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ON FREQUENCY RESPONSE FOR NONLINEAR SYSTEMS

by

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Abstract

In this paper we define a 'generalized frequency response' for a nonlinear input-output map S_{x_0} as the mapping $\mathcal{I} \circ \mathcal{F} \circ S_{x_0} \circ \mathcal{F}^{-1} \circ \mathcal{I}^{-1}$ from ℓ^2 to ℓ^2 where $\mathcal{F}$ is the Fourier transform and $\mathcal{I}$ the usual isomorphism from $L^2[-\infty, \infty]$ to ℓ^2. Realization results relative to linear and bilinear systems are presented. Also sufficient conditions for $\mathcal{L}^2$-stability of bilinear systems are derived.

equivalent.

The paper is organised as follows: In section 2 we shall consider sufficient conditions for a bilinear system to be $\mathcal{L}^2$-stable. In section 3 we define a 'GFR' for a nonlinear input-output map. We shall illustrate the 'GFR' for linear and bilinear systems. Finally a realization theory will be presented in section 4. We shall make use in the sequel of the following notation: The Fourier transform F of a square integrable function f is defined by

$$F(i\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\, e^{-i\omega t}\, dt$$

If x is an n-vector and A is an nxn-matrix, we take as compatible norms:

$$\| x \| = \max_{i=1,n} | x_i | \text{ and } \| A \| = \max_{i=1,n} \sum_{k=1}^{n} | a_{i,k} |$$

$\|\cdot\|_1$ and $\|\cdot\|_{2w}$ are the norms associated with the standard L^1 and L^2_w spaces, the space of absolutely integrable functions and the space of square integrable functions (with weight w) respectively. Whereas, ℓ^2 designates the space of square summable sequences.

2. $\mathcal{L}^2$-stability of bilinear systems:

In this section we shall present sufficient conditions for the $\mathcal{L}^2$-stability of bilinear systems. By $\mathcal{L}^2$-stability we understand that: $\forall u \in \Omega \subset L^2_w[0,\infty] \longrightarrow y \in L^2_w[0,\infty]$ where u and y are respectively the input and the output of the system and Ω is defined by the sufficient conditions. The weight function w is such that $w(t)e^{-2\alpha t} \in L^1[0,\infty]$ for some $\alpha \geq 0$, $w(t) \geq 0$ for $t \geq 0$.

Consider for simplicity the single input-single output system described by

$$\begin{cases} \dot{x} = Ax + uNx + Bu \quad , \quad x(0) = 0 \\ y = Cx \end{cases} \tag{2.1}$$

where A, N, B, C are constant matrices of suitable dimensions.

As is well known [3], the input-output map is given by:

$$\begin{aligned} y(t) &= Ce^{At}x_0 \\ &+ \sum_{j\geq 1} \int_0^t \int_0^{\sigma_j} \cdots \int_0^{\sigma_2} v_j(t,\sigma_1,\cdots,\sigma_j)u(\sigma_1)\cdots u(\sigma_j)d\sigma_1\cdots d\sigma_j \\ &+ \sum_{j\geq 1} \int_0^t \int_0^{\sigma_j} \cdots \int_0^{\sigma_2} w_j(t,\sigma_1,\cdots,\sigma_j)u(\sigma_1)\cdots u(\sigma_j)d\sigma_1\cdots d\sigma_j \end{aligned} \tag{2.2}$$

where

$$v_1(t,\sigma_1) = Ce^{A(t-\sigma_1)}B$$

$$v_j(t,\sigma_1,\cdots,\sigma_j) = Ce^{A(t-\sigma_j)}Ne^{A(\sigma_j-\sigma_{j-1})}N\cdots Ne^{A(\sigma_2-\sigma_1)}B \quad , \quad j>1$$

and

$$w_1(t,\sigma_1) = Ce^{A(t-\sigma_1)}x_0$$

$$w_j(t,\sigma_1,\cdots,\sigma_j) = Ce^{A(t-\sigma_j)}Ne^{A(\sigma_j-\sigma_{j-1})}N\cdots Ne^{A(\sigma_2-\sigma_1)}Ne^{A\sigma_1}x_0 \quad j>1.$$

<u>**Theorem 1 :**</u>

A sufficient condition for the system (2.1) to be $\mathcal{L}^2$-stable is that the following holds:

(i) the eigenvalues of A have real parts less than $-\alpha$,

(ii) u , $e^{\alpha t}u \in L^1[0, \infty]$.

