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# DUALITY IN ALGEBRA AND TOPOLOGY

W. G. DWYER, J. P. C. GREENLEES, AND S. IYENGAR

*Dedicated to Clarence W. Wilkerson, on the occasion of his sixtieth birthday*

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## 1. INTRODUCTION

In this paper we take some classical ideas from commutative algebra, mostly ideas involving duality, and apply them in algebraic topology. To accomplish this we interpret properties of ordinary commutative rings in such a way that they can be extended to the more general rings that come up in homotopy theory. Amongst the rings we work with are the differential graded ring of cochains on a space  $X$ , the differential graded ring of chains on the loop space  $\Omega X$ , and various ring spectra, e.g., the Spanier-Whitehead duals of finite spectra or chromatic localizations of the sphere spectrum.

Maybe the most important contribution of this paper is the conceptual framework, which allows us to view all of the following dualities

- Poincaré duality for manifolds
- Gorenstein duality for commutative rings
- Benson-Carlson duality for cohomology rings of finite groups

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- Poincaré duality for groups
- Gross-Hopkins duality in chromatic stable homotopy theory

as examples of a *single* phenomenon. Beyond setting up this framework, though, we prove some new results, both in algebra and topology, and give new proofs of a number of old results. Some of the rings we look at, such as  $C_*\Omega X$ , are not commutative in any sense, and so implicitly we extend the methods of commutative algebra to certain non-commutative settings. We give a new formula for the dualizing module of a Gorenstein ring (7.1); this formula involves differential graded algebras (or ring spectra) in an essential way and is one instance of a general construction that in another setting gives the Brown-Comenetz dual of the sphere spectrum (7.3). We also prove the local cohomology theorem for  $p$ -compact groups and reprove it for compact Lie groups with orientable adjoint representation (10.2). The previous proof for compact Lie groups [6] uses equivariant topology, but ours does not.

**1.1. Description of results.** The objects we work with are fairly general; briefly, we allow rings, differential graded algebras (DGAs), or ring spectra; these are all covered under the general designation  $\mathbb{S}$ -algebra (see 1.5). We usually work in a derived category or in a homotopy category of module spectra, to the extent that even if  $R$  is a ring, by a *module* over  $R$  we mean a chain complex of ordinary  $R$ -modules. Most of the time we start with a homomorphism  $R \rightarrow k$  of  $\mathbb{S}$ -algebras and let  $\mathcal{E}$  denote the endomorphism  $\mathbb{S}$ -algebra  $\text{End}_R(k)$ . There are three main parts to the paper, which deal with three different but related types of structures: *smallness*, *duality*, and the *Gorenstein condition*.

*Smallness.* There are several different kinds of smallness which the homomorphism  $R \rightarrow k$  might enjoy (4.14); the weakest and most flexible one is called *proxy-smallness*. Any surjection from a commutative Noetherian ring to a regular ring is proxy-small (5.1). One property of a proxy-small homomorphism is particularly interesting to us. Given an  $R$ -module  $M$ , there is an associated module  $\text{Cell}_k(M)$ , which is the closest  $R$ -module approximation to  $M$  which can be cobbled together from shifted copies of  $k$  by using sums and exact triangles. The notation  $\text{Cell}_k(M)$  comes from topology [10], but if  $R$  is a commutative ring and  $k = R/I$  for a finitely generated ideal  $I$ , then  $\pi_* \text{Cell}_k(M)$  is just the local cohomology  $H_I^{-*}(M)$  of  $M$  at  $I$  [13, §6]. It turns out that if  $R \rightarrow k$  is proxy-small, there is a canonical equivalence (4.10)

$$(1.2) \quad \text{Cell}_k M \sim \text{Hom}_R(k, M) \otimes_{\mathcal{E}} k.$$

*Duality.* Given  $R \rightarrow k$ , we look for a notion of ‘‘Pontriagin duality’’ over  $R$  which extends the notion of ordinary duality over  $k$ ; more specifically, we look for an  $R$ -module  $\mathcal{I}$  such that  $\text{Cell}_k(\mathcal{I}) \sim \mathcal{I}$  and such that for any  $k$ -module  $X$ , there is a natural weak equivalence

$$(1.3) \quad \text{Hom}_R(X, \mathcal{I}) \sim \text{Hom}_k(X, k).$$

The associated Pontriagin duality (or *Matlis duality*) for  $R$ -modules sends  $M$  to  $\text{Hom}_R(M, \mathcal{I})$ . If  $R \rightarrow k$  is  $\mathbb{Z} \rightarrow \mathbb{F}_p$ , there is only one such  $\mathcal{I}$ , namely  $\mathbb{Z}/p^\infty (= \mathbb{Z}[1/p]/\mathbb{Z})$ , and  $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/p^\infty)$  is ordinary  $p$ -local Pontriagin duality for abelian groups. Guided by a combination of 1.2 and 1.3, we find that in many circumstances, and in particular if  $R \rightarrow k$  is proxy-small, such dualizing modules  $\mathcal{I}$  are determined by *right*  $\mathcal{E}$ -module structures on  $k$ ; this structure is a new bit of information, since in its state of nature  $\mathcal{E}$  acts on  $k$  from the left. Given a suitable right action, the dualizing module  $\mathcal{I}$  is given by the formula

$$(1.4) \quad \mathcal{I} \sim k \otimes_{\mathcal{E}} k,$$

which mixes the exceptional right action of  $\mathcal{E}$  on  $k$  with the canonical left action. This is a formula which in one setting constructs the injective hull of the residue class field of a local ring (7.1), and in another gives the  $p$ -primary component of the Brown-Comenetz dual of the sphere spectrum (7.3). There are also other examples (§7). We call an  $\mathcal{I}$  which is of the form described in 1.4 a *Matlis lift* of  $k$ .

*The Gorenstein condition.* The homomorphism  $R \rightarrow k$  is said to be *Gorenstein* if, up to a shift,  $\text{Cell}_k(R)$  is a Matlis lift of  $k$ . This amounts to requiring that  $\text{Hom}_R(k, R)$  be equivalent to a shifted module  $\Sigma^a k$ , and that the right action of  $\mathcal{E}$  on  $k$  provided by this equivalence act as in 1.4 to give a dualizing module  $\mathcal{I}$ . There are several consequences of the Gorenstein condition. In the commutative ring case with  $k = R/I$ , the equivalences

$$\mathcal{I} = k \otimes_{\mathcal{E}} k = \Sigma^{-a} \text{Hom}_R(k, R) \otimes_{\mathcal{E}} k \sim \Sigma^{-a} \text{Cell}_k R.$$

give a connection between the dualizing module  $\mathcal{I}$  and the local cohomology object  $\mathbf{R}\Gamma_I(R) = \text{Cell}_k R$ . (Another notation for  $\mathbf{R}\Gamma_I(R)$  might be  $H_I^\bullet(R)$ , since  $\pi_i \mathbf{R}\Gamma_I(R)$  is the local cohomology group  $H_I^{-i}(R)$ .) In this paper we head in a slightly different direction. Suppose that  $R$  is an augmented  $k$ -algebra and  $R \rightarrow k$  is the augmentation; in this case it is possible to compare the two right  $\mathcal{E}$ -modules  $\text{Hom}_R(k, R)$  and  $\text{Hom}_R(k, \text{Hom}_k(R, k))$ . Given that  $R \rightarrow k$  is Gorenstein, the first is abstractly equivalent to  $\Sigma^a k$ ; the second, by an adjointness argument, is always equivalent to  $k$ . If these two objects are the same as  $\mathcal{E}$ -modules

after the appropriate shift, we obtain a formula

$$\Sigma^a \operatorname{Cell}_k \operatorname{Hom}_k(R, k) \sim \operatorname{Cell}_k R,$$

relating duality on the left to local cohomology on the right. In many circumstances  $\operatorname{Cell}_k \operatorname{Hom}_k(R, k)$  is equivalent to  $\operatorname{Hom}_k(R, k)$  itself, and in these cases the above formula becomes

$$\Sigma^a \operatorname{Hom}_k(R, k) \sim \operatorname{Cell}_k R.$$

This leads to spectral sequences relating the local cohomology of a ring to some kind of  $k$ -dual of the ring, for instance, if  $X$  is a suitable space, relating the local cohomology of  $H^*(X; k)$  to  $H_*(X; k)$ . We use this approach to reprove the local cohomology theorem for compact Lie groups and prove it for  $p$ -compact groups.

We intend to treat the special case of chromatic stable homotopy theory in [14]; it turns out that Gross-Hopkins duality is a consequence of the fact that the  $\mathbb{S}$ -algebra map  $L_{K(n)}\mathbb{S} \rightarrow K(n)$  is Gorenstein. In [15] we use our techniques to study derived categories of local rings.

**1.5. Notation and terminology.** In this paper we use the term  *$\mathbb{S}$ -algebra* to mean ring spectrum in the sense of [19] or [29]; the symbol  $\mathbb{S}$  stands for the sphere spectrum. If  $k$  is a commutative  $\mathbb{S}$ -algebra, we refer to algebra spectra over  $k$  as  *$k$ -algebras*. The sphere  $\mathbb{S}$  is itself a commutative ring spectrum, and, as the terminology “ $\mathbb{S}$ -algebra” suggests, any ring spectrum is an algebra spectrum over  $\mathbb{S}$ . There is a brief introduction to the machinery of  $\mathbb{S}$ -algebras in §2; this follows the approach of [29].

Any ring  $R$  gives rise to an  $\mathbb{S}$ -algebra (whose homotopy is  $R$ , concentrated in degree 0), and we do not make a distinction in notation between  $R$  and this associated spectrum. If  $R$  is commutative in the usual sense it is also commutative as an  $\mathbb{S}$ -algebra; the category of  $R$ -algebras (in the way in which we use the term) is then equivalent to the more familiar category of differential graded algebras (DGAs) over  $R$ . For instance,  $\mathbb{Z}$ -algebras are essentially DGAs;  $\mathbb{Q}$ -algebras are DGAs over the rationals. See [42] for a detailed treatment of the relationship between  $\mathbb{Z}$ -algebras and DGAs.

A *module*  $M$  over an  $\mathbb{S}$ -algebra  $R$  is for us a module spectrum over  $R$ ; the category of these is denoted  ${}_R \operatorname{Mod}$ . Note that unspecified modules are left modules. If  $R$  is a ring, then an  $R$ -module in our sense is essentially an unbounded chain complex over  $R$ . More generally, if  $R$  is a  $\mathbb{Z}$ -algebra, an  $R$ -module is essentially the same as a differential graded module over the corresponding DGA [42]. (Unbounded chain

complexes over a ring should be treated homologically as in [44]. Differential graded modules over a DGA are treated very similarly; there are implicit discussions of this in [43, §3] and [42, §2].) If  $R$  is a ring, any ordinary module  $M$  over  $R$  gives rise to an  $R$ -module in our sense by the analog of the usual device of treating  $M$  as a chain complex concentrated in degree 0. We will refer to such an  $M$  as a *discrete* module over  $R$ , and we will not distinguish in notation between  $M$  and its associated spectrum.

*Homotopy/homology.* The homotopy groups of an  $\mathbb{S}$ -algebra  $R$  and an  $R$ -module  $M$  are denoted respectively  $\pi_*R$  and  $\pi_*M$ . The group  $\pi_0R$  is always a ring, and a ring is distinguished among  $\mathbb{S}$ -algebras by the fact that  $\pi_iR \cong 0$  for  $i \neq 0$ . If  $R$  is a  $\mathbb{Z}$ -algebra and  $M$  is an  $R$ -module, the homotopy groups  $\pi_*R$  and  $\pi_*M$  amount to the homology groups of the corresponding differential graded objects. A homomorphism  $R \rightarrow S$  of  $\mathbb{S}$ -algebras or  $M \rightarrow N$  of modules is an *equivalence* (weak equivalence, quasi-isomorphism) if it induces an isomorphism on  $\pi_*$ . In this case we write  $R \sim S$  or  $M \sim N$ .

*Hom and tensor.* Associated to two  $R$ -modules  $M$  and  $N$  is a spectrum  $\mathrm{Hom}_R(M, N)$  of homomorphisms; each  $R$ -module  $M$  also has an endomorphism ring  $\mathrm{End}_R(M)$ . These are derived objects; for instance, in forming  $\mathrm{End}_R(M)$  we always tacitly assume that  $M$  has been replaced by an equivalent  $R$ -module which is cofibrant (projective) in the appropriate sense. If  $M$  and  $N$  are respectively right and left modules over  $R$ , there is a derived smash product, which corresponds to tensor product of differential graded modules, and which we write  $M \otimes_R N$ .

To fix ideas, suppose that  $R$  is a ring,  $M$  is a discrete right module over  $R$ , and  $N, K$  are discrete left modules. Then  $\pi_i(M \otimes_R N) \cong \mathrm{Tor}_i^R(M, N)$ , while  $\pi_i \mathrm{Hom}_R(K, N) \cong \mathrm{Ext}_R^{-i}(K, N)$ . In this situation we sometimes write  $\mathrm{hom}_R(M, N)$  (with a lower-case “h”) for the group  $\mathrm{Ext}_R^0(M, N)$  of ordinary  $R$ -maps  $M \rightarrow N$ .

There are other contexts as above in which we follow the practice of tacitly replacing one object by an equivalent one without changing the notation. For instance, suppose that  $R \rightarrow k$  is a map of  $\mathbb{S}$ -algebras, and let  $\mathcal{E} = \mathrm{End}_R(k)$ . The right action of  $k$  on itself commutes with the left action of  $R$ , and so produces what we refer to as a “homomorphism  $k^{\mathrm{op}} \rightarrow \mathcal{E}$ ”, although in general this homomorphism can be realized as a map of  $\mathbb{S}$ -algebras only after adjusting  $k$  up to weak equivalence. The issue is that in order to form  $\mathrm{End}_R(k)$ , it is necessary to work with a cofibrant (projective) surrogate for  $k$  as a left  $R$ -module, and the right action of  $k$  on itself cannot in general be extended to an action of  $k$  on such a surrogate without tweaking  $k$  to some extent. The reader

might want to consider the example  $R = \mathbb{Z}$ ,  $k = \mathbb{F}_p$  from [13, §3], where it is clear that the ring  $\mathbb{F}_p$  cannot map to the DGA representing  $\mathcal{E}$ , although a DGA weakly equivalent to  $\mathbb{F}_p$  does map to  $\mathcal{E}$ . In general we silently pass over these adjustments and replacements in order to keep the exposition within understandable bounds.

*Derived category.* The *derived category*  $\mathbf{D}(R) = \mathrm{Ho}({}_R\mathrm{Mod})$  of an  $\mathbb{S}$ -algebra  $R$  is obtained from  ${}_R\mathrm{Mod}$  by formally inverting the weak equivalences. A map between  $R$ -modules passes to an isomorphism in  $\mathbf{D}(R)$  if and only if it is a weak equivalence. Sometimes we have to consider a homotopy category  $\mathrm{Ho}(\mathrm{Mod}_R)$  involving right  $R$ -modules; since a right  $R$ -module is the same as a left module over the opposite ring  $R^{\mathrm{op}}$ , we write  $\mathrm{Ho}(\mathrm{Mod}_R)$  as  $\mathbf{D}(R^{\mathrm{op}})$ . If  $R$  is a ring,  $\mathbf{D}(R)$  is categorically equivalent to the usual derived category of  $R$ .

*Augmentations.* Many of the objects we work with are augmented. An *augmented  $k$ -algebra*  $R$  is a  $k$ -algebra together with an augmentation homomorphism  $R \rightarrow k$  which splits the  $k$ -algebra structure map  $k \rightarrow R$ . A map of augmented  $k$ -algebras is a map of  $k$ -algebras which respects the augmentations. If  $R$  is an augmented  $k$ -algebra, we will by default treat  $k$  as an  $R$ -module via the homomorphism  $R \rightarrow k$ .

*Another path.* The advantage of using the term  $\mathbb{S}$ -algebra is that we can refer to rings, DGAs, and ring spectra in one breath. The reader can confidently take  $\mathbb{S} = \mathbb{Z}$ , read DGA for  $\mathbb{S}$ -algebra,  $H_*$  for  $\pi_*$ , and work as in [13] in the algebraic context of [44]; only some examples will be lost. Note however that the loss will include all examples involving commutativity in any essential way, unless the commutative  $\mathbb{S}$ -algebras in question are  $\mathbb{Q}$ -algebras or ordinary commutative rings. This is a consequence of the fact that under the correspondence between  $\mathbb{Z}$ -algebras and DGAs, the notion of commutativity for  $\mathbb{Z}$ -algebras does not carry over to the usual notion of commutativity for DGAs, except in characteristic 0 [36, App. C] [32], or if the homotopy of the  $\mathbb{Z}$ -algebra is concentrated in degree 0.

**1.6. Organization of the paper.** Section 2 has a brief expository introduction to spectra and  $\mathbb{S}$ -algebras, and section 3 describes some elementary properties of  $\mathbb{S}$ -algebras which we use later on. Some readers may wish to skip these sections the first time through. The three main themes, smallness, duality, and the Gorenstein condition, are treated respectively in Sections 4, 6, and 8. Section 9 explains how to set up a local cohomology spectral sequence for a suitable Gorenstein  $\mathbb{S}$ -algebra. We spend a lot of time dealing with examples; §5 has examples relating to smallness, §7 examples related to duality, and §10

examples related to the Gorenstein condition. In particular, Section 10 contains a proof of the local cohomology theorem for  $p$ -compact groups (10.2) and for compact Lie groups with orientable adjoint representation (10.3); following [24] and [6], for finite groups this is one version of Benson-Carlson duality [5].

**1.7. Relationship to previous work.** There is a substantial literature on Gorenstein rings. Our definition of a Gorenstein map  $R \rightarrow k$  of  $\mathbb{S}$ -algebras extends the definition of Avramov-Foxby [4] (see 8.4). Félix, Halperin, and Thomas have considered pretty much this same extension in the topological context of rational homotopy theory and DGAs [20]; we generalize their work and have benefitted from it. Frankild and Jorgensen [21] have also studied an extension of the Gorenstein condition to DGAs, but their intentions are quite different from ours.

## 2. SPECTRA, $\mathbb{S}$ -ALGEBRAS, AND COMMUTATIVE $\mathbb{S}$ -ALGEBRAS

In this section we give a brief introduction to the nuts and bolts of spectra,  $\mathbb{S}$ -algebras, and commutative  $\mathbb{S}$ -algebras; this is purely expository, and so the experts can safely move on to §3. In very rough terms, a spectrum is something like an unbounded chain complex in which the commutative and associative laws for the addition of elements hold only up to coherently specified homotopies. As will become clear later, a spectrum could also reasonably be styled an  $\mathbb{S}$ -module. There is a well-behaved tensor product (AKA smash product) for spectra, and with the help of this it is possible to give simple definitions of  $\mathbb{S}$ -algebras and commutative  $\mathbb{S}$ -algebras.

Spectra are based in one way or another on homotopy-theoretic objects; the specific homotopy-theoretic objects we pick are *pointed simplicial sets* [22]. For the rest of this section, the word *space* taken by itself will mean pointed simplicial set. (In the course of the paper we often refer to topological spaces, but there are standard constructions which make it possible to pass back and forth between topological spaces and simplicial sets without losing homotopical information.) To set up the category of spectra we will rely on the *symmetric spectrum* machinery of Hovey, Shipley and Smith [29]. Both simplicial sets and symmetric spectra are inherently combinatorial objects, and so from the point of view we are taking a spectrum is combinatorial, or even algebraic, in nature.

**2.1. Asymmetric spectra.** We will start with a simple construction, which long predates the notion of symmetric spectrum; for want of a better term we'll call the objects that come up "asymmetric spectra". The category of asymmetric spectra is a good homotopy theoretic



model for the category of spectra, but it lacks a decent tensor product; passing to the more complicated category of symmetric spectra will solve this problem.

