_k(t) = \sum_{j\neq k} u_j N_j$.
Let $U_k(\tau) = \int_0^\tau u_k(\sigma)d\sigma$, Then,

$$\Psi(t) = \Phi^0 + \int_0^t e^{-N_k U_k(\tau)} P_k(\tau) e^{N_k U_k(\tau)}\Psi(\tau)d\tau \qquad (3.7)$$

Hence, we obtain the nonlinear input-output map given by

$$
\begin{aligned}
y(t) &= Ce^{N_k U_k(t)}\Phi^0 + \\
&+ \sum_{l\geq 1}\sum_{j_1\neq k}\cdots\sum_{j_l\neq k}\int_0^t\int_0^{\tau_l}\cdots\int_0^{\tau_2} \\
&\quad Ce^{N_k[U_k(t)-U_k(\tau_l)]}N_{j_l}e^{N_k[U_k(\tau_l)-U_k(\tau_{l-1})]}N_{j_{l-1}} \\
&\quad \ldots N_{j_2}e^{N_k[U_k(\tau_2)-U_k(\tau_1)]}N_{j_1}e^{N_k U_k(\tau_1)}\Phi^0 \\
&\quad u_{j_l}(\tau_l)\ldots u_{j_1}(\tau_1)d\tau_1\ldots d\tau_l
\end{aligned}
\qquad (3.8)
$$

# 4 Stability of linear analytic systems:

In this section we shall present sufficient conditions for a kind of $L^\infty$-stability of nonlinear systems. In [13] we introduced the following

*Definition 1:*

We say that the nonlinear system (3.1) is $(\Omega, L^{\infty}[0,\infty;\mathbb{R}^p])$-stable where $\Omega \subset L^{\infty}[0,\infty;\mathbb{R}^m]$ if

$$\forall u \in \Omega \subset L^{\infty}[0,\infty;\mathbb{R}^m] \longrightarrow y \in L^{\infty}[0,\infty;\mathbb{R}^p]$$

where u and y are respectively the input and output of the system.

We claim the following:

*Theorem 2:*

Suppose there exists at least one $N_j$ $(j = 0,\dots,m)$ (say $N_k$) verifying the diagonal dominance property,

If $\Omega$ is the set of $u \in L^{\infty}[0,\infty;\mathbb{R}^m]$ for which the following conditions hold:

(a) $\lim_{t\to\infty} \int_0^t [-\rho_k u_k(\tau) + \| P_k(\tau) \|] d\tau < \infty$

where $\rho_k > 0$ is such that $\| e^{N_k t} \| \le e^{-\rho_k t}$, $P_k(t) = \sum_{j\neq k} u_j(t) N_j$

(b) $u_k \geq 0$.

Then the system (3.1) is $(\Omega, L^{\infty}[0,\infty;\mathbb{R}^p])$-stable. Furthermore, if the limit is $-\infty$ then $y(t) \to 0$ as $t \to \infty$.

*Proof:*

From the previous section, we know that if $N_k$ satisfies the diagonal dominance property,then there exists a positive constant $\rho_k > 0$ such that $\| e^{N_k t} \| \le e^{-\rho_k t}$ .

Combining (3.5) and (3.3) we obtain

$$\Phi(t) = e^{N_k U_k(t)}\Phi_0 + \int_0^t e^{N_k[U_k(t)-U_k(\tau)]} P_k(\tau)\Phi(\tau) d\tau \tag{4.1}$$

which yields,

$$\| \Phi(t) \| \le \| \Phi_0 \| e^{-\rho_k U_k(t)} + \int_0^t e^{-\rho_k[U_k(t)-U_k(\tau)]} \| P_k(\tau) \| \| \Phi(\tau) \| d\tau \tag{4.2}$$

since $U_k(t) \geq U_k(\tau) \geq 0$ for $t \geq \tau \geq 0$ (because $u_k \geq 0$). Therefore,

$$e^{\rho_k U_k(t)} \| \Phi(t) \| \le \| \Phi_0 \| + \int_0^t \| P_k(\tau) \| e^{\rho_k U_k(\tau)} \| \Phi(\tau) \| d\tau \tag{4.3}$$

Using Gronwall's lemma, we obtain

$$e^{\rho_k U_k(t)} \parallel \Phi(t) \parallel \leq \parallel \Phi_0 \parallel exp[\int_0^t \parallel P_k(\tau) \parallel d\tau] \tag{4.4}$$

Hence,

$$\parallel y(t) \parallel \leq \parallel C \parallel \parallel \Phi_0 \parallel exp \int_0^t [-\rho_k u_k(\tau) + \parallel P_k(\tau) \parallel] d\tau \tag{4.5}$$

Thus the theorem is proved.