<u>**Proof :**</u>

It follows from (i) as a standard result in the theory of differential equations that there exists a positive constant K such that

$$\|e^{At}\| \leq Ke^{-\alpha t} \quad , \quad t \geq 0 .$$

We have,

$$\int_0^\infty w(t)y^2(t)dt \leq \int_0^\infty w(t)\cdot\Big[\|C\|\cdot\|x_0\|\cdot K\cdot e^{-\alpha t}$$

$$+\|C\|\cdot\|B\|\cdot K\cdot\int_0^t e^{-(t-\sigma_1)\alpha}\;|u(\sigma_1)|d\sigma_1$$

$$+\sum_{j\geq 2}\int_0^t\int_0^{\sigma_j}\cdots\int_0^{\sigma_2}\|C\|\cdot\|B\|\cdot\|N\|^{j-1}e^{-(t-\sigma_1)\alpha}$$

$$\cdot K^j|u(\sigma_1)|\cdots|u(\sigma_j)|d\sigma_1\cdots d\sigma_j$$

$$+\|C\|\cdot\|x_0\|\cdot K\cdot\int_0^t e^{-(t-\sigma_1)\alpha}|u(\sigma_1)|d\sigma_1$$

$$+\sum_{j\geq 2}\int_0^t\int_0^{\sigma_j}\cdots\int_0^{\sigma_2}\|C\|\cdot\|x_0\|\cdot\|N\|^j\cdot K^j\cdot e^{-\alpha t}$$

$$\cdot|u(\sigma_1)|\cdots|u(\sigma_j)|d\sigma_1\cdots d\sigma_j\Big]^2\;dt \tag{2.3}$$

therefore,

$$\int_0^\infty w(t)y^2(t)dt \leq \int_0^\infty w(t)e^{-2\alpha t}dt\cdot\Big[\|C\|\cdot\|x_0\|\cdot K$$

$$+\|C\|\cdot\|B\|\cdot K\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1$$

$$+\sum_{j\geq 2}\|C\|\cdot\|B\|\cdot\|N\|^{j-1}\cdot K^j\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1\cdot\frac{\|u\|_1^{j-1}}{(j-1)!}$$

$$+\|C\|\cdot\|x_0\|\cdot K\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1$$

$$+\sum_{j\geq 2}\|C\|\cdot\|x_0\|\cdot\|N\|^j\cdot K^j\cdot\frac{\|u\|_1^{j-2}}{j!}\Big]^2 \tag{2.4}$$

hence,

$$\|y\|_{2w}^2 \leq \int_0^\infty w(t)e^{-2\alpha t}dt\cdot\|C\|^2\cdot\Big[K\cdot\|x_0\| + K\cdot\|B\|\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1$$

$$+K\cdot\|B\|\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1\cdot(e^{K\cdot\|N\|\cdot\|u\|_1}-1)$$

$$+K\cdot\|x_0\|\cdot\int_0^\infty e^{\sigma_1\alpha}|u(\sigma_1)|d\sigma_1$$

$$+\|x_0\|\cdot(e^{K\cdot\|N\|\cdot\|u\|_1}-1-K\cdot\|N\|\cdot\|u\|_1\Big]^2 < \infty \tag{2.5}$$

3. Generalized frequency response of Nonlinear input-output maps:

Consider a system S given in terms of an input-output map

$$S : R^n x L_w^2[0, \infty] \longrightarrow L_w^2[0, \infty] \qquad (3.1)$$

defined by

$$y(t) = S(x_0 , u(\cdot))(t) \qquad (3.2)$$

where u and y are respectively the input and the output of the system and x_0 is the initial state in some given state-space realization. For each fixed initial state x_0 , we have a map

$$S_{x_0} \triangleq S(x_0 , \cdot) : L_w^2[0, \infty] \longrightarrow L_w^2[0, \infty] \qquad (3.3)$$

For simplicity, we have assumed scalar input and scalar output.

In a recent paper [1], we introduced the notion of a 'Generalized Frequency Response' by using the natural isomorphism $\mathcal{J}$ between $L_w^2[0,\infty]$ and ℓ^2 in the time-domain, and then defining the 'G.F.R ' as the induced map from ℓ^2 to ℓ^2 such that the diagram

$$\begin{array}{ccc} L_w^2[0, \infty] & \xrightarrow{S_{x_0}} & L_w^2[0, \infty] \\ \mathcal{J} \downarrow & & \downarrow \mathcal{J} \\ \ell^2 & \xrightarrow{s_{x_0}} & \ell^2 \end{array} \qquad (3.4)$$

commutes.