**2.2. Definition.** A *sequence*  $X$  of spaces is a collection  $\{X_n\}_{n \geq 0}$  of spaces. The (graded) product  $X \boxtimes^a Y$  of two such sequences is the sequence  $Z$  given by

$$Z_n = \coprod_{i+j=n} X_i \wedge Y_j.$$

Here  $\wedge$  denotes the smash product of spaces. The superscript  $a$  in  $\boxtimes^a$  signifies “asymmetric”; later on we will define another kind of  $\boxtimes$ . In spite of its decoration, the operation  $\boxtimes^a$  gives a symmetric monoidal structure on the category of sequences of spaces; the unit is the sequence  $\epsilon$  with  $\epsilon_0 = S^0$  and  $\epsilon_i = *$  for  $i > 0$  (here  $S^0$  is the zero-sphere, i.e., the unit for the smash product operation on the category of spaces). The twist isomorphism  $X \boxtimes^a Y \cong Y \boxtimes^a X$  acts at level  $n$  by using the usual isomorphisms  $X_i \wedge Y_j \rightarrow Y_j \wedge X_i$  ( $i + j = n$ ). Let  $S^1$  denote the simplicial circle, and  $S^n$  ( $n \geq 1$ ) the smash power  $S^1 \wedge \cdots \wedge S^1$  ( $n$  times). There is a sequence  $S$  with  $S_i = S^i$ , and it is easy to produce a pairing map

$$S \boxtimes^a S \rightarrow S$$

which makes  $S$  into a monoid with respect to  $\boxtimes^a$  [29, 2.3.4]. This pairing map is constituted from the obvious isomorphisms  $S^i \wedge S^j \rightarrow S^{i+j}$ .

**2.3. Definition.** An *asymmetric spectrum* is a sequence  $X$  of spaces together with a left action of  $S$  on  $X$ , i.e, a map  $S \boxtimes^a X \rightarrow X$  which satisfies appropriate associativity and unital identities.

**2.4. Remark.** Since  $S$  is the free monoid with respect to  $\boxtimes^a$  on a copy of  $S^1$  at level 1, an asymmetric spectrum amounts to a sequence  $X$  of spaces together with structure maps  $S^1 \wedge X_i \rightarrow X_{i+1}$ . This is exactly a spectrum in the sense of Whitehead [47] (although he worked with topological spaces instead of with simplicial sets).

If  $X$  is an asymmetric spectrum, the structure maps  $S^1 \wedge X_i \rightarrow X_{i+1}$  induce homotopy group maps  $\pi_n X_i \rightarrow \pi_{n+1} X_{i+1}$ .

**2.5. Definition.** The *homotopy groups* of an asymmetric spectrum  $X$  are the groups  $\pi_n X = \operatorname{colim}_i \pi_{n+i} X_i$ . A map  $X \rightarrow Y$  of asymmetric spectra is a *weak equivalence* (quasi-isomorphism) if it induces isomorphisms  $\pi_i X \cong \pi_i Y$ ,  $i \in \mathbb{Z}$ .

*Additivity, associativity, commutativity.* There is an elaborate homotopy theory of asymmetric spectra based upon this definition of weak equivalence. On its own, the definition of weak equivalence subtly imposes the additivity, associativity, and commutativity structures referred to in the introduction to the section. For instance, it follows from 2.5 that any asymmetric spectrum  $X$  is weakly equivalent to  $\Omega^n(S^n \wedge X)$  (where the loop functor  $\Omega^n$  and the smash functor  $S^n \wedge -$  are applied levelwise). However, for any space  $A$ ,  $\Omega^n(S^n \wedge A)$  ( $n \geq 1$ ) has up to homotopy an associative multiplication, and these multiplications enjoy ever richer commutativity properties as  $n$  increases.

2.6. *Relationship to chain complexes.* As defined in 2.5, asymmetric spectra have both positive and negative dimensional homotopy groups, just as unbounded chain complexes have both positive and negative dimensional homology groups. An unbounded chain complex  $C$  can be converted into an asymmetric spectrum  $X$  by setting  $X_i = N^{-1}(t \Sigma^i C)$ , where  $\Sigma^i$  shifts the complex upward  $i$  times, “ $t$ ” cuts off the negative dimensional components, and  $N^{-1}$  is the Dold-Kan denormalization functor, which converts a nonnegative chain complex into a simplicial abelian group. Then  $\pi_i X \cong H_i C$ .

2.7. *Homotopy category, shifting, triangulated structure, suspension spectra.* The *homotopy category* of asymmetric spectra is constructed by formally inverting weak equivalences, just as the derived category of a ring is constructed by formally inverting quasi-isomorphisms between chain complexes. For any  $i$  there is a shift map  $\Sigma^i$  defined on the category of asymmetric spectra, given by  $(\Sigma^i X)_n = X_{n-i}$ ; the formula is to be interpreted to mean that if  $n - i < 0$ , then  $(\Sigma^i X)_n = *$ . This is parallel to the usual shift operation on chain complexes: in particular,  $\pi_k \Sigma^i X = \pi_{k-i} X$ , and up to weak equivalence  $\Sigma^i \Sigma^j = \Sigma^{i+j}$ . (The words “up to weak equivalence” are needed here because  $i$  and  $j$  can be negative.) Given a map  $f : X \rightarrow Y$  of asymmetric spectra, the fibre  $F$  of  $f$  is an asymmetric spectrum with  $F_i$  given by the homotopy fibre of  $X_i \rightarrow Y_i$ ; the cofibre  $C$  is the asymmetric spectrum with  $C_i$  given by the mapping cone of  $X_i \rightarrow Y_i$ . It turns out that  $\Sigma F$  and  $C$  are naturally weakly equivalent, and that the cofibre of  $Y \rightarrow C$  is naturally weakly equivalent to  $\Sigma X$ . This allows the homotopy category of asymmetric spectra to be given a triangulated structure, in which  $F \rightarrow X \rightarrow Y$  or  $X \rightarrow Y \rightarrow C$  are distinguished triangles. These distinguished triangles give long exact sequences on homotopy groups.

Any space gives rise to a suspension spectrum  $\Sigma^\infty X$ , where  $(\Sigma^\infty X)_n = S^n \wedge X$ ; in some sense this is the free  $S$ -module on  $X$ . The homotopy groups of  $\Sigma^\infty X$  are the stable homotopy groups of  $X$ .

In spite of these encouraging signs, the category of asymmetric spectra has one serious shortcoming: there is no obvious way to define an internal tensor product on the category. Given two asymmetric spectra  $X, Y$ , one would like to define  $X \otimes Y$  to be  $X \boxtimes_S^a Y$ . There is a real difficulty in making a definition like this, stemming from the fact that  $S$  is *not* a commutative monoid with respect to  $\boxtimes^a$ . This is exactly the same difficulty that comes up in trying to form  $M \otimes_R N$  when  $M$  and  $N$  are left modules over the noncommutative ring  $R$ , and the intention is that  $M \otimes_R N$  be another left  $R$ -module. The remedy for this is to provide a little extra structure in the underlying objects, enough structure so that  $S$  becomes a commutative monoid.

**2.8. Symmetric spectra.** The trick is to build in symmetric group actions, hence the name, *symmetric spectra*.

**2.9. Definition.** A *symmetric sequence*  $X$  of spaces is a collection  $\{X_n\}_{n \geq 0}$  of spaces, together with, for each  $n$ , a left action of the symmetric group  $\Sigma_n$  on  $X_n$ . The (graded) product  $X \boxtimes Y$  of two such sequences is the sequence  $Z$  given by

$$Z_n = \coprod_{i+j=n} (\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_j} X_i \wedge Y_j.$$

Here  $(\Sigma_n)_+$  denotes the union of  $\Sigma_n$  with a disjoint basepoint.

**2.10. Remark.** A symmetric sequence can equally well be thought of as a functor from the category of finite sets and isomorphisms to the category of spaces. From this point of view the graded product has the more elegant description

$$(X \boxtimes Y)(C) = \coprod_{A \cup B = C, A \cap B = \emptyset} X(A) \wedge Y(B).$$

The definition of  $\boxtimes$  differs from that of  $\boxtimes^a$  (2.2) because of the need to have symmetric group actions on the constituents of the result. The product  $\boxtimes$  gives a symmetric monoidal structure on the category of symmetric sequences of spaces; the unit for  $\boxtimes$  is again the sequence  $\epsilon$  mentioned after 2.2, promoted to a symmetric sequence in the only possible way. The twist isomorphism  $X \boxtimes Y \cong Y \boxtimes X$  is composed, in the formulation from 2.10, of the usual isomorphisms

$$X(A) \wedge Y(B) \cong Y(B) \wedge X(A).$$

In the formulation of 2.9, the twist isomorphism combines isomorphisms  $X_i \wedge Y_j \cong Y_j \wedge X_i$  with right multiplication on  $\Sigma_n$  by an element of the symmetric group which conjugates  $\Sigma_i \times \Sigma_j$  to  $\Sigma_j \times \Sigma_i$ .

The sequence  $S$  described above after 2.2 extends to a symmetric sequence in a natural way, where  $\Sigma_n$  acts on  $S^1 \wedge \cdots \wedge S^1$  by permuting the factors. We will denote this symmetric sequence by  $\mathbb{S}$ , since it will correspond to the sphere spectrum. The natural  $\Sigma_i \times \Sigma_j$ -equivariant isomorphisms  $S^i \wedge S^j \cong S^{i+j}$  combine to give a natural map  $m : \mathbb{S} \boxtimes \mathbb{S} \rightarrow \mathbb{S}$ . The key property of this map is the following one.

**2.11. Lemma.** *The map  $m$  gives  $\mathbb{S}$  the structure of a commutative monoid (with respect to  $\boxtimes$ ) in the category of symmetric sequences of spaces.*

**2.12. Remark.** Commutativity means that  $m\tau = m$ , where  $\tau : \mathbb{S} \boxtimes \mathbb{S} \rightarrow \mathbb{S} \boxtimes \mathbb{S}$  is the twist isomorphism of the symmetric monoidal structure.

The basic definitions are now clear.

**2.13. Definition.** A *symmetric spectrum*  $X$  is a symmetric sequence of pointed spaces which is a left module over  $\mathbb{S}$  (i.e., has been provided with a map  $m_X : \mathbb{S} \boxtimes X \rightarrow X$  with appropriate unital and associativity properties).

**2.14. Remark.** Since  $\mathbb{S}$  is a commutative monoid, there is no real distinction between the notions of left and right modules; if  $m_X : \mathbb{S} \boxtimes X \rightarrow X$  gives  $X$  the structure of a left  $\mathbb{S}$ -module, then  $m_X \cdot \tau : X \boxtimes \mathbb{S} \rightarrow X$  gives  $X$  the structure of a right  $\mathbb{S}$ -module. Moreover, in this case  $m_X \cdot \tau$  is a map of left  $\mathbb{S}$ -modules.

**2.15. Definition.** The *tensor product* (or *smash product*) of two symmetric spectra  $X$  and  $Y$  is the symmetric spectrum  $X \boxtimes_{\mathbb{S}} Y$  defined by the coequalizer diagram

$$X \boxtimes \mathbb{S} \boxtimes Y \rightrightarrows X \boxtimes Y \rightarrow X \boxtimes_{\mathbb{S}} Y,$$

where the two maps on the left are induced by the  $\mathbb{S}$ -module structures of  $X$  and  $Y$ .

From now on we will drop the word “symmetric” and call a symmetric spectrum a spectrum. The tensor product  $X \boxtimes_{\mathbb{S}} Y$  is denoted  $X \otimes Y$ , or, in topological contexts,  $X \wedge Y$ . The tensor product gives a symmetric monoidal structure on the category of spectra; the unit for this structure is  $\mathbb{S}$ .

**2.16. Definition.** An  $\mathbb{S}$ -*algebra* (or *ring spectrum*)  $R$  is a spectrum together with maps  $S \rightarrow R$  and  $R \otimes R \rightarrow R$  with appropriate unital and associativity properties. The  $\mathbb{S}$ -algebra  $R$  is commutative if the multiplication  $m_R : R \otimes R \rightarrow R$  is *commutative*, i.e., if  $m_R \cdot \tau = m_R$ , where  $\tau$  is the twist automorphism of  $R \otimes R$ .

We leave it to the reader to define modules over an  $\mathbb{S}$ -algebra, tensor products of modules, algebras over a commutative  $\mathbb{S}$ -algebra, etc. Note that  $\mathbb{S}$  is an  $\mathbb{S}$ -algebra, that every spectrum is a module over  $\mathbb{S}$ , and that every  $\mathbb{S}$ -algebra is, in fact, an algebra over  $\mathbb{S}$ . From this point of view  $\mathbb{S}$  plays the role of the ground ring for the category of spectra, just as  $\mathbb{Z}$  is the ground ring for the category of chain complexes.

*Mapping spectra.* Given two spectra  $X, Y$ , it is possible to define a mapping spectrum  $\mathrm{Hom}(X, Y)$ . If the two spectra are modules over the  $\mathbb{S}$ -algebra  $R$ , there is also a spectrum  $\mathrm{Hom}_R(X, Y)$  of  $R$ -module maps; if in addition  $R$  is commutative,  $\mathrm{Hom}_R(X, Y)$  is an  $R$ -module spectrum.

*Homotopy theory and derived constructions.* There is quite a bit of work to be done in setting up the homotopy theory of spectra; in particular, it is tricky to define the homotopy groups  $\pi_i X$  of a spectrum  $X$ , or what comes to the same thing, to define weak equivalences (quasi-isomorphisms) between spectra [29, §3]. Familiar issues of a homological algebra nature come up: for instance, tensor products or mapping spectra do not necessarily preserve weak equivalences unless the objects involved have freeness (cofibrancy) or injectivity (fibrancy) properties. These issues are handled in the non-additive context of spectra by Quillen's model category machinery [16] [28], which essentially allows a great deal of homological algebra to be extended to sufficiently structured non-additive settings. Invoking this machinery leads to notions of derived tensor product and derived mapping spectrum.

After the dust has settled, it is possible to prove that the homotopy theory of spectra is equivalent to the homotopy theory of asymmetric spectra. There is a shift operation as in 2.7, as well as a triangulated structure on the homotopy category. This homotopy category is obtained as usual from the category of spectra by inverting the weak equivalences. Any space  $X$  gives rise to a suspension (symmetric) spectrum, which we continue to denote  $\Sigma^\infty X$ ; as in 2.7,  $(\Sigma^\infty X)_n$  is  $S^n \wedge X$ , but now the symmetric group acts on  $S^n = S^1 \wedge \cdots \wedge S^1$  by permuting the factors.

2.17. *Rings and DGAs vs.  $\mathbb{S}$ -algebras.* The construction of 2.6 can be extended to convert any ring  $R$  into an  $\mathbb{S}$ -algebra  $R_{\mathbb{S}}$  and any ordinary module over  $R$  to a module over  $R_{\mathbb{S}}$ . More generally, any chain complex over  $R$  gives a module over  $R_{\mathbb{S}}$ , and this correspondence provides an equivalence between the derived category of  $R$  and the homotopy category of modules over  $R_{\mathbb{S}}$  (this is actually part of a Quillen equivalence between two model categories). In this paper we work consistently with

$\mathbb{S}$ -algebras, and when an ordinary ring  $R$  comes into play we do not usually distinguish in notation between  $R$  and  $R_{\mathbb{S}}$ . Note that  $\pi_i R_{\mathbb{S}}$  is  $R$  if  $i = 0$ , and 0 otherwise.

These considerations apply more generally if  $R$  is a DGA [42]; there is an associated  $\mathbb{S}$ -algebra  $R_{\mathbb{S}}$ , and a Quillen equivalence which induces an equivalence between the derived category of  $R$  (i.e. the category obtained from DG  $R$ -modules by inverting the quasi-isomorphisms) and the homotopy category of  $R_{\mathbb{S}}$ -modules. Note that  $\pi_i R_{\mathbb{S}} = H_i R$ .

Any ring or DGA is an algebra over  $\mathbb{Z}$ , and the corresponding  $\mathbb{S}$ -algebra  $R_{\mathbb{S}}$  is an algebra over  $\mathbb{Z}_{\mathbb{S}}$ . The correspondence  $R \mapsto R_{\mathbb{S}}$  gives a bijection up to equivalence between DGAs and  $\mathbb{Z}_{\mathbb{S}}$ -algebras, or between rings and  $\mathbb{Z}_{\mathbb{S}}$  algebras whose homotopy is concentrated in degree 0. (Actually, any  $\mathbb{S}$ -algebra whose homotopy is concentrated in degree 0 is canonically a  $\mathbb{Z}_{\mathbb{S}}$ -algebra, and so amounts to an ordinary ring.)

The situation with commutativity is more complicated; commutative  $\mathbb{Z}_{\mathbb{S}}$ -algebras correspond to  $E_{\infty}$  DGAs (these are DGAs with a multiplication which is homotopy commutative up to explicit higher homotopies which are parametrized by the cells of an  $E_{\infty}$  operad.) The prime example of such an  $E_{\infty}$  algebra is the integral cochain algebra on a space  $X$  [36, App. C] (since all of our DGAs have differential which decreases dimension by 1, the cochain algebra is treated as a DGA by placing the  $i$ -dimensional cochains in degree  $-i$ ). From a homotopical point of view,  $C^*(X; \mathbb{Z})$  is  $\text{Hom}_{\mathbb{S}}(\Sigma^{\infty} X_+, \mathbb{Z}_{\mathbb{S}})$ , where  $X_+$  is  $X$  with a disjoint basepoint adjoined. The commutative  $\mathbb{S}$ -algebra structure is derived from the multiplication on  $\mathbb{Z}_{\mathbb{S}}$  and the diagonal map on  $X$ .

To repeat, commutative  $\mathbb{Z}_{\mathbb{S}}$ -algebras do *not* correspond to commutative DGAs. However, there is one bright note: commutative  $\mathbb{Z}_{\mathbb{S}}$ -algebras with homotopy concentrated in degree 0 do correspond bijectively up to equivalence to commutative rings.

*Terminological caveat.* In the literature,  $\mathbb{S}$ -algebras are sometimes styled  $A_{\infty}$  ring spectra (or structured ring spectra) and commutative  $\mathbb{S}$ -algebras  $E_{\infty}$  ring spectra. The terms *ring spectrum* and *commutative ring spectrum* are occasionally used even today for a much weaker notion in which various diagrams involving the multiplication map are only required to commute up to homotopy.

### 3. SOME BASIC CONSTRUCTIONS WITH MODULES

This section looks into some constructions with  $\mathbb{S}$ -algebras and modules which we refer to in the rest of the paper. We first describe some Postnikov constructions which allow modules to be filtered in such a way that the successive (co)fibrations are “Eilenberg-MacLane objects”, in

the sense that they have nonvanishing homotopy in only a single dimension (3.2, 3.3). Next, we show that in many cases these Eilenberg-MacLane objects are determined by the homotopy group which appears (3.9), although there are surprises (3.11). We end by formulating “directionality” properties for modules (3.12), and studying finiteness conditions (3.15).

First, some terminology. An  $\mathbb{S}$ -algebra  $R$  is *connective* if  $\pi_i R = 0$  for  $i < 0$  and *coconnective* if  $\pi_i R = 0$  for  $i > 0$ . An  $R$ -module  $M$  is *bounded below* if  $\pi_i M = 0$  for  $i \ll 0$ , and *bounded above* if  $\pi_i M = 0$  for  $i \gg 0$ .

**3.1. Postnikov constructions.** There are subtle differences between the connective and coconnective cases.

**3.2. Proposition.** *Suppose that  $R$  is connective, that  $M$  is an  $R$ -module, and that  $n$  is an integer. Then there is a natural  $R$ -module  $P_n M$  with  $\pi_i(P_n M) \cong 0$  for  $i > n$ , together with a natural map  $M \rightarrow P_n M$  inducing isomorphisms on  $\pi_i$  for  $i \leq n$ .*

**3.3. Proposition.** *Suppose that  $R$  is coconnective with  $\pi_0 R$  a field, that  $M$  is an  $R$ -module, and that  $n$  is an integer. Then there is an  $R$ -module  $Q_n M$  with  $\pi_i(Q_n M) = 0$  for  $i < n$ , together with a map  $M \rightarrow Q_n M$  inducing isomorphisms on  $\pi_i$  for  $i \geq n$ .*

**3.4. Remark.** In the above situations there are maps  $P_n M \rightarrow P_{n-1} M$  or  $Q_n M \rightarrow Q_{n+1} M$  inducing isomorphisms on all appropriate nonzero homotopy groups. In the first case  $M \sim \text{holim}_n P_n M$ , while in the second  $M \sim \text{holim}_n Q_n M$ . The fibre of the map  $P_n M \rightarrow P_{n-1} M$  or  $Q_n M \rightarrow Q_{n+1} M$  is a module with only one nonvanishing homotopy group.