*Corollary 2:*

Assume that there exists at least one $N_j$ $(j = 0, \ldots, m)$ (say $N_k$) satisfying the diagonal dominance property, and let $\Omega$ be the set of $u \in L^\infty[0, \infty; I\!R^m]$ for which
(a) $\lim_{t \to \infty} \int_0^t [-\rho_k u_k(\tau) + \sum_{j \neq k} \mid u_j(\tau) \mid \parallel N_j \parallel] d\tau < \infty$
where $\rho_k > 0$ is such that $\parallel e^{N_k t} \parallel \leq e^{-\rho_k t}$
and
(b) $u_k \geq 0$.
Then the system (3.1) is $(\Omega, L^\infty[0, \infty; I\!R^p])$ -stable. Furthermore, if the limit is $-\infty$ then $y(t) \to 0$ as $t \to \infty$ .

# 5 Examples:

In this section we shall present few examples to illustrate the theory.

*Example 1:* All the solutions of the system

$$\begin{cases} \dot{x_1} &= -5x_1 + 2x_1^3 x_2^2 - x_1^2 \\ \dot{x_2} &= -2x_2 + x_1^4 \end{cases}$$

starting in the unit ball $\{(a, b) \in I\!R^2 \ / \ max(\mid a \mid, \mid b \mid) \leq 1\}$ will go to zero exponentially as $t \longrightarrow \infty$ and $\mid x_i(t) \mid \leq \mid x_i(0) \mid e^{-t}$ , $i = 1, 2$. Indeed, the sum of a diagonal coefficient -5 (resp. -2) and the absolute values of the corresponding off-diagonal elements 2, 1 (resp. 1) is -2 (resp. -1).

*Example 2:* Consider the system

$$\begin{cases} \dot{x_1} &= -5x_1 + \frac{1}{4}x_2^3x_1 \\ \dot{x_2} &= -6x_1 - \frac{1}{8}x_1^2x_2^3 \end{cases}$$

such that $x_1(0) = x_{10}, x_2(0) = x_{20}$. Let $\alpha = max\{\mid x_{10} \mid, \mid x_{20} \mid\} \neq 0$. Define $z = \frac{1}{\alpha}\mathbf{x}$. We obtain,

$$\begin{cases} \dot{z_1} &= -5z_1 + \frac{\alpha^3}{4}z_2^3z_1 \\ \dot{z_2} &= -6z_1 - \frac{\alpha^4}{8}z_1^2z_2^3 \end{cases}$$

A sufficient condition for exponential stability is given by the diagonal dominance inequalities from which we get $\alpha = max\{\mid x_{10} \mid, \mid x_{20} \mid\} \leq 2(3)^{\frac{1}{4}}$ .

*Example 3:* Consider the single-input system

$$\dot{\mathbf{x}} = f(\mathbf{x}) + ug(\mathbf{x}) \quad , \quad \mathbf{x}(0) = \mathbf{x}_0 \quad , \quad \mathbf{y} = \mathbf{x} \tag{5.1}$$

where $f(x_1, x_2) = (x_1 + x_2^2, -x_1^2 + 2x_2)^T$ , $g(x_1, x_2) = (-2x_1 + x_1x_2, -3x_2 + x_1^2)^T$ . The tensor operator associated with $g$ is diagonally dominant ($\rho = 1$). Let $\Omega$ be the set of $u \in L^\infty[0, \infty; I\!R^m]$ for which the following conditions hold: (a)$lim_{t \longrightarrow \infty} \int_0^t [-u(\tau) + \parallel N_0 \parallel d\tau < \infty$ , (b)$u \geq 0$. From theorem 1, the system is $(\Omega, L^\infty[0, \infty; I\!R])$-stable, and if the limit is $-\infty$ then $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$. $N_0$ is the tensor operator associated with $f$.

# 6 Conclusion:

We have presented in this paper a functional series representation for linear analytic systems using the tensor operator approach introduced in [6]. This result extends the one derived for bilinear systems [13]. We also derived a condition under which a free nonlinear system is globally exponentially stable. The condition is based on the diagonal dominance for tensor operators.

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