**3.5. Remark.** The construction of  $Q_n M$  cannot be made functorial in any reasonable sense. Consider the DGA  $\mathcal{E}$  of [13, §3];  $\mathcal{E}$  is coconnective and  $\pi_0 \mathcal{E} \cong \mathbb{F}_p$ . Then  $\pi_0 \text{Hom}_{\mathcal{E}}(\mathcal{E}, \mathcal{E}) \cong \pi_0 \mathcal{E} \cong \mathbb{F}_p$ , while  $\pi_0 \text{Hom}_{\mathcal{E}}(Q_0 \mathcal{E}, Q_0 \mathcal{E}) = \pi_0 \text{Hom}_{\mathcal{E}}(\mathbb{F}_p, \mathbb{F}_p) \sim \mathbb{Z}_p$ . We are using topological notation:  $\mathbb{Z}_p$  is the ring of  $p$ -adic integers. Since there is no additive map  $\mathbb{F}_p \rightarrow \mathbb{Z}_p$ , there is no way to form  $Q_0 \mathcal{E}$  functorially from  $\mathcal{E}$ .

**3.6. Remark.** We have used the notion of “holim” above, and later on we will use “hocolim”. If  $M_0 \rightarrow M_1 \rightarrow \cdots$  is an inductive system of spectra, then  $\text{hocolim}_n M_n$  is the fibre of  $1 - \sigma$ , where  $\sigma : \coprod M_n \rightarrow \coprod M_n$  is the shift map. There are isomorphisms  $\pi_i \text{hocolim}_n M_n \cong \text{colim}_n \pi_i M_n$ . Dually, if  $\cdots \rightarrow N_1 \rightarrow N_0$  is an inverse system of spectra,  $\text{holim}_n N_n$  is defined to be the fibre of  $1 - \sigma$ , where  $\sigma : \prod N_n \rightarrow \prod N_n$

is the shift map. In this case there are short exact sequences

$$0 \rightarrow \lim_n^1 \pi_{i+1} N_n \rightarrow \pi_i \operatorname{holim}_n N_n \rightarrow \lim_n \pi_i N_n \rightarrow 0.$$

Homotopy colimits and limits are defined for arbitrary small diagrams of spectra [27, 18.1], but we will not need them in this generality.

For the proofs we need to make attaching constructions.

**3.7. Definition.** Suppose that  $R$  is an  $\mathbb{S}$ -algebra and that  $X$  and  $Y$  are  $R$ -modules. Then  $Y$  is obtained from  $X$  by *attaching an  $R$ -module  $A$*  if there is a cofibration sequence  $A \rightarrow X \rightarrow Y$ . If  $\{A_\alpha\}$  is a collection of  $R$ -modules, then  $Y$  is obtained from  $X$  by *attaching copies of the  $A_\alpha$*  if there is a cofibration sequence  $U \rightarrow X \rightarrow Y$  in which  $U$  is a coproduct of elements from  $\{A_\alpha\}$ . Even more generally,  $Y$  is obtained from  $X$  by *iteratively attaching copies of  $\{A_\alpha\}$*  if  $Y$  is the colimit of a directed system  $Y_0 \rightarrow Y_1 \rightarrow \dots$ , such that  $X_0 = X$  and  $X_{n+1}$  is obtained from  $X_n$  by attaching copies of the  $A_\alpha$ .

*Proof of 3.2.* Form  $P_n M$  by iteratively attaching copies of  $\Sigma^i R$ ,  $i > n$  to  $M$  (3.7) to kill off the homotopy of  $M$  above dimension  $n$ . More specifically, for each element  $x \in \pi_{i_x} M$  with  $i_x > n$ , construct a map  $\Sigma^{i_x} R \rightarrow M$  which takes the unit in  $\pi_0 R$  to  $x$ . Let  $C(M)$  be the cofibre of the resulting map  $\coprod_x \Sigma^{i_x} R \rightarrow M$ , and observe that the map  $M \rightarrow C(M)$  induces isomorphisms on  $\pi_i$  for  $i \leq n$  and is zero on  $\pi_i$  for  $i > n$ . Repeat the process, and let  $P_n M = \operatorname{hocolim}_k C^k(M)$ , where  $C^k(M) = C(C^{k-1}M)$ . The construction can be made functorial by doing the attachments over all maps with domain  $\Sigma^i R$ ,  $i > n$ , and not bothering to choose representative maps from each homotopy class.  $\square$

*Proof of 3.3.* Given an  $R$ -module  $X$  and an integer  $m$ , choose a basis for  $\pi_m X$  over  $\pi_0 R$ , and let  $V_m X$  be a sum of copies of  $\Sigma^m R$ , one for each basis element. There is a map  $V_m X \rightarrow X$  which induces an isomorphism on  $\pi_m$ . Let  $C(M)$  be the cofibre of the map  $\coprod_{m < n} V_m X \rightarrow X$ , and observe that the map  $M \rightarrow C(M)$  induces an isomorphism on  $\pi_i$  for  $i \geq n$ , and is zero on  $\pi_i$  for  $i < n$ . Repeat the process, and let  $Q_n M = \operatorname{hocolim}_k C^k(M)$ .

It is the fact that  $\pi_0 R$  is a field which guarantees that the attachment producing  $C(M)$  can be done without introducing new homotopy in dimensions  $\geq n$ . However, the attachment must be done minimally, and it is this requirement that prevents the construction from being functorial.  $\square$

**3.8. Uniqueness of module structures.** We first aim for the following elementary uniqueness result.



**3.9. Proposition.** *Suppose that  $R$  is connective or that  $R$  is coconnective with  $\pi_0 R$  a field, and that  $M$  and  $N$  are  $R$ -modules with nonvanishing homotopy only in a single dimension  $n$ . Then  $M$  and  $N$  are equivalent as  $R$ -modules if and only if  $\pi_n M$  and  $\pi_n N$  are isomorphic over  $\pi_0 R$ .*

3.10. *Remark.* It follows easily from the proof below that if  $R$  is as in 3.9,  $A$  is a discrete module over  $\pi_0 R$ , and  $n$  is an integer, then there exists an  $R$ -module  $K(A, n)$  with  $\pi_n K(A, n) \cong A$  (over  $\pi_0 R$ ) and  $\pi_i K(A, n) \cong 0$  for  $i \neq n$ . If  $R$  is connective the construction of  $K(A, n)$  can be made functorially in  $A$ , otherwise in general not. If  $A$  and  $B$  are two discrete  $\pi_0 R$ -modules, the natural map

$$\pi_0 \operatorname{Hom}_R(K(A, n), K(B, n)) \rightarrow \operatorname{hom}_{\pi_0 R}(A, B)$$

is an isomorphism if  $R$  is connective but only a surjection in general if  $R$  is coconnective and  $\pi_0 R$  is a field.

3.11. *Remark.* A startling cautionary note is struck by the fact that if  $R$  is coconnective and  $\pi_0 R$  is not a field, the conclusion of 3.9 is *not* necessarily true. We sketch an example of an  $\mathbb{S}$ -algebra  $R$  and two  $R$ -modules  $M$  and  $N$  with homotopy concentrated in degree 0, such that  $\pi_0 M \cong \pi_0 N$  as modules over  $\pi_0 R$ , but  $M$  is not equivalent to  $N$  as an  $R$ -module. Let  $S$  be the ring  $\mathbb{Z}[t]$ ; make  $\mathbb{Z}$  into a discrete  $S$ -module by letting  $t$  act as multiplication by zero, and  $\mathbb{Z}/p^\infty$  into a discrete  $S$ -module by letting  $t$  act by multiplication by  $p$ . Let  $\mathbb{F}_p[t]/t^\infty$  be the discrete  $S$ -module  $\mathbb{F}_p[t, t^{-1}]/\mathbb{F}_p[t]$ . We let  $R = \operatorname{End}_S(\mathbb{Z})$ ,  $M = \operatorname{Hom}_S(\mathbb{Z}, \mathbb{Z}/p^\infty)$ , and  $N = \operatorname{Hom}_S(\mathbb{Z}, \mathbb{F}_p[t, t^{-1}]/\mathbb{F}_p[t])$ . Then  $\pi_* R = \operatorname{Ext}_S^{-*}(\mathbb{Z}, \mathbb{Z})$  is an exterior algebra over  $\mathbb{Z}$  on a generator of degree  $-1$ ;  $\pi_* M = \operatorname{Ext}_S^{-*}(\mathbb{Z}, \mathbb{Z}/p^\infty)$  is a copy of  $\mathbb{Z}/p$  in degree 0, and  $\pi_* N$  is isomorphic to  $\pi_* M$ . Both  $M$  and  $N$  are right  $R$ -modules, but they are not equivalent as right  $R$ -modules, since for instance by [13, 2.1] (cf. 4.9) there are equivalences

$$M \otimes_R \mathbb{Z} \sim \mathbb{Z}/p^\infty \quad \text{and} \quad N \otimes_R \mathbb{Z} \sim \mathbb{F}_p[t]/t^\infty.$$

*Proof of 3.9.* One way to prove this is to construct a suitable spectral sequence converging to  $\pi_* \operatorname{Hom}_R(M, N)$ ; under the connectivity assumptions on  $R$ ,  $\operatorname{hom}_{\pi_0 R}(\pi_n M, \pi_n N)$  will appear in one corner of the  $E_2$ -page and subsequently remain undisturbed for positional reasons. This implies that any map  $\pi_n M \rightarrow \pi_n N$  of  $\pi_0 R$ -modules, in particular any isomorphism, can be realized by an  $R$ -map  $M \rightarrow N$ . We will take a more elementary approach. Assume without loss of generality that  $n = 0$  and suppose that there are isomorphisms  $\pi_0 M \cong \pi_0 N \cong A$  over  $\pi_0 R$ . First we treat the case in which  $R$  is connective. Find a free

presentation

$$\phi_1 \rightarrow \phi_0 \rightarrow A \rightarrow 0$$

of  $A$  over  $\pi_0 R$  and construct a map  $F_1 \rightarrow F_0$  of  $R$ -modules such that each  $F_i$  is a sum of copies of  $R$ , and such that  $\pi_0 F_1 \rightarrow \pi_0 F_0$  is  $\phi_1 \rightarrow \phi_0$ . Let  $C$  be the cofibre of  $F_1 \rightarrow F_0$ . By inspection  $\pi_0 C \cong A$  and there are isomorphisms  $\pi_0 \text{Hom}_R(C, M) \cong \text{hom}_{\pi_0 R}(A, A)$  and  $\pi_0 \text{Hom}_R(C, N) \cong \text{hom}_{\pi_0 R}(A, A)$ . Choose maps  $C \rightarrow M$  and  $C \rightarrow N$  which induce isomorphisms on  $\pi_0$ , and apply the functor  $P_0$  (3.2) to conclude  $M \sim N$ . Now suppose that  $R$  is coconnective, and that  $\pi_0 R$  is a field. Write  $A \cong \bigoplus_{\alpha} \pi_0 R$  over  $\pi_0 R$ , let  $F = \bigoplus_{\alpha} R$ , and construct maps  $F \rightarrow M$  and  $F \rightarrow N$  inducing isomorphisms on  $\pi_0$ . Consider  $Q_0 F$  (3.3). Since  $Q_0 F$  is obtained from  $F$  by attaching copies of  $\Sigma^i R$ ,  $i < 0$ , there are surjections (not necessarily isomorphisms)  $\pi_0 \text{Hom}_R(Q_0 F, M) \twoheadrightarrow \text{hom}_{\pi_0 R}(A, A)$  and  $\pi_0 \text{Hom}_R(Q_0 F, N) \twoheadrightarrow \text{hom}_{\pi_0 R}(A, A)$ . Clearly, then, there are equivalences  $Q_0 F \rightarrow M$  and  $Q_0 F \rightarrow N$ .  $\square$

**3.12. Upward and downward (finite) type.** Suppose that  $M$  is an  $R$ -module. We say that  $M$  is *of upward type* if there is some integer  $n$  such that up to equivalence  $M$  can be built by starting with the zero module and iteratively attaching copies of  $\Sigma^i R$ ,  $i \geq n$ ;  $M$  is *of upward finite type* if the construction can be done in such a way that for any single  $i$  only a finite number of copies of  $\Sigma^i R$  are employed. Similarly,  $M$  is *of downward type* if there is some integer  $n$  such that  $M$  can be built by starting with the zero module and iteratively attaching copies of  $\Sigma^i R$ ,  $i \leq n$ ;  $M$  is *of downward finite type* if the construction can be done in such a way that for any single  $i$  only a finite number of copies of  $\Sigma^i R$  are employed.

We look for conditions under which an  $R$ -module has upward or downward (finite) type.

**3.13. Proposition.** *Suppose that  $R$  is a connective  $\mathbb{S}$ -algebra, and that  $M$  is a module over  $R$  which is bounded below. Then  $M$  is of upward type. If in addition  $\pi_0 R$  is Noetherian and the groups  $\pi_i R$  and  $\pi_i M$  ( $i \in \mathbb{Z}$ ) are individually finitely generated over  $\pi_0 R$ , then  $M$  is of upward finite type.*

*Proof.* Suppose for definiteness that  $\pi_i M = 0$  for  $i < 0$ . We inductively construct maps  $X_n \rightarrow M$  such that  $\pi_i X_n \rightarrow \pi_i M$  is an isomorphism for  $i < n$  and an epimorphism for  $i = n$ , and such that  $X_n$  is constructed from 0 by attaching copies of  $\Sigma^i R$  for  $0 \leq i \leq n$ . Let  $X_{-1} = 0$ . Given  $X_n \rightarrow M$ , construct a surjection

$$\coprod_{\alpha} \pi_0 R \twoheadrightarrow \ker(\pi_n X_n \rightarrow \pi_n M)$$

of modules over  $\pi_0 R$ , realize this surjection by a map  $\coprod_{\alpha} \Sigma^n R \rightarrow X_n$ , and let  $X'_n$  be the cofibre of this map. The map  $X_n \rightarrow M$  extends to a map  $X'_n \rightarrow M$  which is an isomorphism on  $\pi_i$  for  $i \leq n$ . Choose a surjection  $\coprod_{\beta} \pi_0 R \rightarrow \pi_{n+1} M$  of  $\pi_0 R$ -modules, realize this by a map  $Y = \coprod_{\beta} \Sigma^{n+1} R \rightarrow M$ , and let  $X_{n+1} = X'_n \coprod Y$ . The map  $X'_n \rightarrow M$  then extends to a map  $X_{n+1} \rightarrow M$  with the desired properties. A homotopy group calculation gives that  $\text{hocolim}_n X_n \sim M$ , and so it is clear that  $M$  is of upward type over  $R$ . Under the stated finiteness assumptions, it is easy to prove inductively that the homotopy groups of  $X_n$  are finitely generated over  $\pi_0 R$ , and consequently that the above coproducts of suspensions of  $R$  can be chosen to be finite.  $\square$

**3.14. Proposition.** *Suppose that  $R$  is a coconnective  $\mathbb{S}$ -algebra such that  $\pi_0 R$  is a field, and that  $M$  is an  $R$ -module which is bounded above. Then  $M$  is of downward type. If in addition  $\pi_{-1} R = 0$  and the groups  $\pi_i R$  and  $\pi_i M$  ( $i \in \mathbb{Z}$ ) are individually finitely generated over  $\pi_0 R$ , then  $M$  is of downward finite type.*

*Proof.* Given an  $R$ -module  $X$ , let  $V_m X$  be as in the proof of 3.3 and let  $W_m X$  be the cofibre of  $V_m X \rightarrow X$ . Now suppose that  $M$  is non-trivial and bounded above, let  $n$  be the greatest integer such that  $\pi_n M \neq 0$ , and let  $W_n M$  be the cofibre of the map  $V_n M \rightarrow M$ . Iteration gives a sequence of maps  $M \rightarrow W_n M \rightarrow W_n^2 M \rightarrow \dots$ , and we let  $W_n^\infty M = \text{hocolim}_k W_n^k M$ . Then  $\pi_n W_n^\infty M = \text{colim}_k \pi_n W_n^k M = 0$ . Define modules  $U_i$  inductively by  $U_0 = M$ ,  $U_{i+1} = W_{n-i}^\infty U_i$ . There are maps  $U_i \rightarrow U_{i+1}$  and it is clear that  $\text{hocolim} U_i \sim 0$ . Let  $F_i$  be the homotopy fibre of  $M \rightarrow U_i$ . Then  $\text{hocolim} F_i$  is equivalent to  $M$ , and  $F_{i+1}$  is obtained from  $F_i$  by repeatedly attaching copies of  $\Sigma^{n-i-1} R$ . This shows that  $M$  is of downward type. If  $\pi_{-1} R = 0$ , then  $\pi_{n-i} W_{n-i} U_i \cong 0$ , so that  $W_{n-i}^\infty U_i \sim W_{n-i} U_i$ . Under the stated finiteness hypotheses, one sees by an inductive argument that the groups  $\pi_j U_i$ ,  $j \in \mathbb{Z}$ , are finite dimensional over  $k$ , and so  $F_{i+1}$  is obtained from  $F_i$  by attaching a finite number of copies of  $\Sigma^{n-i-1} R$ . This shows that  $M$  is of downward finite type.  $\square$

**3.15. (Finitely) built.** A subcategory of  ${}_R \text{Mod}$  is *thick* if it is closed under equivalences, triangles, and retracts; here closure under triangles means that given any distinguished triangle (i.e. cofibration sequence) with two of its terms in the category, the third is in the category as well. The subcategory is *localizing* if in addition it is closed under arbitrary coproducts (or equivalently, under arbitrary homotopy colimits). If  $A$  and  $B$  are  $R$ -modules, we say that  $A$  is *finitely built from  $B$*  if  $A$  is in the smallest thick subcategory of  ${}_R \text{Mod}$  which contains  $B$ ;  $A$  is *built*

from  $B$  if it is contained in the smallest localizing subcategory of  ${}_R \text{Mod}$  which contains  $B$ .

Given an augmented  $k$ -algebra  $R$ , we look at the question of when an  $R$ -module  $M$  is (finitely) built from  $R$  itself or from  $k$ . We have already touched on related issues. It is in fact not hard to see that *any*  $R$ -module is built from  $R$ ; propositions 3.13 and 3.14 amount to statements that sometimes this building can be done in a controlled way.

**3.16. Proposition.** *Suppose that  $k$  is a field, that  $R$  is an augmented  $k$ -algebra, and that  $M$  is an  $R$ -module. Assume either that  $R$  is connective and the kernel of the augmentation  $\pi_0 R \rightarrow k$  is a nilpotent ideal, or that  $R$  is coconnective and  $\pi_0 R \cong k$ . Then an  $R$ -module  $M$  is finitely built from  $k$  over  $R$  if and only if  $\pi_* M$  is finite dimensional over  $k$ .*

**3.17. Remark.** A similar argument shows that if  $R$  is coconnective and  $\pi_0 R \cong k$ , then any  $R$ -module  $M$  which is bounded below is built from  $k$  over  $R$ . It is only necessary to note that the fibre  $F_n$  of  $M \rightarrow Q_n M$  is built from  $k$  (it has only a finite number of nontrivial homotopy groups) and that  $M \sim \text{hocolim } F_n$ . Along the same lines, if  $R$  is connective and  $\pi_0 R$  is as in 3.16, then any  $R$ -module  $M$  which is bounded above is built from  $k$  over  $R$ .

*Proof of 3.16.* It is clear that if  $M$  is finitely built from  $k$  then  $\pi_* M$  is finite dimensional. Suppose then that  $\pi_* M$  is finite-dimensional, so that in particular  $\pi_i M$  vanishes for all but a finite number of  $i$ . By using the Postnikov constructions  $P_*$  (3.2) or  $Q_*$  (3.3), we can find a finite filtration of  $M$  such that the successive cofibres are of the form  $K(\pi_n M, n)$  (3.10). It is enough to show that if  $A$  is a discrete module over  $\pi_0 R$  which is finite-dimensional over  $k$ , then  $K(A, n)$  is finitely built from  $k$  over  $R$ . But this follows from 3.10 and that fact that under the given assumptions,  $A$  has a finite filtration by  $\pi_0 R$ -submodules such that the successive quotients are isomorphic to  $k$ .  $\square$

We need one final result, in which  $k$  and  $R$  play reciprocal roles. In the following proposition, there is a certain arbitrariness to the choice of which  $\mathbb{S}$ -algebra is named  $\mathcal{E}$  and which is named  $R$ ; we have picked the notation so that the formulation is parallel to 3.14.

**3.18. Proposition.** *Suppose that  $k$  is a field, and that  $\mathcal{E}$  is a connective augmented  $k$ -algebra such that the kernel of the augmentation  $\pi_0 \mathcal{E} \rightarrow k$  is a nilpotent ideal. Let  $M$  be an  $\mathcal{E}$ -module, let  $N = \text{Hom}_{\mathcal{E}}(M, k)$ , and let  $R = \text{End}_{\mathcal{E}}(k)$ . If  $M$  is finitely built from  $k$  over  $\mathcal{E}$  (i.e.  $\pi_* M$  is finite-dimensional over  $k$ ) then  $N$  is finitely built from  $R$  over  $R$ . If  $M$*

is bounded below and each  $\pi_i M$  is finite-dimensional over  $k$ , then  $N$  is of downward finite type over  $R$ .

3.19. *Remark.* This proposition will let us derive the conclusion of 3.14 in some cases in which  $\pi_{-1}R \neq 0$ .

*Proof of 3.18.* Suppose that  $X$  is some  $\mathcal{E}$ -module. It is elementary that if  $M$  is finitely built from  $X$  over  $\mathcal{E}$ , then  $\mathrm{Hom}_{\mathcal{E}}(M, k)$  is finitely built from  $\mathrm{Hom}_{\mathcal{E}}(X, k)$  over  $\mathrm{End}_{\mathcal{E}}(k)$ . Taking  $X = k$  gives the first statement of the proposition.

Suppose then that  $M$  is bounded below, and that each  $\pi_i M$  is finite-dimensional over  $k$ . Let  $M_i$  denote the Postnikov stage  $P_i M$  (3.2), so that  $M$  is equivalent to  $\mathrm{holim}_i M_i$ , and let  $N_i = \mathrm{Hom}_{\mathcal{E}}(M_i, k)$ . We claim that  $N$  is equivalent to  $\mathrm{hocolim}_i N_i$ . This follows from the fact that  $M_n$  is obtained from  $M$  by attaching copies of  $\Sigma^i \mathcal{E}$  for  $i > n$ , and so the natural map  $N_n \rightarrow N$  induces isomorphisms on  $\pi_i$  for  $i \geq -n$ . The triangle  $M_n \rightarrow M_{n-1} \rightarrow K(\pi_n M, n+1)$  (3.10) dualizes to give a triangle

$$\mathrm{Hom}_{\mathcal{E}}(K(\pi_n M, n+1), k) \rightarrow N_{n-1} \rightarrow N_n.$$

Since (as in the proof of 3.16)  $\pi_n M$  has a finite filtration by  $\pi_0 \mathcal{E}$ -submodules in which the successive quotients are isomorphic to  $k$ , it follows that  $N_n$  is obtained from  $N_{n-1}$  by attaching a finite number of copies of  $\mathrm{Hom}_{\mathcal{E}}(\Sigma^{n+1} k, k) \sim \Sigma^{-(n+1)} R$ . Since  $M_i \sim 0$  for  $i \ll 0$ , the proposition follows.  $\square$

#### 4. SMALLNESS

In this section we describe the main setting that we work in. We start with a pair  $(R, k)$ , where  $R$  is an  $\mathbb{S}$ -algebra and  $k$  is an  $R$ -module. Eventually we assume that  $k$  is an  $R$ -module via an  $\mathbb{S}$ -algebra homomorphism  $R \rightarrow k$ .

We begin by discussing cellularity (4.1) and then describing some smallness hypotheses under which cellular approximations are given by a simple formula (4.10). These smallness hypotheses lead to various homotopical formulations of smallness for an  $\mathbb{S}$ -algebra homomorphism  $R \rightarrow k$  (4.14). Let  $\mathcal{E} = \mathrm{End}_R(k)$ . We show that the smallness conditions have a certain symmetry under the interchange  $R \leftrightarrow \mathcal{E}$ , at least if  $R$  is complete in an appropriate sense (4.17), and that the smallness conditions also behave well with respect to “short exact sequences” of  $\mathbb{S}$ -algebras (4.18). Finally, we point out that in some algebraic situations the notion of completeness from 4.17 amounts to ordinary completeness with respect to powers of an ideal, and that in topological situations it amounts to convergence of the Eilenberg-Moore spectral sequence (4.22).

**4.1. Cellular modules.** A map  $U \rightarrow V$  of  $R$ -modules is a  $k$ -equivalence if the induced map  $\mathrm{Hom}_R(k, U) \rightarrow \mathrm{Hom}_R(k, V)$  is an equivalence. An  $R$ -module  $M$  is said to be  $k$ -cellular or  $k$ -torsion ([13, §4], [10]) if any  $k$ -equivalence  $U \rightarrow V$  induces an equivalence  $\mathrm{Hom}_R(M, U) \rightarrow \mathrm{Hom}_R(M, V)$ . A  $k$ -equivalence between  $k$ -cellular objects is necessarily an equivalence.

It is not hard to see that any  $R$ -module which is built from  $k$  in the sense of 3.15 is  $k$ -cellular, and in fact it turns out that an  $R$ -module is  $k$ -cellular if and only if it is built from  $k$  (cf. [27, 5.1.5]). The proof involves using a version of Quillen's small object argument to show that for any  $R$ -module  $M$ , there exists a  $k$ -equivalence  $M' \rightarrow M$  in which  $M'$  is built from  $k$ . If  $M$  is  $k$ -cellular, this  $k$ -equivalence must be an equivalence.

We let  $\mathbf{D}\mathrm{Cell}(R, k)$  denote the full subcategory of the derived category  $\mathbf{D}(R)$  containing the  $k$ -cellular objects. For any  $R$ -module  $X$  there is a  $k$ -cellular object  $\mathrm{Cell}_k(X)$  together with a  $k$ -equivalence  $\mathrm{Cell}_k(X) \rightarrow X$ ; such an object is unique up to a canonical equivalence and is called the  $k$ -cellular approximation to  $X$ . If we want to emphasize the role of  $R$  we write  $\mathrm{Cell}_k^R(X)$ .

**4.2. Remark.** If  $R$  is a commutative ring and  $k = R/I$  for a finitely generated ideal  $I \subset R$ , then an  $R$ -module  $X$  is  $k$ -cellular if and only if each element of  $\pi_* X$  is annihilated by some power of  $I$  [13, 6.12]. The chain complex incarnation (1.5) of  $\mathrm{Cell}_k^R(X)$  is the local cohomology object  $\mathbf{R}\Gamma_I(R)$  [13, 6.11].

We are interested in a particular approach to constructing  $k$ -cellular approximations, and it is convenient to have some terminology to describe it.

**4.3. Definition.** Suppose that  $k$  is an  $R$ -module and that  $\mathcal{E} = \mathrm{End}_R(k)$ . An  $R$ -module  $M$  is said to be *effectively constructible from  $k$*  if the natural map

$$\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \rightarrow M$$

is an equivalence.

Note that  $\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k$  is always  $k$ -cellular over  $R$ , because  $\mathrm{Hom}_R(k, M)$  is  $\mathcal{E}$ -cellular over  $\mathcal{E}$  (3.15). The following lemma is easy to deduce from the fact that the map  $\mathrm{Cell}_k(M) \rightarrow M$  is a  $k$ -equivalence.

**4.4. Lemma.** *In the situation of 4.3, the following conditions are equivalent:*

- (1)  $\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \rightarrow M$  is a  $k$ -equivalence.
- (2)  $\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \rightarrow M$  is a  $k$ -cellular approximation.

(3)  $\text{Cell}_k(M)$  is effectively constructible from  $k$ .

**4.5. Smallness.** In the context above of  $R$  and  $k$ , we will consider three finiteness conditions derived from the notion of “being finitely built” (3.15).

**4.6. Definition.** The  $R$ -module  $k$  is *small* if  $k$  is finitely built from  $R$ , and *cosmall* if  $R$  is finitely built from  $k$ . Finally,  $k$  is *proxy-small* if there exists an  $R$ -module  $K$ , such that  $K$  is finitely built from  $R$ ,  $K$  is finitely built from  $k$ , and  $K$  builds  $k$ . The object  $K$  is then called a *Koszul complex* associated to  $k$  (cf. 5.1).

**4.7. Remark.** The  $R$ -module  $k$  is small if and only if  $\text{Hom}_R(k, -)$  commutes with arbitrary coproducts; if  $R$  is a ring this is equivalent to requiring that  $k$  be a perfect complex, i.e., isomorphic in  $\mathbf{D}(R)$  to a chain complex of finite length whose constituents are finitely generated projective  $R$ -modules.

**4.8. Remark.** The condition in 4.6 that  $k$  and  $K$  can be built from one another implies that an  $R$ -module  $M$  is built from  $k$  if and only if it is built from  $K$ ; in particular,  $\mathbf{D}\text{Cell}(R, k) = \mathbf{D}\text{Cell}(R, K)$ . If  $k$  is either small or cosmall it is also proxy-small; in the former case take  $K = k$  and in the latter  $K = R$ .

In [15] we explore the relevance of the concept of proxy-smallness to commutative rings.

One of the main results of [13] is the following; although in [13] it is phrased for DGAs, the proof for general  $\mathbb{S}$ -algebras is the same. If  $k$  is an  $R$ -module, let  $\mathcal{E} = \text{End}_R(k)$ , let  $E$  be the functor which assigns to an  $R$ -module  $M$  the right  $\mathcal{E}$ -module  $\text{Hom}_R(k, M)$ , and let  $T$  be the functor which assigns to a right  $\mathcal{E}$ -module  $X$  the  $R$ -module  $X \otimes_{\mathcal{E}} k$ .

**4.9. Theorem.** [13, 2.1, 4.3] *If  $k$  is a small  $R$ -module, then the functors  $E$  and  $T$  above induce adjoint categorical equivalences*

$$T : \mathbf{D}(\mathcal{E}^{op}) \leftrightarrow \mathbf{D}\text{Cell}(R, k) : \mathcal{E}.$$

*All  $k$ -cellular  $R$ -modules are effectively constructible from  $k$ .*

There is a partial generalization of this to the proxy-small case.

**4.10. Theorem.** *Suppose that  $k$  is a proxy-small  $R$ -module with Koszul complex  $K$ . Let  $\mathcal{E} = \text{End}_R(k)$ ,  $J = \text{Hom}_R(k, K)$ , and  $\mathcal{E}_K = \text{End}_R(K)$ . Then the three categories*

$$\mathbf{D}\text{Cell}(R, k), \mathbf{D}\text{Cell}(\mathcal{E}^{op}, J), \mathbf{D}(\mathcal{E}_K^{op})$$

*are all equivalent to one another. All  $k$ -cellular  $R$ -modules are effectively constructible from  $k$ .*

4.11. *Remark.* We leave it to the reader to inspect the proof below and write down the functors that induce the various categorical equivalences.

*Proof of 4.10.* We will show that  $J$  is a small  $\mathcal{E}^{\text{op}}$ -module, and that the natural map  $\mathcal{E}_K \rightarrow \text{End}_{\mathcal{E}^{\text{op}}}(J)$  is an equivalence. The first statement is then proved by applying 4.9 serially to the pairs  $(\mathcal{E}_K^{\text{op}}, J)$  and  $(R, K)$  whilst keeping 4.8 in mind. For the smallness, observe that since  $K$  is finitely built from  $k$  as an  $R$ -module,  $J = \text{Hom}_R(k, K)$  is finitely built from  $\mathcal{E} = \text{Hom}_R(k, k)$  as a right  $\mathcal{E}$ -module. Next, consider all  $R$ -modules  $X$  with the property that for any  $R$ -module  $M$  the natural map

$$(4.12) \quad \text{Hom}_R(X, M) \rightarrow \text{Hom}_{\mathcal{E}^{\text{op}}}(\text{Hom}_R(k, X), \text{Hom}_R(k, M))$$

is an equivalence. The class includes  $X = k$  by inspection, and hence by triangle arguments any  $X$  finitely built from  $k$ , in particular  $X = K$ .

For the second statement, suppose that  $M$  is  $k$ -cellular. By 4.8,  $M$  is also  $K$ -cellular and hence (4.9) effectively constructible from  $K$ . In other words, the natural map

$$\text{Hom}_R(K, M) \otimes_{\mathcal{E}_K} K \rightarrow M$$

is an equivalence. We wish to analyze the domain of the map. As above (4.12),  $\text{Hom}_R(K, M)$  is equivalent to  $\text{Hom}_{\mathcal{E}^{\text{op}}}(J, \text{Hom}_R(k, M))$ , which, because  $J$  is small as a right  $\mathcal{E}$ -module, is itself equivalent to  $\text{Hom}_R(k, M) \otimes_{\mathcal{E}} \text{Hom}_{\mathcal{E}^{\text{op}}}(J, \mathcal{E})$ . Since  $\mathcal{E} \sim \text{Hom}_R(k, k)$ , the second factor of the tensor product is (again as with 4.12) equivalent to  $\text{Hom}_R(K, k)$ . We conclude that the natural map

$$\text{Hom}_R(k, M) \otimes_{\mathcal{E}} (\text{Hom}_R(K, k) \otimes_{\mathcal{E}_K} K) \rightarrow M$$

is an equivalence. But the factor  $\text{Hom}_R(K, k) \otimes_{\mathcal{E}_K} K$  is equivalent to  $k$ , since by 4.9 the  $K$ -cellular module  $k$  is effectively constructible from  $K$ . Hence  $M$  is effectively constructible from  $k$ .  $\square$

4.13. **Smallness conditions on an  $\mathbb{S}$ -algebra homomorphism.** Now we identify certain  $\mathbb{S}$ -algebra homomorphisms which are particularly convenient to work with. See 5.1 for the main motivating example.

4.14. **Definition.** An  $\mathbb{S}$ -algebra homomorphism  $R \rightarrow k$  is *small* if  $k$  is small as an  $R$ -module, *cosmall* if  $k$  is cosmall, and *proxy-small* if  $k$  is proxy-small.

4.15. *Remark.* As in 4.8, if  $R \rightarrow k$  is either small or cosmall it is also proxy-small. These are three very different conditions to put on the map  $R \rightarrow k$ , with proxy-smallness being by far the weakest one (see 5.1).



Our notion of smallness is related to the notion of regularity from commutative algebra. For instance, a commutative ring  $R$  is *regular* (in the absolute sense) if and only if every finitely-generated discrete  $R$ -module  $M$  is small, i.e., has a finite length resolution by finitely generated projectives. Suppose that  $f : R \rightarrow k$  is a surjection of commutative Noetherian rings. If  $f$  is regular as a map of rings it is small as a map of  $\mathbb{S}$ -algebras, but the converse does not hold in general. The point is that for  $f$  to be regular in the ring-theoretic sense, certain additional conditions must be satisfied by the fibres of  $R \rightarrow k$ .

4.16. *Relationships between types of smallness.* Suppose that  $k$  is an  $R$ -module and that  $\mathcal{E} = \text{End}_R(k)$ . The *double centralizer* of  $R$  is the ring  $\hat{R} = \text{End}_{\mathcal{E}}(k)$ . Left multiplication gives a ring homomorphism  $R \rightarrow \hat{R}$ , and the pair  $(R, k)$  is said to be *dc-complete* if the homomorphism  $R \rightarrow \hat{R}$  is an equivalence. We show below (4.20) that if  $R \rightarrow k$  is a surjective map of Noetherian commutative rings with kernel  $I \subset R$ , then, as long as  $k$  is a regular ring,  $(R, k)$  is dc-complete if and only if  $R$  is isomorphic to its  $I$ -adic completion.

If  $R$  is an augmented  $k$ -algebra, then  $\mathcal{E} = \text{End}_R(k)$  is also an augmented  $k$ -algebra. The augmentation is provided by the natural map  $\text{End}_R(k) \rightarrow \text{End}_k(k) \sim k$  induced by the  $k$ -algebra structure homomorphism  $k \rightarrow R$ .

4.17. **Proposition.** *Suppose that  $R$  is an augmented  $k$ -algebra, and let  $\mathcal{E} = \text{End}_R(k)$ . Assume that the pair  $(R, k)$  is dc-complete. Then  $R \rightarrow k$  is small if and only if  $\mathcal{E} \rightarrow k$  is cosmall. Similarly,  $R \rightarrow k$  is proxy-small if and only if  $\mathcal{E} \rightarrow k$  is proxy-small.*

*Proof.* If  $k$  is finitely built from  $R$  as an  $R$ -module, then by applying  $\text{Hom}_R(-, k)$  to the construction process, we see that  $\mathcal{E} = \text{Hom}_R(k, k)$  is finitely built from  $k = \text{Hom}_R(R, k)$  as an  $\mathcal{E}$ -module. Conversely, if  $\mathcal{E}$  is finitely built from  $k$  as an  $\mathcal{E}$ -module, it follows that  $k = \text{Hom}_{\mathcal{E}}(\mathcal{E}, k)$  is finitely built from  $\hat{R} \sim \text{Hom}_{\mathcal{E}}(k, k)$  as an  $R$ -module. If  $R \sim \hat{R}$ , this implies that  $k$  is finitely built from  $R$ .

For the rest, it is enough by symmetry to show that if  $R \rightarrow k$  is proxy-small, then so is  $\mathcal{E} \rightarrow k$ . Suppose then that  $k$  is proxy-small over  $R$  with Koszul complex  $K$ . Let  $L = \text{Hom}_R(K, k)$ . Arguments as above show that  $L$  is finitely built both from  $\text{Hom}_R(R, k) \sim k$  and from  $\text{Hom}_R(k, k) \sim \mathcal{E}$  as an  $\mathcal{E}$ -module. This means that  $L$  will serve as a Koszul complex for  $k$  over  $\mathcal{E}$ , as long as  $L$  builds  $k$  over  $\mathcal{E}$ . Let  $\mathcal{E}_K = \text{End}_R(K)$ . By 4.10, the natural map  $L \otimes_{\mathcal{E}_K} K \rightarrow k$  is an equivalence; it is evidently a map of  $\mathcal{E}$ -modules. Since  $\mathcal{E}_K$  builds  $K$  over  $\mathcal{E}_K$ ,  $L \sim L \otimes_{\mathcal{E}_K} \mathcal{E}_K$  builds  $k$  over  $\mathcal{E}$ .  $\square$

In the following proposition, we think of  $S \rightarrow R \rightarrow Q$  as a “short exact sequence” of commutative  $\mathbb{S}$ -algebras; often, such a sequence is obtained by applying a cochain functor to a fibration sequence of spaces (cf. 5.7 or 10.3).

**4.18. Proposition.** *Suppose that  $S \rightarrow R$  and  $R \rightarrow k$  are homomorphisms of commutative  $\mathbb{S}$ -algebras, and let  $Q = R \otimes_S k$ . Note that  $Q$  is a commutative  $\mathbb{S}$ -algebra and that there is a natural homomorphism  $Q \rightarrow k$  which extends  $R \rightarrow k$ . Assume that one of the following holds:*

- (1)  $S \rightarrow k$  is proxy-small and  $Q \rightarrow k$  is cosmall, or
- (2)  $S \rightarrow k$  is small and  $Q \rightarrow k$  is proxy-small.

*Then  $R \rightarrow k$  is proxy-small.*

*Proof.* Note that there is an  $\mathbb{S}$ -algebra homomorphism  $R \rightarrow Q$ . In case (1), suppose that  $K$  is a Koszul complex for  $k$  over  $S$ . We will show that  $R \otimes_S K$  is a Koszul complex for  $k$  over  $R$ . Since  $K$  is small over  $S$ ,  $R \otimes_S K$  is small over  $R$ . Since  $k$  finitely builds  $K$  over  $S$ ,  $R \otimes_S k = Q$  finitely builds  $R \otimes_S K$  over  $R$ . But  $k$  finitely builds  $Q$  over  $Q$ , and hence over  $R$ ; it follows that  $k$  finitely builds  $R \otimes_S K$  over  $R$ . Finally,  $K$  builds  $k$  over  $S$ , and so  $R \otimes_S K$  builds  $Q$  over  $R$ ; however,  $Q$  clearly builds  $k$  as a  $Q$ -module, and so *a fortiori* builds  $k$  over  $R$ .

In case (2), let  $K$  be a Koszul complex for  $k$  over  $Q$ . We will show that  $K$  is also a Koszul complex for  $k$  over  $R$ . Note that  $S \rightarrow k$  is small, so that  $k$  is small over  $S$  and hence  $Q = R \otimes_S k$  is small over  $R$ . But  $K$  is finitely built from  $Q$  over  $Q$  and hence over  $R$ ; it follows that  $K$  is small over  $R$ . Since  $k$  finitely builds  $K$  over  $Q$ , it does so over  $R$ ; for a similar reason  $K$  builds  $k$  over  $R$ .  $\square$

**4.19. DC-completeness in algebra.** Recall that a ring  $k$  is said to be *regular* if every finitely generated discrete module over  $k$  has a finite projective resolution, i.e., if every finitely generated discrete module over  $k$  is small over  $k$ .

**4.20. Proposition.** *Suppose that  $R \rightarrow k$  is a surjection of commutative Noetherian rings with kernel ideal  $I \subset R$ . Assume that  $k$  is a regular ring. Then the double centralizer map  $R \rightarrow \hat{R}$  is can be identified with the  $I$ -adic completion map  $R \rightarrow \lim_s R/I^s$ . In particular,  $(R, k)$  is dc-complete (4.16) if and only if  $R$  is  $I$ -adically complete.*

*Proof.* Let  $\mathcal{E} = \text{End}_R(k)$ , and  $\hat{R} = \text{End}_{\mathcal{E}}(k)$ , so that there is a natural homomorphism  $R \rightarrow \hat{R}$  which is an equivalence if and only if  $(R, k)$  is dc-complete. We will show that  $\hat{R}$  is equivalent to  $R_{\hat{I}} = \lim_s R/I^s$ , so that  $(R, k)$  is dc-complete if and only if  $R \rightarrow R_{\hat{I}}$  is an isomorphism.

Consider the class of all  $R$ -modules  $X$  with the property that the natural map

$$(4.21) \quad X \rightarrow \mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(X, k), k)$$

is an equivalence. The class includes  $k$ , and hence all  $R$ -modules finitely built from  $k$ . Each quotient  $I^s/I^{s+1}$  is finitely generated over  $k$ , hence small over  $k$ , and hence finitely built from  $k$  over  $R$ . It follows from an inductive argument that the modules  $R/I^s$  are finitely built from  $k$  over  $R$ , and consequently that 4.21 is an equivalence for  $X = R/I^s$ . By a theorem of Grothendieck [26, 2.8], there are isomorphisms

$$\mathrm{colim}_s \mathrm{Ext}_R^i(R/I^s, k) \cong \begin{cases} k & i = 0 \\ 0 & i > 0 \end{cases}$$

which (1.5) assemble into an equivalence

$$\mathrm{hocolim}_s \mathrm{Hom}_R(R/I^s, k) \sim \mathrm{Hom}_R(R, k) \sim k.$$

This allows for the calculation

$$\begin{aligned} \hat{R} &\sim \mathrm{Hom}_{\mathcal{E}}(k, k) \sim \mathrm{Hom}_{\mathcal{E}}(\mathrm{hocolim}_s \mathrm{Hom}_R(R/I^s, k), k) \\ &\sim \mathrm{holim}_s \mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(R/I^s, k), k) \\ &\sim \mathrm{holim}_s R/I^s \sim R_{\hat{I}}. \end{aligned}$$

It is easy to check that under this chain of equivalences the map  $R \rightarrow \hat{R}$  corresponds to the completion map  $R \rightarrow R_{\hat{I}}$ .  $\square$

**4.22. DC-completeness in topology.** Suppose that  $X$  is a connected pointed topological space, and that  $k$  is a commutative  $\mathbb{S}$ -algebra. For any space  $Y$ , pointed or not, let  $Y_+$  denote  $Y$  with a disjoint basepoint added, and  $\Sigma^\infty Y_+$  the associated suspension spectrum. We will consider two  $k$ -algebras associated to the pair  $(X, k)$ : the chain algebra  $R = C_*(\Omega X; k) = k \otimes_{\mathbb{S}} \Sigma^\infty(\Omega X)_+$  and the cochain algebra  $S = C^*(X; k) = \mathrm{Map}_{\mathbb{S}}(\Sigma^\infty X_+, k)$ . Here  $\Omega X$  is the loop space on  $X$ , and  $R$  is an  $\mathbb{S}$ -algebra because  $\Omega X$  can be constructed as a topological or simplicial group;  $R$  is essentially the group ring  $k[\Omega X]$ . The multiplication on  $S$  is cup product coming from the diagonal map on  $X$ , and so  $S$  is a commutative  $k$ -algebra. Both of these objects are augmented, one by the map  $R \rightarrow k$  induced by the map  $\Omega X \rightarrow \mathrm{pt}$ , the other by the map  $S \rightarrow k$  induced by the basepoint inclusion  $\mathrm{pt} \rightarrow X$ . If  $k$  is a ring, then  $\pi_i R \cong H_i(\Omega X; k)$  and  $\pi_i S \cong H^{-i}(X; k)$ .

The Rothenberg-Steenrod construction [41] shows that for any  $X$  and  $k$  there is an equivalence  $S \sim \mathrm{End}_R(k)$ . We will say that the pair  $(X, k)$  is of *Eilenberg-Moore type* if  $k$  is a field, each homology group  $H_i(X; k)$  is finite dimensional over  $k$ , and either

- (1)  $X$  is simply connected, or
- (2)  $k$  is of characteristic  $p$  and  $\pi_1 X$  is a finite  $p$ -group.

If  $(X, k)$  is of Eilenberg-Moore type, then by the Eilenberg-Moore spectral sequence construction ([18], [11], [36, Appendix C]),  $R \sim \text{End}_S(k)$  and both of the pairs  $(R, k)$  and  $(S, k)$  are dc-complete (4.16).

Keep in mind that if  $(X, k)$  is of Eilenberg-Moore type, then the augmentation ideal of  $\pi_0 C_*(\Omega X; k) = k[\pi_1 X]$  is nilpotent (cf. 3.18).

## 5. EXAMPLES OF SMALLNESS

In this section we look at some sample cases in which the smallness conditions of §4 are or are not satisfied.

**5.1. Commutative rings.** If  $R$  is a commutative Noetherian ring and  $I \subset R$  is an ideal such that the quotient  $R/I = k$  is a regular ring (4.15), then  $R \rightarrow k$  is proxy-small [13, §6]; the complex  $K$  can be chosen to be the Koszul complex associated to any finite set of generators for  $I$ . The construction of the Koszul complex is sketched below in the proof of 9.3. The pair  $(R, k)$  is dc-complete if and only if  $R$  is complete and Hausdorff with respect to the  $I$ -adic topology (4.20).

For example, if  $R$  is a Noetherian local ring with residue field  $k$ , then the map  $R \rightarrow k$  is proxy-small; this map is small if and only if  $R$  is a regular ring (Serre's Theorem) and cosmall if and only if  $R$  is artinian.

**5.2. The sphere spectrum.** Consider the map  $\mathbb{S} \rightarrow \mathbb{F}_p$  of commutative  $\mathbb{S}$ -algebras; here as usual  $\mathbb{S}$  is the sphere spectrum and the ring  $\mathbb{F}_p$  is identified with the associated Eilenberg-MacLane spectrum. This map is *not* proxy-small. A Koszul complex  $K$  for  $\mathbb{S} \rightarrow \mathbb{F}_p$  would be a stable finite complex with nontrivial mod  $p$  homology (because  $K$  would build  $\mathbb{F}_p$ ), and only a finite number of non-trivial homotopy groups, each one a finite  $p$ -group (because  $\mathbb{F}_p$  would finitely build  $K$ ). We leave it to the reader to show that no such  $K$  exists, for instance because of Lin's theorem [35] that  $\text{Map}_{\mathbb{S}}(\mathbb{F}_p, \mathbb{S}) \sim 0$ .

Let  $\mathbb{S}_p$  denote the  $p$ -completion of the sphere spectrum. The map  $\mathbb{S} \rightarrow \mathbb{F}_p$  is not dc-complete, but  $\mathbb{S}_p \rightarrow \mathbb{F}_p$  is; this can be interpreted in terms of the convergence of the classical mod  $p$  Adams spectral sequence.

The next two examples refer to the following proposition.

**5.3. Proposition.** *Suppose that  $X$  is a pointed connected finite complex, that  $k$  is a commutative  $\mathbb{S}$ -algebra, that  $R$  is the augmented  $k$ -algebra  $C_*(\Omega X; k)$ , and that  $S$  is the augmented  $k$ -algebra  $C^*(X; k) = \text{End}_R(k)$ . Then  $k$  is small as an  $R$ -module and cosmall as an  $S$ -module.*

*Proof.* Let  $E$  be the total space of the universal principal bundle over  $X$  with fibre  $\Omega X$ , so that  $E$  is contractible and  $M = C_*(E; k) \sim k$ . The action of  $\Omega X$  on  $E$  induces an action of  $R$  on  $M$  which amounts to the augmentation action of  $R$  on  $k$ . Let  $E_i$  be the inverse image in  $E$  of the  $i$ -skeleton of  $X$ , and let  $M_i$  be the  $R$ -module  $C_*(E_i; k)$ . Then  $M_i/M_{i-1}$  is equivalent to a finite sum  $\bigoplus \Sigma^i R$  indexed by the  $i$ -cells of  $X$ . Since  $k \sim M = M_n$  (where  $n$  is the dimension of  $X$ ), it follows that  $k$  is small as an  $R$ -module. The last statement is immediate, as in the proof of 3.18.  $\square$

5.4. *Remark.* The argument above also shows that if  $X$  is merely of finite type (i.e., has a finite number of cells in each dimension), then the augmentation module  $k$  is of upward finite type over  $R = C_*(\Omega X; k)$ .

5.5. **Cochains.** Suppose that  $X$  is a pointed connected topological space, that  $k$  is a commutative  $\mathbb{S}$ -algebra, and that  $R$  is the augmented  $k$ -algebra  $C^*(X; k)$ .

- (1) The map  $R \rightarrow k$  is cosmall if  $X$  is a finite complex (5.3).
- (2) If  $k$  is a field, then  $R \rightarrow k$  is cosmall if and only if  $H^*(X; k)$  is finite-dimensional (3.16).
- (3) If  $(X, k)$  is of Eilenberg-Moore type, then  $R \rightarrow k$  is small if and only if  $H_*(\Omega X; k)$  is finite-dimensional (3.16, 4.17).

5.6. **Chains.** Suppose that  $X$  is a pointed connected topological space, that  $k$  is a commutative  $\mathbb{S}$ -algebra, and that  $R$  is the augmented  $k$ -algebra  $C_*(\Omega X; k)$ .

- (1) The map  $R \rightarrow k$  is small if  $X$  is a finite complex (5.3).
- (2) If  $(X, k)$  is of Eilenberg-Moore type, then  $R \rightarrow k$  is small if and only if  $H^*(X; k)$  is finite-dimensional (4.17, 5.5).
- (3) If  $(X, k)$  is of Eilenberg-Moore type, then  $R \rightarrow k$  is cosmall if and only if  $H_*(\Omega X; k)$  is finite-dimensional (3.16).

The parallels between 5.5 and 5.6 are explained by 4.17.

5.7. **Completed classifying spaces.** Suppose that  $G$  is a compact Lie group (e.g., a finite group), that  $k = \mathbb{F}_p$ , and that  $X$  is the  $p$ -completion of the classifying space  $BG$  in the sense of Bousfield-Kan [8]. Let

$$R = C^*(X; k) \text{ and } \mathcal{E} = C_*(\Omega X; k).$$

We will show in the following paragraph that  $R \rightarrow k$  and  $\mathcal{E} \rightarrow k$  are both proxy-small, and that the pair  $(X, k)$  is of Eilenberg-Moore type. There are many  $G$  for which neither  $H_*(\Omega X; k)$  nor  $H^*(X; k)$  is finite dimensional [34]; by 5.5 and 5.6, in such cases the maps  $R \rightarrow k$

and  $\mathcal{E} \rightarrow k$  are neither small nor cosmall. We are interested in these examples for the sake of local cohomology theorems (10.3).

By elementary representation theory there is a faithful embedding  $\rho : G \rightarrow SU(n)$  for some  $n$ , where  $SU(n)$  is the special unitary group of  $n \times n$  Hermitian matrices of determinant one. Consider the associated fibration sequence

$$(5.8) \quad M = SU(n)/G \rightarrow BG \rightarrow BSU(n).$$

The fibre  $M$  is a finite complex. Recall that  $R = C^*(BG; k)$ ; write  $S = C^*(BSU(n); k)$  and  $Q = C^*(M; k)$ . Since  $BSU(n)$  is simply-connected, the Eilenberg-Moore spectral sequence of 5.8 converges and  $Q \sim k \otimes_S R$  (cf. [36, 5.2]). The map  $S \rightarrow k$  is small by 5.5(3) and  $Q \rightarrow k$  is cosmall by 5.5(2); it follows from 4.18 that  $R \rightarrow k$  is proxy-small. Since  $\pi_1 BG = \pi_0 G$  is finite,  $BG$  is  $\mathbb{F}_p$ -good (i.e.,  $C^*(X; k) \sim C^*(BG; k)$ ), and  $\pi_1 X$  is a finite  $p$ -group [8, VII.5]. In particular,  $(X, k)$  is of Eilenberg-Moore type. Since  $\mathcal{E} = C_*(\Omega X; k)$  is thus equivalent to  $\text{End}_R(k)$ , we conclude from 4.17 that  $\mathcal{E} \rightarrow k$  is also proxy-small.

**5.9. Group rings.** If  $G$  is a finite group and  $k$  is a commutative ring, then the augmentation map  $k[G] \rightarrow k$  is proxy-small. We will prove this by producing a Koszul complex  $K$  for  $\mathbb{Z}$  over  $\mathbb{Z}[G]$ ; it is then easy to argue that  $k \otimes_{\mathbb{Z}} K$  is a Koszul complex for  $k$  over  $k[G]$ . Embed  $G$  as above into a unitary group  $SU(n)$  and let  $K = C_*(SU(n); \mathbb{Z})$ . The space  $SU(n)$  with the induced left  $G$ -action is a compact manifold on which  $G$  acts smoothly and freely, and so by transformation group theory [30] can be constructed from a finite number of  $G$ -cells of the form  $(G \times D^i, G \times S^{i-1})$ . This implies that  $K$  is small over  $\mathbb{Z}[G]$ , since, up to equivalence over  $\mathbb{Z}[G]$ ,  $K$  can be identified with the  $G$ -cellular chains on  $SU(n)$ . Note that  $G$  acts trivially on  $\pi_* K = H_*(SU(n); \mathbb{Z})$  (because  $SU(n)$  is connected) and that, since  $H_*(SU(n); \mathbb{Z})$  is torsion free, each group  $\pi_i K$  is isomorphic over  $G$  to a finite direct sum of copies of the augmentation module  $\mathbb{Z}$ . The Postnikov argument in the proof of 3.16 thus shows that  $K$  is finitely built from  $\mathbb{Z}$  over  $\mathbb{Z}[G]$ . Finally,  $K$  itself is an  $\mathbb{S}$ -algebra, the action of  $\mathbb{Z}[G]$  on  $K$  is induced by a homomorphism  $\mathbb{Z}[G] \rightarrow K$ , and the augmentation  $\mathbb{Z}[G] \rightarrow \mathbb{Z}$  extends to an augmentation  $K \rightarrow \mathbb{Z}$ . Since  $K$  builds  $\mathbb{Z}$  over  $K$  (see 3.15),  $K$  certainly builds  $\mathbb{Z}$  over  $\mathbb{Z}[G]$ .

## 6. MATLIS LIFTS

Suppose that  $R$  is a commutative Noetherian local ring, and that  $R \rightarrow k$  is reduction modulo the maximal ideal. Let  $\mathcal{I}(k)$  be the injective hull of  $k$  as an  $R$ -module. The starting point of this section is the

isomorphism

$$(6.1) \quad \mathrm{Hom}_k(X, k) \sim \mathrm{Hom}_R(X, \mathcal{I}(k)),$$

which holds for any  $k$ -module  $X$ . We think of  $\mathcal{I}(k)$  as a lift of  $k$  to an  $R$ -module, not the obvious lift obtained by using the homomorphism  $R \rightarrow k$ , but a more mysterious construction that allows for 6.1. If we define the *Pontriagin dual* of an  $R$ -module  $M$  to be  $\mathrm{Hom}_R(M, \mathcal{I}(k))$ , then Pontriagin duality is a correspondingly mysterious construction for  $R$ -modules which extends ordinary  $k$ -duality for  $k$ -modules.

We generalize this in the following way.

**6.2. Definition.** Suppose that  $R \rightarrow k$  is a map of  $\mathbb{S}$ -algebras, and that  $N$  is a  $k$ -module. An  $R$ -module  $\mathcal{I}$  is said to be a *Matlis lift* of  $N$  if the following two conditions hold:

- (1)  $\mathrm{Hom}_R(k, \mathcal{I})$  is equivalent to  $N$  as a left  $k$ -module, and
- (2)  $\mathcal{I}$  is effectively constructible from  $k$ .

**6.3. Remark.** If  $\mathcal{I}$  is a Matlis lift of  $N$  and  $X$  is an arbitrary  $k$ -module, then the adjunction equivalence

$$\mathrm{Hom}_R(X, \mathcal{I}) \sim \mathrm{Hom}_k(X, \mathrm{Hom}_R(k, \mathcal{I}))$$

implies, by 6.2(1), that there is an equivalence

$$(6.4) \quad \mathrm{Hom}_k(X, N) \sim \mathrm{Hom}_R(X, \mathcal{I}).$$

This is the crucial property of a Matlis lift (cf. 6.1). Condition 6.2(2) tightens things up a bit. There's no real reason not to assume that  $\mathcal{I}$  is  $k$ -cellular, since if  $\mathcal{I}$  satisfies 6.2(1), so does  $\mathrm{Cell}_k \mathcal{I}$ . The somewhat stronger assumption that  $\mathcal{I}$  is effectively constructible from  $k$  will allow Matlis lifts to be constructed and enumerated (6.9). In many situations, the assumption that  $\mathcal{I}$  is  $k$ -cellular implies 6.2(2).

**6.5. Remark.** If  $R$  is a commutative Noetherian local ring, and  $R \rightarrow k$  is reduction modulo the maximal ideal, then the injective hull  $\mathcal{I}(k)$  is a Matlis lift of  $k$  (7.1).

For the rest of the section we assume that  $R \rightarrow k$  is a map of  $\mathbb{S}$ -algebras, and that  $N$  is a  $k$ -module. Let  $\mathcal{E} = \mathrm{End}_R(k)$ . Observe that the right multiplication action of  $k$  on itself gives a homomorphism  $k^{\mathrm{op}} \rightarrow \mathcal{E}$ , or equivalently  $k \rightarrow \mathcal{E}^{\mathrm{op}}$ , so it makes sense to look at right  $\mathcal{E}$ -actions on  $N$  which extend the left  $k$ -action.

**6.6. Definition.** An  $\mathcal{E}$ -lift of  $N$  is a right  $\mathcal{E}$ -module structure on  $N$  which extends the left  $k$ -action. An  $\mathcal{E}$ -lift of  $N$  is said to be of *Matlis type* if the natural map

$$(6.7) \quad N \sim N \otimes_{\mathcal{E}} \mathrm{Hom}_R(k, k) \rightarrow \mathrm{Hom}_R(k, N \otimes_{\mathcal{E}} k)$$

is an equivalence. (Here the action of  $R$  on  $N \otimes_{\mathcal{E}} k$  is obtained from the left action of  $R$  on  $k$ .)

6.8. *Remark.* More generally, an arbitrary right  $\mathcal{E}$ -module  $N$  is said to be of *Matlis type* if the map 6.7 is an equivalence.

The following proposition gives a classification of Matlis lifts.

6.9. **Proposition.** *The correspondences*

$$\mathcal{I} \mapsto \mathrm{Hom}_R(k, \mathcal{I}) \quad N \mapsto N \otimes_{\mathcal{E}} k$$

*give inverse bijections, up to equivalence, between Matlis lifts  $\mathcal{I}$  of  $N$  and  $\mathcal{E}$ -lifts of  $N$  which are of Matlis type.*

*Proof.* If  $\mathcal{I}$  is a Matlis lift of  $N$ , then  $\mathrm{Hom}_R(k, \mathcal{I})$  is equivalent to  $N$  as a  $k$ -module (6.2), and so the natural right action of  $\mathcal{E}$  on  $\mathrm{Hom}_R(k, \mathcal{I})$  provides an  $\mathcal{E}$ -lift of  $N$ . Equivalence 6.7 holds because  $\mathcal{I}$  is effectively constructible from  $k$ ; consequently, this  $\mathcal{E}$ -lift is of Matlis type.

Conversely, given an  $\mathcal{E}$ -lift of  $N$  which is of Matlis type, let  $\mathcal{I} = N \otimes_{\mathcal{E}} k$ . Equivalence 6.7 guarantees that  $\mathcal{I}$  satisfies 6.2(1), and the same formula leads to the conclusion that  $\mathcal{I}$  is effectively constructible from  $k$ .  $\square$

The following observation is useful for recognizing Matlis lifts.

6.10. **Proposition.** *Suppose that  $R \rightarrow k$  is a map of  $\mathbb{S}$ -algebras, that  $\mathcal{E} = \mathrm{End}_R(k)$ , and that  $M$  is an  $R$ -module. Then the right  $\mathcal{E}$ -module  $\mathrm{Hom}_R(k, M)$  is of Matlis type if and only if the evaluation map*

$$\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \rightarrow M$$

*is a  $k$ -cellular approximation, i.e., if and only if  $\mathrm{Cell}_k M$  is effectively constructible from  $k$ .*

*Proof.* Let  $N = \mathrm{Hom}_R(k, M)$ . Since  $N$  is  $\mathcal{E}^{\mathrm{op}}$ -cellular over  $\mathcal{E}^{\mathrm{op}}$ ,  $N \otimes_{\mathcal{E}} k$  is  $k$ -cellular over  $R$ . This implies that the evaluation map  $\epsilon$  is a  $k$ -cellular approximation if and only if it is a  $k$ -equivalence. Consider the maps

$$N \otimes_{\mathcal{E}} \mathrm{Hom}_R(k, k) \rightarrow \mathrm{Hom}_R(k, N \otimes_{\mathcal{E}} k) \xrightarrow{\mathrm{Hom}_R(k, \epsilon)} N.$$

It is easy to check that the composite is the obvious equivalence, so the left hand map is an equivalence ( $N$  is of Matlis type) if and only if the right-hand map is an equivalence ( $\epsilon$  is a  $k$ -equivalence). The final statement is from 4.4.  $\square$

6.11. **Matlis duality.** In the situation of 6.6, let  $N = k$  and let  $\mathcal{I} = k \otimes_{\mathcal{E}} k$  be a Matlis lift of  $k$ . The *Pontriagin dual* or *Matlis dual* of



an  $R$ -module  $M$  (with respect to  $\mathcal{I}$ ) is defined to be  $\mathrm{Hom}_R(M, \mathcal{I})$ . By 6.3, Matlis duality is a construction for  $R$ -modules which extends ordinary  $k$ -duality for  $k$ -modules. Note, however, that in the absence of additional structure (e.g., commutativity of  $R$ ) it is not clear that  $\mathrm{Hom}_R(M, \mathcal{I})$  is a right  $R$ -module. We will come up with one way to remedy this later on (8.3).

**6.12. Existence of Matlis lifts.** We give four conditions under which a right  $\mathcal{E}$ -module is of Matlis type (6.8), and so gives rise to a Matlis lift of the underlying  $k$ -module. The first two conditions are of an algebraic nature; the second two may seem technical, but they apply to many ring spectra, chain algebras, and cochain algebras. In all of the statements below,  $R \rightarrow k$  is a map of  $\mathbb{S}$ -algebras,  $\mathcal{E} = \mathrm{End}_R(k)$ , and  $N$  is a right  $\mathcal{E}$ -module.

**6.13. Proposition.** *If  $R \rightarrow k$  is small, then any  $N$  is of Matlis type.*

*Proof.* Calculate

$$\mathrm{Hom}_R(k, N \otimes_{\mathcal{E}} k) \sim N \otimes_{\mathcal{E}} \mathrm{Hom}_R(k, k) \sim N \otimes_{\mathcal{E}} \mathcal{E} \sim N$$

where the first weak equivalence comes from the fact that  $k$  is small as an  $R$ -module.  $\square$

**6.14. Proposition.** *If  $R \rightarrow k$  is proxy-small, then  $N$  is of Matlis type if and only if there exists an  $R$ -module  $M$  such that  $N$  is equivalent to  $\mathrm{Hom}_R(k, M)$  as a right  $\mathcal{E}$ -module.*

*Proof.* If  $N$  is of Matlis type, then  $M = N \otimes_{\mathcal{E}} k$  will do. Given  $M$ , the fact that  $\mathrm{Hom}_R(k, M)$  is of Matlis type follows from 6.10 and 4.10 (cf. 4.4).  $\square$

**6.15. Proposition.** *Suppose that  $k$  and  $N$  are bounded above, that  $k$  is of upward finite type as an  $R$ -module, and that  $N$  is of downward type as an  $\mathcal{E}^{\mathrm{op}}$ -module. Then  $N$  is of Matlis type.*

**6.16. Proposition.** *Suppose that  $k$  and  $N$  are bounded below, that  $k$  is of downward finite type as an  $R$ -module and that  $N$  is of upward type as an  $\mathcal{E}^{\mathrm{op}}$ -module. Then  $N$  is of Matlis type.*

*Proof of 6.15.* Note first that since  $k$  and  $N$  are bounded above and  $N$  is of downward type as an  $\mathcal{E}^{\mathrm{op}}$ -module,  $N \otimes_{\mathcal{E}} k$  is also bounded above. Consider the class of all  $R$ -modules  $X$  such that the natural map

$$N \otimes_{\mathcal{E}} \mathrm{Hom}_R(X, k) \rightarrow \mathrm{Hom}_R(X, N \otimes_{\mathcal{E}} k)$$

is an equivalence. This certainly includes  $R$ , and so by triangle arguments includes everything that can be finitely built from  $R$ . We must

show that the class contains  $k$ . Pick an integer  $B$ , and suppose that  $A$  is another integer. Since  $k$  is of upward finite type as an  $R$ -module and both  $k$  and  $N \otimes_{\mathcal{E}} k$  are bounded above, there exists an  $R$ -module  $X$ , finitely built from  $R$ , and a map  $X \rightarrow k$  which induces isomorphisms

$$(6.17) \quad \begin{aligned} \pi_i \operatorname{Hom}_R(k, k) &\xrightarrow{\cong} \pi_i \operatorname{Hom}_R(X, k) \\ \pi_i \operatorname{Hom}_R(k, N \otimes_{\mathcal{E}} k) &\xrightarrow{\cong} \pi_i \operatorname{Hom}_R(X, N \otimes_{\mathcal{E}} k) \end{aligned} \quad i > A.$$

Now  $N$  is of downward type as a right  $\mathcal{E}$ -module, so if we choose  $A$  small enough we can guarantee that the map

$$\pi_i(N \otimes_{\mathcal{E}} \operatorname{Hom}_R(k, k)) \rightarrow \pi_i(N \otimes_{\mathcal{E}} \operatorname{Hom}_R(X, k))$$

is an isomorphism for  $i > B$ . By reducing  $A$  if necessary (which of course affects the choice of  $X$ ), we can assume  $A \leq B$ . Now consider the commutative diagram

$$(6.18) \quad \begin{array}{ccc} N \otimes_{\mathcal{E}} \operatorname{Hom}_R(k, k) & \longrightarrow & \operatorname{Hom}_R(k, N \otimes_{\mathcal{E}} k) \\ \downarrow & & \downarrow \\ N \otimes_{\mathcal{E}} \operatorname{Hom}_R(X, k) & \longrightarrow & \operatorname{Hom}_R(X, N \otimes_{\mathcal{E}} k) \end{array}$$

The lower arrow is an equivalence, because  $X$  is finitely built from  $R$ , and the vertical arrows are isomorphisms on  $\pi_i$  for  $i > B$ . Since  $B$  is arbitrary, it follows that the upper arrow is an equivalence.  $\square$

*Proof of 6.16.* This is very similar to the proof above, but with the inequalities reversed. Observe that since  $k$  and  $N$  are bounded below, and  $N$  is of upward type as an  $\mathcal{E}$ -module,  $N \otimes_{\mathcal{E}} k$  is also bounded below. Pick an integer  $B$ , and let  $A$  be another integer. Since  $k$  is of downward finite type as an  $R$ -module and both  $k$  and  $N \otimes_{\mathcal{E}} k$  are bounded below, there exists an  $X$  finitely built from  $R$  such that the maps in 6.17 are isomorphisms for  $i < A$ . Now  $N$  is of upward type as a right  $\mathcal{E}$ -module, so if we choose  $A$  large enough we can guarantee that the map

$$\pi_i(N \otimes_{\mathcal{E}} \operatorname{Hom}_R(k, k)) \rightarrow \pi_i(N \otimes_{\mathcal{E}} \operatorname{Hom}_R(X, k))$$

is an isomorphism for  $i < B$ . By making  $A$  larger if necessary, we can assume  $A > B$ . The proof is now completed by using the commutative diagram 6.18.  $\square$

## 7. EXAMPLES OF MATLIS LIFTING

In this section we look at particular examples of Matlis lifting (§6). In each case we start with a morphism  $R \rightarrow k$  of rings, and look for Matlis lifts of  $k$ . As usual,  $\mathcal{E}$  denotes  $\operatorname{End}_R(k)$ .

**7.1. Local rings.** Suppose that  $R$  is a commutative Noetherian local ring with maximal ideal  $I$  and residue field  $R/I = k$ , and that  $R \rightarrow k$  is the quotient map. Let  $\mathcal{I} = \mathcal{I}(k)$  be the injective hull of  $k$  (as an  $R$ -module). We will show that  $\mathcal{I}$  is a Matlis lift of  $k$ .

To see this, first note that  $\mathcal{I}$  is  $k$ -cellular, or equivalently [13, 6.12], that each element of  $\mathcal{I}$  is annihilated by some power of  $I$ . Pick an element  $x \in \mathcal{I}$ ; by Krull's Theorem [2, 10.20] the intersection  $\bigcap_j I^j x$  is trivial. But each submodule  $I^j x$  of  $\mathcal{I}$  is either trivial itself or contains  $k \subset \mathcal{I}$  [37, p. 281]. The conclusion is that  $I^j x = 0$  for  $j \gg 0$ . Since  $\mathrm{Hom}_R(k, \mathcal{I}) \sim k$  (again, for instance, by [37]) and  $\mathcal{I}$  is effectively constructible from  $k$  (5.1, 4.10),  $\mathcal{I}$  provides an  $\mathcal{E}$ -lift of  $k$ . Up to equivalence there is exactly one  $\mathcal{E}$ -lift of  $k$  (3.9), and so in fact  $\mathcal{I}$  is the only Matlis lift of  $k$ .

For instance, if  $R \rightarrow k$  is  $\mathbb{Z}_{(p)} \rightarrow \mathbb{F}_p$ , then  $\mathcal{I} \sim k \otimes_{\mathcal{E}} k$  is  $\mathbb{Z}/p^\infty$  (cf. [13, §3]), and Matlis duality (6.11) for  $R$ -modules is Pontriagin duality for  $p$ -local abelian groups.

**7.2.  $k$ -algebras.** Suppose that  $R$  is an augmented  $k$ -algebra, and let  $M$  be the  $R$ -module  $\mathrm{Hom}_k(R, k)$ . The left  $R$ -action on  $M$  is induced by the right  $R$ -action of  $R$  on itself. By an adjointness calculation,  $\mathrm{Hom}_R(k, M)$  is equivalent to  $k$ , and so in this way  $M$  provides an  $\mathcal{E}$ -lift of  $k$ . If this  $\mathcal{E}$ -lift is of Matlis type, then the  $R$ -module  $k \otimes_{\mathcal{E}} k$ , which by 6.10 is equivalent to  $\mathrm{Cell}_k^R \mathrm{Hom}_k(R, k)$ , is a Matlis lift of  $k$ . There are equivalences

$$\mathrm{Hom}_k(k \otimes_{\mathcal{E}} k, k) \sim \mathrm{Hom}_{\mathcal{E}}(k, \mathrm{Hom}_k(k, k)) \sim \mathrm{Hom}_{\mathcal{E}}(k, k) \sim \hat{R},$$

so that if  $(R, k)$  is dc-complete, the Matlis lift  $k \otimes_{\mathcal{E}} k$  is pre-dual over  $k$  to  $R$  (i.e., the  $k$ -dual of  $k \otimes_{\mathcal{E}} k$  is  $R$ ). Note that this calculation does not depend on assuming that  $R$  is small in any sense as a  $k$ -module; there is an interesting example below in 7.6.

**7.3. The sphere spectrum.** Let  $R \rightarrow k$  be the unit map  $\mathbb{S} \rightarrow \mathbb{F}_p$ . (Recall that we identify the ring  $\mathbb{F}_p$  with the corresponding Eilenberg-MacLane  $\mathbb{S}$ -algebra.) The endomorphism  $\mathbb{S}$ -algebra  $\mathcal{E}$  is the Steenrod algebra spectrum, with  $\pi_{-i} \mathcal{E}$  isomorphic to the degree  $i$  homogeneous component of the Steenrod algebra. Since  $k$  has a unique  $\mathcal{E}$ -lift (3.9) and the conditions of 6.15 are satisfied (3.13, 3.14),  $k$  has a unique Matlis lift given by  $k \otimes_{\mathcal{E}} k$ . Let  $J$  be the Brown-Comenetz dual of  $\mathbb{S}$  [9] and  $J_p$  its  $p$ -primary summand. We argue below that  $J_p$  is  $k$ -cellular; by the basic property of Brown-Comenetz duality,  $\mathrm{Hom}_R(k, J_p) \sim k$ . By 6.10 the evaluation map  $k \otimes_{\mathcal{E}} k \rightarrow J_p$  is a  $k$ -cellular approximation and hence, because  $J_p$  is  $k$ -cellular, an equivalence. Matlis duality amounts to the  $p$ -primary part of Brown-Comenetz duality. Arguments

parallel to those in the proof of 6.15 show that if  $X$  is spectrum which is bounded below and of finite type then the natural map

$$k \otimes_{\mathcal{E}} \mathrm{Hom}_R(X, k) \rightarrow \mathrm{Hom}_R(X, k \otimes_{\mathcal{E}} k)$$

is an equivalence. Suppose that  $X_*$  is an Adams resolution of the sphere. Taking the Brown-Comenetz dual  $\mathrm{Hom}_R(X_*, k \otimes_{\mathcal{E}} k)$  gives a spectral sequence which is the  $\mathbb{F}_p$ -dual of the mod  $p$  Adams spectral sequence. On the other hand, computing  $\pi_* \mathrm{Hom}_R(X_*, k)$  amounts to taking the cohomology of  $X_*$  and so gives a free resolution of  $k$  over the Steenrod Algebra; the spectral sequence associated to  $k \otimes_{\mathcal{E}} \mathrm{Hom}_R(X_*, k)$  is then the Kunnetth spectral sequence

$$\mathrm{Tor}_*^{\pi_* \mathcal{E}}(\pi_* k, \pi_* k) \Rightarrow \pi_*(k \otimes_{\mathcal{E}} k) \cong \pi_* J_p.$$

It follows that these two spectral sequences are isomorphic.

To see that  $J_p$  is  $k$ -cellular, write  $J_p = \mathrm{hocolim} J_p(-i)$ , where  $J_p(-i)$  is the  $(-i)$ -connective cover of  $J_p$ . Each  $J_p(-i)$  has only a finite number of homotopy groups, each of which is a finite  $p$ -primary torsion group, and it follows immediately that  $J_p$  can be finitely built from  $k$ . Thus  $J_p$ , as a homotopy colimit of  $k$ -cellular objects, is itself  $k$ -cellular (cf. 3.17).

**7.4. Cochains.** Suppose that  $X$  is a pointed connected space and  $k$  is a field such that  $(X, k)$  is of Eilenberg-Moore type (4.22). Let  $R = C^*(X; k)$  and  $\mathcal{E} = \mathrm{End}_R(k) \sim C_*(\Omega X; k)$ , and suppose that some  $\mathcal{E}$ -lift of  $k$  is given. By 3.13,  $k$  is of upward finite type over  $\mathcal{E}^{\mathrm{op}}$ , and hence of downward finite type over  $R$  (3.18), the conditions of 6.16 are satisfied, and  $\mathcal{I} = k \otimes_{\mathcal{E}} k$  is a Matlis lift of  $k$ . In fact there is only one  $\mathcal{E}$ -lift of  $k$ ; this follows from 3.9 and the fact that if  $k$  is a field of characteristic  $p$  and  $G$  is a finite  $p$ -group, any homomorphism  $G \rightarrow k^\times$  is trivial. In these cases the Matlis lift  $\mathcal{I} = k \otimes_{\mathcal{E}} k$  is equivalent by the Rothenberg-Steenrod construction to  $C_*(X; k) = \mathrm{Hom}_k(R, k)$ . Observe in particular that  $\mathrm{Hom}_k(R, k)$  is  $k$ -cellular as an  $R$ -module; this also follows from 3.17.

**7.5. Chains.** Let  $X$  be a connected pointed space,  $k$  a field, and  $R$  the chain algebra  $C_*(\Omega X; k)$ , so that  $\mathcal{E} \sim C^*(X; k)$ . By 3.9 there is only one  $\mathcal{E}$ -lift of  $k$ , necessarily given by the augmentation action of  $\mathcal{E}$  on  $k$ . Suppose that  $k$  has upward finite type as an  $R$ -module, for instance, suppose that the conditions of 3.13 hold, or that  $X$  has finite skeleta (5.4). Then, by 3.14 and 6.15,  $k$  has a unique Matlis lift, given by  $k \otimes_{\mathcal{E}} k$ , or alternatively (7.2) by  $\mathrm{Cell}_k \mathrm{Hom}_k(R, k) \sim \mathrm{Cell}_k C^*(\Omega X; k)$ . We have *not* assumed that  $(X, k)$  is of Eilenberg-Moore type, and so the identification

$$k \otimes_{\mathcal{E}} k \sim \mathrm{Cell}_k C^*(\Omega X; k)$$

gives an interpretation of the abutment of the cohomology Eilenberg-Moore spectral sequence associated to the path fibration over  $X$ ; this is in some sense dual to the interpretation of the abutment of the corresponding homology spectral as a suitable completion of  $C_*(\Omega X)$  [12].

**7.6. Suspension spectra of loop spaces.** Suppose that  $X$  is a pointed finite complex, let  $k = \mathbb{S}$ , and let  $R$  be the augmented  $k$ -algebra  $C_*(\Omega X; k)$ . Then  $\mathcal{E}$  is equivalent to  $C^*(X; k)$ , i.e., to the Spanier-Whitehead dual of  $X$  (4.22). Since  $X$  is finite,  $k$  is small as an  $R$ -module (5.3). It follows from 6.13 that Matlis lifts of  $k$  correspond bijectively to  $\mathcal{E}$ -lifts of  $k$ . Note that since the augmentation action of  $\mathcal{E}$  on  $k$  factors through  $\mathcal{E} \rightarrow k$ , and  $k$  is commutative, this augmentation action amounts in itself to an  $\mathcal{E}$ -lift. (It is possible to show that this is the only  $\mathcal{E}$ -lift of  $k$ , but we will not do that here.) By inspection, this augmentation  $\mathcal{E}$ -lift of  $k$  is the same as the  $\mathcal{E}$ -lift obtained by letting  $\mathcal{E}$  act in the natural way on  $\mathrm{Hom}_R(k, \mathrm{Hom}_k(R, k)) \sim k$  as in 7.2. By 6.10, the corresponding Matlis lift  $k \otimes_{\mathcal{E}} k$  is  $\mathrm{Cell}_k \mathrm{Hom}_k(R, k)$ .

Suppose in addition that  $X$  is 1-connected, and write  $k \otimes_{\mathcal{E}} k$  as the realization of the ordinary simplicial bar construction

$$k \otimes_{\mathbb{S}} k \leftarrow k \otimes_{\mathbb{S}} \mathcal{E} \otimes_{\mathbb{S}} k \leftarrow k \otimes_{\mathbb{S}} \mathcal{E} \otimes_{\mathbb{S}} \mathcal{E} \otimes_{\mathbb{S}} k \cdots .$$

The spectrum  $\mathrm{Hom}_{\mathbb{S}}(k \otimes_{\mathbb{S}} k, \mathbb{S})$  is then the total complex of the corresponding cosimplicial object

$$\mathrm{Hom}_{\mathbb{S}}(k \otimes_{\mathbb{S}} k, \mathbb{S}) \Rightarrow \mathrm{Hom}_{\mathbb{S}}(k \otimes_{\mathbb{S}} \mathcal{E} \otimes_{\mathbb{S}} k, \mathbb{S}) \Rrightarrow \cdots .$$

This is the cosimplicial object obtained by applying the unpointed suspension spectrum functor to the cobar construction on  $X$ , and by a theorem of Bousfield [7] its total complex is the suspension spectrum of  $\Omega X$ , i.e.,  $R$ . Equivalently, Bousfield's theorem shows that in this case  $(R, k)$  is dc-complete. In this way if  $X$  is 1-connected the Matlis lift of  $k$  is a Spanier-Whitehead pre-dual of  $R$  (cf. 7.2). This object has come up in a different way in work of N. Kuhn [33].

## 8. GORENSTEIN $\mathbb{S}$ -ALGEBRAS

If  $R$  is a commutative Noetherian local ring with maximal ideal  $I$  and residue field  $R/I = k$ , one says that  $R$  is *Gorenstein* if  $\mathrm{Ext}_R^*(k, R)$  is concentrated in a single degree, and is isomorphic to  $k$  there. We give a similar definition for  $\mathbb{S}$ -algebras, with an extra technical condition added on.

**8.1. Definition.** Suppose that  $R \rightarrow k$  is a map of  $\mathbb{S}$ -algebras, and let  $\mathcal{E} = \text{End}_R(k)$ . Then  $R \rightarrow k$  is *Gorenstein* of shift  $a$  if the following two conditions hold:

- (1) as a left  $k$ -module,  $\text{Hom}_R(k, R)$  is equivalent to  $\Sigma^a k$ , and
- (2) as a right  $\mathcal{E}$ -module,  $\text{Hom}_R(k, R)$  is of Matlis type (6.8).

**8.2. Remark.** Suppose that  $R \rightarrow k$  is Gorenstein of shift  $a$ , and give  $\Sigma^a k$  the right  $\mathcal{E}$ -module structure from 8.1(1). Then by 6.10,  $\text{Cell}_k(R)$  is equivalent to  $\Sigma^a k \otimes_{\mathcal{E}} k$ . It follows from 8.4 and 5.1 that if  $R \rightarrow k$  is a map from a commutative Noetherian local ring to its residue field, then  $R \rightarrow k$  is Gorenstein in our sense if and only if  $R$  is Gorenstein in the classical sense of commutative algebra.

**8.3. Remark.** Definition 8.1 does not exhaust all of the structure in  $\text{Hom}_R(k, R)$ ; in fact, the right action of  $R$  on itself gives a right  $R$ -action on  $\text{Hom}_R(k, R)$  which commutes with the right  $\mathcal{E}$ -action (since  $\mathcal{E}$  acts through  $k$ ). This implies that if  $R \rightarrow k$  is Gorenstein and  $k$  is given the right  $\mathcal{E}$ -action obtained from  $k \sim \Sigma^{-a} \text{Hom}_R(k, R)$ , then the Matlis lift  $\mathcal{I} = k \otimes_{\mathcal{E}} k$  of  $k$  inherits a right  $R$ -action. In this case the Matlis dual  $\text{Hom}_R(M, \mathcal{I})$  of a left  $R$ -module is naturally a right  $R$ -module.

In the proxy-small case it is possible to simplify definition 8.1. We record the following, which is a consequence of 6.14.

**8.4. Proposition.** *Suppose that the map  $R \rightarrow k$  of  $\mathbb{S}$ -algebras is proxy-small. Then  $R \rightarrow k$  is Gorenstein of shift  $a$  if and only if  $\text{Hom}_R(k, R)$  is equivalent to  $\Sigma^a k$  as a left  $k$ -module.*

The rest of the section provides techniques for recognizing Gorenstein homomorphisms  $R \rightarrow k$ .

**8.5. Proposition.** *Suppose that  $R$  is an augmented  $k$ -algebra, and let  $\mathcal{E} = \text{End}_R(k)$ . Assume that  $(R, k)$  is dc-complete, and that  $R \rightarrow k$  is proxy-small. Then  $R \rightarrow k$  is Gorenstein (of shift  $a$ ) if and only if  $\mathcal{E} \rightarrow k$  is Gorenstein (of shift  $a$ ).  $\square$*

See [20, 2.1] for a differential graded version of this.

*Proof.* Compute

$$\begin{aligned} \text{Hom}_R(k, R) &\sim \text{Hom}_R(k, \text{Hom}_{\mathcal{E}}(k, k)) \sim \text{Hom}_{R \otimes_k \mathcal{E}}(k \otimes_k k, k) \\ \text{Hom}_{\mathcal{E}}(k, \mathcal{E}) &\sim \text{Hom}_{\mathcal{E}}(k, \text{Hom}_R(k, k)) \sim \text{Hom}_{\mathcal{E} \otimes_k R}(k \otimes_k k, k). \end{aligned}$$

What we are using is the fact that if  $A$  and  $B$  are left modules over the  $k$ -algebras  $S$  and  $T$ , respectively, and  $C$  is a left module over  $S \otimes_k T$ ,

then there is an equivalence

$$\mathrm{Hom}_S(A, \mathrm{Hom}_T(B, C)) \sim \mathrm{Hom}_{S \otimes_k T}(A \otimes_k B, C).$$

In our case  $A = B = C = k$ . This reveals a subtlety:  $k \otimes_k k$  is certainly equivalent to  $k$ , but not necessarily in a way which relates the tensor product action of  $R \otimes_k \mathcal{E}$  on  $k \otimes_k k$  to the action of  $R \otimes_k \mathcal{E}$  on  $k$  given by  $\mathcal{E} = \mathrm{End}_R(k)$ . Nevertheless, it is clear that  $\mathrm{Hom}_R(k, R)$  is equivalent to a shift of  $k$  if and only if  $\mathrm{Hom}_{\mathcal{E}}(k, \mathcal{E})$  is. If  $\mathcal{E}$  is Gorenstein, then  $R$  is Gorenstein by 8.4. If  $R$  is Gorenstein,  $\mathcal{E}$  is Gorenstein by 4.17 and 8.4.  $\square$

**8.6. Proposition.** *Suppose that  $S \rightarrow R$  is a map of augmented commutative  $k$ -algebras such that  $R$  is small as an  $S$ -module. Let  $Q$  be the augmented  $k$ -algebra  $k \otimes_S R$ . Then there is an equivalence of  $k$ -modules*

$$\mathrm{Hom}_R(k, R) \sim \mathrm{Hom}_Q(k, \mathrm{Hom}_S(k, S) \otimes_k Q),$$

where the action of  $Q$  on  $\mathrm{Hom}_S(k, S) \otimes_k Q$  is induced by the usual action of  $Q$  on itself.

There is a rational version in [20, 4.3]. The argument below depends on the following general lemma, whose proof we leave to the reader.

**8.7. Lemma.** *Suppose that  $R$  is a  $k$ -algebra, that  $A$  is a right  $R$ -module, and that  $B$  and  $C$  are left  $R$ -modules. Then there are natural equivalences*

$$\begin{aligned} \mathrm{Hom}_R(B, C) &\sim \mathrm{Hom}_{R \otimes_k R^{\mathrm{op}}}(R, \mathrm{Hom}_k(B, C)) \\ A \otimes_R B &\sim R \otimes_{R \otimes_k R^{\mathrm{op}}}(A \otimes_k B). \end{aligned}$$

*Proof of 8.6.* Since  $R$  is commutative, we do not distinguish in notation between  $R$  and  $R^{\mathrm{op}}$ . First note that

$$\mathrm{Hom}_R(k, R) \sim \mathrm{Hom}_{R \otimes_S R}(R, \mathrm{Hom}_S(k, R))$$

as in 8.7. Now observe that  $R$  is small over  $S$ , so that

$$(8.8) \quad \mathrm{Hom}_S(k, R) \sim \mathrm{Hom}_S(k, S) \otimes_S R.$$

Under this equivalence, the left action of  $R$  on  $\mathrm{Hom}_S(k, S) \otimes_S R$  is induced by the left action of  $R$  on itself, and the right action of  $R$  by the left action of  $R$  on  $k$ . Now since  $S$  is commutative, the right and left actions of  $S$  on  $\mathrm{Hom}_S(k, S)$  are the same. In particular, the right action (which is used in forming  $\mathrm{Hom}_S(k, S) \otimes_S R$ ) factors through the homomorphism  $S \rightarrow k$ , and we obtain an equivalence

$$(8.9) \quad \mathrm{Hom}_S(k, S) \otimes_S R \sim \mathrm{Hom}_S(k, S) \otimes_k (k \otimes_S R) \sim \mathrm{Hom}_S(k, S) \otimes_k Q.$$

Let  $M = \mathrm{Hom}_S(k, S) \otimes_k Q$ . Under 8.8 and 8.9 the left action of  $R$  on  $M$  is induced by the left action of  $R$  on  $Q$ , while the right action of

$R$  is induced by the left action of  $R$  on  $k$ . In particular, the action of  $R \otimes_S R$  on  $M$  factors through an action of  $k \otimes_S R \sim Q$  on  $M$ , and so by adjointness we have

$$\begin{aligned} \mathrm{Hom}_{R \otimes_S R}(R, M) &\sim \mathrm{Hom}_Q(Q \otimes_{R \otimes_S R} R, M) \\ &\sim \mathrm{Hom}_Q(k, M), \end{aligned}$$

where the last equivalence depends on the calculation (8.7)

$$(k \otimes_S R) \otimes_{R \otimes_S R} R \sim k \otimes_R R \sim k.$$

The action of  $Q$  on this object is the obvious one that factors through  $Q \rightarrow k$ . Combining the above gives the desired statement.  $\square$

**8.10. Proposition.** *Let  $S \rightarrow R$  be a homomorphism of commutative augmented  $k$ -algebras, and set  $Q = k \otimes_S R$ . Suppose that  $R$  is small as an  $S$ -module, and that  $R \rightarrow k$  is proxy-small. Then if the maps  $S \rightarrow k$  and  $Q \rightarrow k$  are Gorenstein, so is  $R \rightarrow k$ .*

*Proof.* By 8.6,  $\mathrm{Hom}_R(k, R) \sim \Sigma^a k$ . It follows from 8.4 that  $R \rightarrow k$  is Gorenstein.  $\square$

**8.11. Poincaré Duality.** A  $k$ -algebra  $R$  is said to *satisfy Poincaré duality* of dimension  $a$  if there is an equivalence  $R \rightarrow \Sigma^a \mathrm{Hom}_k(R, k)$  of  $R$ -modules; note that here we give  $\mathrm{Hom}_k(R, k)$  the left  $R$ -module structure induced by the right action of  $R$  on itself. The algebra  $R$  satisfies this condition if and only if there is an orientation class  $\omega \in \pi_{-a} \mathrm{Hom}_k(R, k)$  with the property that  $\pi_* \mathrm{Hom}_k(R, k)$  is a free module of rank one over  $\pi_* R$  with generator  $\omega$ . If  $k$  is a field, then  $\pi_* \mathrm{Hom}_k(R, k) = \mathrm{hom}_k(\pi_* R, k)$ , and  $R$  satisfies Poincaré duality if and only if  $\pi_* R$  satisfies Poincaré duality in the simplest algebraic sense.

**8.12. Proposition.** *Suppose that  $R$  is an augmented  $k$  algebra such that the map  $R \rightarrow k$  is proxy-small. If  $R$  satisfies Poincaré duality of dimension  $a$ , then  $R \rightarrow k$  is Gorenstein of shift  $a$ .*

*Proof.* As in 7.2, compute

$$\mathrm{Hom}_R(k, R) \sim \mathrm{Hom}_R(k, \Sigma^a \mathrm{Hom}_k(R, k)) \sim \Sigma^a \mathrm{Hom}_k(R \otimes_R k, k) \sim \Sigma^a k.$$

The fact that  $R \rightarrow k$  is Gorenstein follows from 8.4.  $\square$

We now give a version of the result from commutative ring theory that “regular implies Gorenstein”.

**8.13. Proposition.** *Suppose that  $k$  is a field,  $R$  is a connective commutative  $\mathbb{S}$ -algebra, and  $R \rightarrow k$  is a small homomorphism which is surjective on  $\pi_0$ . Assume that the pair  $(R, k)$  is dc-complete. Then  $R \rightarrow k$  is Gorenstein.*



8.14. *Remark.* It is possible to omit the dc-completeness hypothesis from 8.13 in the commutative ring case. Suppose that  $R$  is a commutative Noetherian ring,  $I \subset R$  is a maximal ideal,  $k = R/I$  is the residue field, and  $R \rightarrow k$  is small. We show that  $R \rightarrow k$  is also Gorenstein. To see this, let  $S = \lim_s R/I^s$  be the  $I$ -adic completion of  $R$ . As in the proof of 4.20,  $S$  is flat over  $R$  and  $\mathrm{Tor}_0^R(S, k) \cong k$ ; in addition, the map  $R \rightarrow S$  is a  $k$ -equivalence (of  $R$ -modules). This gives a chain of equivalences

$$\mathrm{Hom}_R(k, R) \sim \mathrm{Hom}_R(k, S) \sim \mathrm{Hom}_S(S \otimes_R k, S) \sim \mathrm{Hom}_S(k, S).$$

The flatness easily implies that  $S \rightarrow k$  is also small, and so  $R \rightarrow k$  is Gorenstein if and only if  $S \rightarrow k$  is Gorenstein. But it follows from 4.20 that the pair  $(S, k)$  is dc-complete, and so  $S \rightarrow k$  is Gorenstein by 8.13.

8.15. *Remark.* As the arguments below suggest, 8.13 fails without the commutativity assumption. For instance, let  $k = \mathbb{F}_p$ , let  $X$  be a simply-connected finite complex which does not satisfy Poincaré duality, and let  $R \rightarrow k$  be the augmentation map  $C_*(\Omega X; k) \rightarrow k$ . Then  $R \rightarrow k$  is small (5.3),  $(R, k)$  is dc-complete (4.22), but it is easy to see that  $R \rightarrow k$  is not Gorenstein.

8.16. **Lemma.** *Suppose that  $k$  is a field,  $R$  is a connective commutative  $\mathbb{S}$ -algebra, and  $R \rightarrow k$  is a homomorphism which is surjective on  $\pi_0$ . Assume that  $k$  is of upward finite type over  $R$ . Then  $\pi_* \mathrm{End}_R(k)$  is in a natural way a cocommutative Hopf algebra over  $k$ .*

*Proof.* The diagram chasing necessary to prove this is described in detail in [1, pp. 56–76], with a focus at the end on the case in which  $R = \mathbb{S}$ ,  $k = \mathbb{F}_p$ , and  $\pi_* \mathrm{End}_R(k)$  is the mod  $p$  Steenrod algebra. Let  $\mathcal{E} = \mathrm{End}_R(k)$ . The key idea is that  $\pi_* \mathcal{E}$  is the  $k$ -dual of the commutative  $k$ -algebra  $\pi_*(k \otimes_R k)$ : as in 7.2 there are equivalences

$$\mathrm{Hom}_k(k \otimes_R k, k) \sim \mathrm{Hom}_R(k, \mathrm{Hom}_k(k, k)) \sim \mathrm{End}_R(k).$$

The  $k$ -dual of the multiplication on  $\pi_*(k \otimes_R k)$  then provides the comultiplication on  $\pi_* \mathrm{End}_R(k)$ . The fact that  $k$  is of upward finite type over  $R$  guarantees that the groups  $\pi_i(k \otimes_R k)$  are finite-dimensional over  $k$ .

There is a technicality:  $k \otimes_R k$  is a bimodule over  $k$ , not an algebra over  $k$ . However  $k \otimes_R k$  is an algebra over  $R$ , so that the surjection  $\pi_0 R \rightarrow k$  guarantees that the left and right action of  $k$  on  $\pi_*(k \otimes_R k)$  agree. For the same reason, the left and right actions of  $k$  on  $\pi_* \mathrm{End}_R(k)$  agree, and this graded ring becomes a Hopf algebra over  $k$ .  $\square$

*Proof of 8.13.* Let  $\mathcal{E} = \text{End}_R(k)$ . The connectivity assumptions on  $R$  imply that  $\pi_0\mathcal{E} \cong k$  and that  $\mathcal{E}$  is coconnective; by 8.16,  $\mathcal{E}$  is a Hopf algebra over  $k$ . In fact,  $\mathcal{E}$  is finitely built from  $k$  (4.17), and so  $\pi_*\mathcal{E}$  is a finite dimensional Hopf algebra over  $k$ . Sweedler has remarked that a connected finite-dimensional Hopf algebra over  $k$  with commutative comultiplication and involution satisfies algebraic Poincaré duality [39]; see also [45, 5.1.6]. The map  $\mathcal{E} \rightarrow k$  is thus Gorenstein by 8.12, and  $R \rightarrow k$  by 8.5.  $\square$

8.17. *Remark.* The above arguments are related to those of Avramov and Golod [3], who show that a commutative Noetherian local ring  $R$  is Gorenstein if and only if the homology of the associated Koszul complex is a Poincaré duality algebra.

## 9. A LOCAL COHOMOLOGY THEOREM

One of the attractions of the Gorenstein condition on an  $\mathbb{S}$ -algebra  $R$  is that it has structural implications for  $\pi_*R$ , which can sometimes be thought of as a duality property. To illustrate this, we look at the special case in which  $R \rightarrow k$  is a Gorenstein map of augmented  $k$ -algebras, where  $k$  is a field. Let  $\mathcal{E} = \text{End}_R(k)$ . By 8.2, the Gorenstein condition gives

$$\Sigma^a k \otimes_{\mathcal{E}} k \sim \text{Cell}_k R.$$

We next assume that the right  $\mathcal{E}$ -structure on  $\Sigma^a k$  given by  $\Sigma^a k \sim \text{Hom}_R(k, R)$  is equivalent to the right  $\mathcal{E}$ -structure given by

$$\Sigma^a k \sim \text{Hom}_R(k, \Sigma^a \text{Hom}_k(R, k)).$$

By 6.10 this gives an equivalence

$$\Sigma^a k \otimes_{\mathcal{E}} k \sim \Sigma^a \text{Cell}_k \text{Hom}_k(R, k).$$

Assume in addition that  $\text{Hom}_k(R, k)$  is itself  $k$ -cellular as an  $R$ -module. Combining the above then gives

$$(9.1) \quad \Sigma^a \text{Hom}_k(R, k) \sim \text{Cell}_k R.$$

Now in some reasonable circumstances we might expect a spectral sequence

$$(9.2) \quad E_{i,j}^2 = \pi_i \text{Cell}_k^{\pi_*R}(\pi_*R)_j \Rightarrow \pi_{i+j} \text{Cell}_k^R(R)$$

which in the special situation we are considering would give

$$E_{i,j}^2 = \pi_i \text{Cell}_k^{\pi_*R}(\pi_*R)_j \Rightarrow \pi_{i+j-a} \text{Hom}_k(R, k).$$

(The subscript  $j$  refers to the  $j$ 'th homogeneous component of an appropriate grading on  $\pi_i \text{Cell}_k^{\pi_*R}(\pi_*R)$ .) This is what we mean by a

duality property for  $\pi_*R$ : a spectral sequence starting from some covariant algebraic data associated to  $\pi_*R$  and abutting to the dual object  $\pi_* \operatorname{Hom}_k(R, k) \cong \operatorname{Hom}_k(\pi_*R, k)$ . If  $R$  is  $k$ -cellular as a module over itself, then 9.1 gives  $\Sigma^a \operatorname{Hom}_k(R, k) \sim R$ , and we obtain ordinary Poincaré duality.

The problematic point here is the existence of the spectral sequence 9.2. Rather than trying to construct this spectral sequence in general and study its convergence properties, we concentrate on a special case in which it is possible to identify  $\operatorname{Cell}_k^R(R)$  explicitly. To connect the following statement with 9.2, recall [13, §6] that if  $S$  is a commutative ring and  $I \subset S$  a finitely generated ideal with quotient ring  $k = S/I$ , then for any discrete  $S$ -module  $M$  the local cohomology group  $H_I^i(M)$  can be identified with  $\pi_{-i} \operatorname{Cell}_k^S(M)$ .

**9.3. Proposition.** *Suppose that  $k$  is a field, and that  $R$  is a coconnective commutative augmented  $k$ -algebra. Assume that  $\pi_*R$  is Noetherian, and that the augmentation map induces an isomorphism  $\pi_0R \cong k$ . Then for any  $R$ -module  $M$  there is a spectral sequence*

$$E_{i,j}^2 = H_I^{-i}(\pi_*M)_j \Rightarrow \pi_{i+j} \operatorname{Cell}_k^R(M).$$

Under the assumptions of 9.3,  $\operatorname{Hom}_R(R, k)$  is  $k$ -cellular as an  $R$ -module (3.17). Given the above discussion, this leads to the following local cohomology theorem.

**9.4. Proposition.** *In the situation of 9.3, assume in addition that  $R \rightarrow k$  is Gorenstein of shift  $a$ , and that  $k$  has a unique  $\mathcal{E}$ -lift (where  $\mathcal{E} = \operatorname{End}_R(k)$ ). Then there is a spectral sequence*

$$E_{i,j}^2 = H_I^{-i}(\pi_*R)_j \Rightarrow \pi_{i+j-a} \operatorname{Hom}_k(R, k).$$

**9.5. Remark.** The structural implications of this spectral sequence for the geometry of the ring  $\pi_*R$  are investigated in [25].

*Proof of 9.3.* We first copy some constructions from [13, §6]. For any  $x \in \pi_{|x|}R$  we can form an  $R$ -module  $R[1/x]$  by taking the homotopy colimit of the sequence

$$R \xrightarrow{x} \Sigma^{-|x|}R \xrightarrow{x} \Sigma^{-2|x|}R \xrightarrow{x} \dots$$

(Actually,  $R[1/x]$  can also be given the structure of a commutative  $\mathbb{S}$ -algebra, in such a way that  $R \rightarrow R[1/x]$  is a homomorphism.) Write  $K_m(x)$  for the fibre of  $x^m : R \rightarrow \Sigma^{-m|x|}R$ , and  $K_\infty(x)$  for the fibre of the map  $R \rightarrow R[1/x]$ . Now choose a finite sequence  $x_1, \dots, x_n$  of generators for  $I \subset \pi_*R$ , and let

$$\begin{aligned} K_m &= K_m(x_1) \otimes_R \cdots \otimes_R K_m(x_n) \\ K_\infty &= K_\infty(x_1) \otimes_R \cdots \otimes_R K_\infty(x_n). \end{aligned}$$

Recall that  $R$  is commutative, so that right and left  $R$ -module structures are interchangeable, and tensoring two  $R$ -modules over  $R$  produces a third  $R$ -module. Write  $K = K_1$ . It is easy to see that  $\pi_*K$  is finitely built from  $k$  as a module over  $\pi_*R$ , and hence (3.16) that  $K$  is finitely built from  $k$  as a module over  $R$ . An inductive argument (using cofibration sequences  $K_m(x_i) \rightarrow K_{m+1}(x_i) \rightarrow \Sigma^{m|x_i|}K_1(x_i)$ ) shows that  $K$  builds  $K_m$  and hence also builds  $K_\infty \sim \text{hocolim } K_m$  (cf. [13, 6.6]). It is easy to see that the evident map  $K_\infty \rightarrow R$  gives equivalences

$$(9.6) \quad k \otimes_R K_\infty \sim k \quad K \otimes_R K_\infty \sim K.$$

See [13, proof of 6.9]; the second equivalence follows from the first because  $K$  is built from  $k$ . The first equivalence implies that  $K_\infty$  builds  $k$  and this in turn shows that  $K$  builds  $k$ . Since  $K$  is small over  $R$ , we see that  $R \rightarrow k$  is proxy-small with Koszul complex  $K$ . In particular, a map  $A \rightarrow B$  of  $R$ -modules is a  $k$ -equivalence if and only if it is a  $K$ -equivalence, or (since  $\text{Hom}_R(K(x_i), R) \sim \Sigma K(x_i)$  and hence  $\text{Hom}_R(K, R) \sim \Sigma^n K$ ) if and only if it induces an equivalence  $K \otimes_R A \rightarrow K \otimes_R B$ . Since  $K_\infty$  is built from  $k$  as an  $R$ -module, so is  $K_\infty \otimes_R M$ . The right hand equivalence in 9.6 implies that the map  $K_\infty \otimes_R M \rightarrow M$  induces an equivalence

$$K \otimes_R K_\infty \otimes_R M \rightarrow K \otimes_R M,$$

and it follows that  $K_\infty \otimes_R M$  is  $\text{Cell}_k^R(M)$ . Each module  $K_\infty(x_i)$  lies in a cofibration sequence

$$\Sigma^{-1}R[1/x_i] \rightarrow K_\infty(x_i) \rightarrow R$$

which can be interpreted as a one-step increasing filtration of  $K_\infty(x_i)$ . Tensoring these together gives an  $n$ -step filtration of  $K_\infty$ ,

$$0 = F_{n+1} \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 = K_\infty,$$

with the property that there are equivalences

$$F_s/F_{s+1} \sim \bigoplus_{\{i_1, \dots, i_s\}} R[1/x_{i_1}] \otimes_R \cdots \otimes_R R[1/x_{i_s}].$$

The sums here are indexed over subsets of cardinality  $s$  from  $\{1, \dots, n\}$ . Tensoring this filtration with  $M$  gives a finite filtration of  $\text{Cell}_k^R(M)$ , and the spectral sequence of the proposition is the homotopy spectral sequence associated to the filtration. The identification of the  $E^2$ -page as local cohomology is standard [13, §6] [26]; the main point here is to notice that since  $\pi_*R[1/x_i]$  is flat over  $\pi_*R$ , there are isomorphisms  $\pi_*(R[1/x_i] \otimes_R M) \cong \pi_*(R[1/x_i]) \otimes_{\pi_*R} \pi_*M \cong (\pi_*R)[1/x_i] \otimes_{\pi_*R} \pi_*M$ .  $\square$

## 10. GORENSTEIN EXAMPLES

We give several examples of  $\mathbb{S}$ -algebras which are Gorenstein, and at least one example of an  $\mathbb{S}$ -algebra which is not. Of course, commutative Noetherian local rings which are not Gorenstein are easy to come by; these also provide examples of non-Gorenstein  $\mathbb{S}$ -algebras.

**10.1. Small chains.** Suppose that  $X$  is a pointed connected topological space and that  $k$  is a field such that the pair  $(X, k)$  is of Eilenberg-Moore type (4.22), and such that  $H^*(X, k)$  is finite-dimensional and satisfies Poincaré duality of formal dimension  $b$ . Let  $a = -b$ . We point out that the augmentation map  $C_*(\Omega X; k) \rightarrow k$  is small and Gorenstein of shift  $a$ ; note that the shift  $a$  is negative in this case.

Let  $R = C^*(X; k)$ ,  $\mathcal{E} = C_*(\Omega X; k) \sim \text{End}_R(k)$ . By 5.5,  $R \rightarrow k$  is cosmall and  $\mathcal{E} \rightarrow k$  is small. The map  $R \rightarrow k$  is Gorenstein of shift  $a$  (8.12) and  $\mathcal{E} \rightarrow k$  is also Gorenstein with the same shift (8.5). The ring  $R$  has a local cohomology spectral sequence (9.4), but this collapses to a restatement of Poincaré duality:

$$E^2 = \pi_* R \cong \text{Cell}_k^{\pi_* R}(\pi_* R) \cong \pi_* \Sigma^a \text{Hom}_k(R, k).$$

In the absence of the hypothetical spectral sequence 9.2, there is nothing like a local cohomology theorem for the noncommutative  $\mathbb{S}$ -algebra  $\mathcal{E}$ .

**10.2. Small cochains.** Suppose that  $k$  is a field and  $G$  is a topological group such that  $H_*(G; k)$  is finite dimensional. Assume in addition that  $(BG, k)$  is of Eilenberg-Moore type; this covers the cases in which  $k = \mathbb{F}_p$ , and  $G$  is a finite  $p$ -group, a compact Lie group with  $\pi_0 G$  a finite  $p$ -group, or a  $p$ -compact group [17]. We point out that the augmentation map  $C^*(BG; k) \rightarrow k$  is small and Gorenstein, and satisfies a local cohomology theorem.

Let  $R = C^*(BG; k)$  and  $\mathcal{E} = C_*(G; k)$ . The map  $\mathcal{E} \rightarrow k$  is cosmall (3.16), and hence  $R \rightarrow k$  is small (5.6). The graded ring  $H_*(G; k)$  is a finite dimensional group-like Hopf algebra over  $k$ , and so by Sweedler (cf. [45, 5.1.6]) satisfies algebraic Poincaré duality of some dimension, say  $a$ . If  $G$  is a connected compact Lie group, then  $a = \dim G$ ; the “fundamental class”  $\omega$  (see 8.11) lies in

$$H^a(G; k) = \pi_{-a} C^*(G; k) = \pi_{-a}(k \otimes_R k).$$

By 8.12,  $\mathcal{E} \rightarrow k$  is Gorenstein of shift  $a$ , and so  $R \rightarrow k$  is also Gorenstein with the same shift (8.5). The graded ring  $H^*(BG; k) = \pi_* R$  is Noetherian. If  $k$  is of characteristic zero, this follows from the fact that the ring is a finitely generated polynomial algebra over  $k$ ; see [38, 7.20]. If  $k = \mathbb{F}_p$  and  $G$  is a compact Lie group, the finite generation

statement is a classical theorem of Golod [23] and Venkov [46]; in the  $p$ -compact group case it amounts to the main result of [17]. By 3.17 and 9.4 there is a local cohomology theorem for  $R$ .

**10.3. Completed classifying spaces.** Suppose that  $G$  is a compact Lie group such that the adjoint action of  $G$  on its Lie algebra is orientable (e.g.,  $G$  might be a finite group). Let  $k = \mathbb{F}_p$ . We point out that the augmentation map  $C^*(BG; k) \rightarrow k$  is proxy-small and Gorenstein, and that  $C^*(BG; k)$  has a local cohomology theorem. Note that we are not assuming that  $(BG, k)$  is of Eilenberg-Moore type, and in particular we are not assuming that  $\pi_0 G$  is a  $p$ -group; the present case contrasts with 10.2 in that  $C^*(BG; k) \rightarrow k$  need not be small.

We continue the discussion in 5.7, with the same notation. Recall that  $X$  is the  $p$ -completion of  $BG$ ,  $R = C^*(X; k) \sim C^*(BG; k)$ , and  $\mathcal{E} = C_*(\Omega X; k)$ ; the space  $\Omega X$  plays the role of  $G$  above in 10.2, but we do not have that  $H_*(\Omega X; k)$  is finite dimensional. The fibre  $M$  in 5.8 is a compact manifold; it is orientable because its tangent bundle is the bundle associated to the conjugation action of  $G$  on the quotient of the Lie algebra of  $SU(n)$  by the Lie algebra of  $G$ . (Note that since  $SU(n)$  is connected, the conjugation action of  $G$  on the Lie algebra of  $SU(n)$  preserves orientation.) As in 10.1,  $Q = C^*(M; k)$  is cosmall and Gorenstein. Similarly,  $S = C^*(BSU(n); k)$  is small and Gorenstein by 10.2. Let  $S' = C_*(SU(n); k)$ , so that  $S \sim \text{End}_{S'}(k)$ . The group  $SU(n)$  acts on  $M = SU(n)/G$ , so that  $S'$  acts on  $C_*(M; k)$ ; by an Eilenberg-Moore spectral sequence argument, there is an equivalence  $R \sim \text{Hom}_{S'}(C_*(M; k), k)$ . It follows from 3.18 that  $R$  is small as a module over  $S$ . By 4.18 and 8.10,  $R \rightarrow k$  is proxy-small and Gorenstein, as is  $\mathcal{E} \rightarrow k$  (8.5). Since  $\text{Hom}_k(R, k)$  is  $k$ -cellular over  $R$  (3.17), there is a local cohomology spectral sequence for  $R$  (9.4).

**10.4. Finite complexes.** Suppose that  $X$  is a pointed connected finite complex which is a Poincaré duality complex over  $k$  of formal dimension  $b$ ; in other words, assume that  $X$  satisfies possibly unoriented Poincaré duality with arbitrary (twisted)  $k$ -module coefficients. To be specific, assume that  $k$  is a finite field, the field  $\mathbb{Q}$ , or the ring  $\mathbb{Z}$  of integers. We point out that the augmentation map  $C_*(\Omega X; k) \rightarrow k$  is small and Gorenstein. This is related to 10.1 but slightly different: here we assume that  $X$  is finite, we do not insist that  $(X, k)$  be of Eilenberg-Moore type, but we require (possibly twisted) Poincaré duality with arbitrary  $k$ -module coefficients.

Let  $R$  denote the augmented  $k$ -algebra  $C_*(\Omega X; k)$ , so that  $\pi_0 R \cong k[\pi_1 X]$ . Note that  $R \rightarrow k$  is small (5.3). Any module  $M$  over  $k[\pi_1 X]$  gives a module over  $R$ , and (by a version of the Rothenberg-Steenrod

construction) there are isomorphisms

$$H_i(X; M) \cong \pi_i(k \otimes_R M) \quad H^i(X; M) \cong \pi_{-i} \text{Hom}_R(k, M) .$$

Let  $a = -b$ . The duality condition on  $X$  can be expressed by saying that there is a module  $\lambda$  over  $k[\pi_1 X]$  whose underlying  $k$ -module is isomorphic to  $k$  itself, and an orientation class  $\omega \in \pi_{-a}(\lambda \otimes_R k)$ , such that, for any  $k[\pi_1 X]$ -module  $M$ , evaluation on  $\omega$  gives an equivalence

$$(10.5) \quad \text{Hom}_R(k, M) \rightarrow \Sigma^a \lambda \otimes_R M .$$

By 3.9, it follows that 10.5 is an equivalence for any  $R$ -module  $M$  which has only one nonvanishing homotopy group. By triangle arguments (cf. 3.2) it is easy to conclude that 10.5 is an equivalence for all  $M$  which have only a finite number of nonvanishing homotopy groups, and by passing to a limit (cf. proof of 6.15) that 10.5 is actually an equivalence for all  $R$ -modules  $M$ . Note that this passage to the limit depends on the fact that  $k$  is small over  $R$ . The case  $M = R$  of 10.5 gives

$$\text{Hom}_R(k, R) \sim \Sigma^a \lambda \otimes_R R \sim \Sigma^a \lambda \sim \Sigma^a k ,$$

and so by 8.4,  $R \rightarrow k$  is Gorenstein of shift  $a$ . Let  $\mathcal{E} = \text{End}_R(k)$ . The pair  $(R, k)$  is not necessarily dc-complete, and so  $\mathcal{E} \rightarrow k$  is not necessarily Gorenstein; for example, it is clear that  $\pi_* \mathcal{E} \cong H^*(X; k)$  need not satisfy algebraic Poincaré duality in the nonorientable case.

The equivalence  $\text{Hom}_R(k, R) \sim \lambda$  is an  $R$ -module equivalence as long as  $\text{Hom}_R(k, R)$  is given the right  $R$ -module structure obtained from the right action of  $R$  on itself. In this way the orientation character of the Poincaré complex  $X$  is derived from the one bit of structure on  $\text{Hom}_R(k, R)$  that does not play a role in the definition of what it means for  $R \rightarrow k$  to be Gorenstein (8.3).

**10.6. Suspension spectra of loop spaces.** This is a continuation of 10.4 and 7.6:  $X$  is a pointed connected finite complex which is a Poincaré duality complex over  $k$  of formal dimension  $a$ . Let  $k = \mathbb{S}$ . We observe that the augmentation map  $C_*(\Omega X; k) \rightarrow k$  is small and Gorenstein, and point out how this Gorenstein condition leads to a homotopical construction of the Spivak normal bundle of  $X$ .

Let  $R$  denote the augmented  $k$ -algebra  $C_*(\Omega X; k)$ , and let  $\mathcal{E}$  denote  $C^*(X; k) \sim \text{End}_R(k)$ . The map  $R \rightarrow k$  is small by 5.3. Note that  $S = \mathbb{Z} \otimes_{\mathbb{S}} R \sim C_*(\Omega X; \mathbb{Z})$  (4.22). We wish to show that  $R \rightarrow k$  is Gorenstein of shift  $-a$ , or equivalently, that  $\text{Hom}_R(k, R) \sim \Sigma^{-a} k$ . The spectrum  $Y = \Sigma^{-a} k$  is characterized by a combination of the homotopical property that  $Y$  is bounded below, and the homological property that  $\mathbb{Z} \otimes_k Y \sim \Sigma^{-a} \mathbb{Z}$ . The spectrum  $\text{Hom}_R(k, R)$  is bounded below because  $R$  is bounded below and  $k$  is small over  $R$ . Similarly, the fact that

$k$  is small over  $R$  implies that  $\mathbb{Z} \otimes_k \text{Hom}_R(k, R) \sim \text{Hom}_R(k, \mathbb{Z} \otimes_k R)$ . Now compute

$$\text{Hom}_R(k, \mathbb{Z} \otimes_k R) \sim \text{Hom}_{\mathbb{Z} \otimes_k R}(\mathbb{Z}, \mathbb{Z} \otimes_k R) \sim \Sigma^{-a} \mathbb{Z} ,$$

where the first equivalence comes from adjointness, and the second from 10.4. It follows that  $R \rightarrow k$  is Gorenstein. (If  $X$  is simply connected, then  $\mathcal{E} \rightarrow k$  is cosmall and Gorenstein, since in this case  $(R, k)$  is dc-complete.)

The stable homotopy orientation character of  $X$  is given by the action of  $R$  on  $k \sim \mathbb{S}$  obtained via  $\Sigma^{-a} k \sim \text{Hom}_R(k, R)$  from the right action of  $k$  on itself; see 10.5 for the homological version of this. It is not too far off to interpret this character as a homomorphism  $\Omega X \rightarrow \mathbb{S}^\times$ ; in any case it determines a stable spherical fibration over  $X$  which can be identified with the Spivak normal bundle. (To see this, note that the Thom complex of this spherical fibration is  $\text{Hom}_R(k, R) \otimes_R k \sim \text{Hom}_R(k, k) = \mathcal{E}$ , and the top cell has a spherical reduction given by the unit homomorphism  $\mathbb{S} \rightarrow \mathcal{E}$ .) For some more details see [31].

**10.7. The sphere spectrum.** Let  $R = \mathbb{S}$  and  $k = \mathbb{F}_p$ . The map  $R \rightarrow k$  is not Gorenstein; in fact, by Lin's theorem [35],  $\text{Hom}_R(k, R)$  is trivial. Neeman has suggested that  $\mathbb{S}$  should be considered to be a "fairly ordinary non-Noetherian ring" [40, §0]. If so, then Lin's theorem is perhaps analogous to the classical calculation that if  $R = \mathbb{F}_p[x_1, x_2, \dots]$  and  $k = R/I$  for  $I = \langle x_1, x_2, \dots \rangle$ , then  $\text{Hom}_R(k, R) = 0$ .

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