Alternatively, we can use the natural isomorphism between $L^2[-\infty,\infty]$ and ℓ^2 in the frequency domain (i.e., we operate on the Fourier transforms of the input and output) and then defining the 'G.F.R ' as the induced map from ℓ^2 to ℓ^2 such that the diagram

$$
\begin{array}{ccc}
L^2[0,\infty] & \xrightarrow{S_{x_0}} & L^2[0,\infty] \\
\mathcal{F}\downarrow & & \downarrow\mathcal{F} \\
L^2[-\infty,\infty] & \xrightarrow{\hat{S}_{x_0}} & L^2[-\infty,\infty] \\
\mathcal{I}\downarrow & & \downarrow\mathcal{I} \\
\ell^2 & \xrightarrow{s_{x_0}} & \ell^2
\end{array}
\tag{3.5}
$$

commutes.

Let $\{e_j\}_{j\geq 0}$ be a basis of $L^2[0,\infty]$ and $E_j=\mathcal{F}\{e_j\}$, $j\geq 0$. Recalling the fact that the scalar product is invariant under the Fourier transform, we deduce that $\{E_j\}_{j\geq 0}$ is a basis of $L^2[-\infty,\infty]$. Let $\mathcal{I}$ denotes the usual isomorphism

$$\mathcal{I}:L^2[-\infty,\infty] \longrightarrow \ell^2$$

given by $$\mathcal{I}(F) = \{F_j\}_{j\geq 0} \tag{3.6}$$

where $$F \in L^2[-\infty,\infty] \ , \ F = \sum_{j\geq 0} F_j E_j$$

explicitly s_{x_0} is given by

$$s_{x_0} = \mathcal{I} o \mathcal{F} o S_{x_0} o \mathcal{F}^{-1} o \mathcal{I}^{-1} \tag{3.7}$$

that is $$s_{x_0}(\{U_k\}_{k\geq 0}) = \{Y_j\}_{j\geq 0} \tag{3.8}$$

where $U_k = \langle \mathcal{F}(u), E_k \rangle$, $Y_j = \langle \mathcal{F}(S_{x_0}(u)), E_j \rangle$

and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2[-\infty,\infty]$. We have

$$U_k = \langle u, e_k \rangle \text{ and } Y_j = \langle S_{x_0}(u), e_j \rangle$$

Let assume the systems at hand are $\mathcal{L}^2$-stable. We shall illustrate the expression (3.7) for the linear and the bilinear input-output maps.

Example 1: Linear systems

Consider the linear system

$$\begin{cases} \dot{x} = Ax + Bu & , \quad x(0) = x_0 \\ y = Cx \end{cases} \tag{3.9}$$

then the input-output map is given by

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \qquad (3.10)$$

In this case

$$S(x_0,u(\cdot))(t) = g_0(t) + (g*u)(t) \qquad (3.11)$$

where $g_0(t) = Ce^{At}x_0$, $g(t) = Ce^{At}B$

and $*$ denotes the convolution operator.

Taking the Fourier transform of both sides of (3.10) we obtain

$$Y(i\omega) = G_0(i\omega) + G(i\omega)\cdot U(i\omega) \qquad (3.12)$$

Let $x_0 = 0$ and introduce a basis $\{E_j\}_{j\geq 0}$ of $L^2[-\infty,\infty]$.

We obtain,
$$Y_l = \sum_{j\geq 0} G_{lj}\cdot U_j \quad , \; l\geq 0 \qquad (3.13)$$

where $Y(i\omega) = \sum_{l\geq 0} Y_l\cdot E_l(i\omega)$, $U(i\omega) = \sum_{j\geq 0} U_j\cdot E_j(i\omega)$

$$G_{lj} = \langle G(i\omega)\cdot E_j(i\omega), E_l(i\omega) \rangle$$

We therefore see that the matrix representation of the linear operator $s : \ell^2 \longrightarrow \ell^2$ for the linear system above, with respect to the basis $(1,0,0,\cdots),(0,1,0,\cdots),\cdots$ of ℓ^2 is $(G_{lj})_{l,j\geq 0}$.

Example 2: Bilinear systems

Consider the bilinear system (2.1). Let $N = N_1\cdot N_2$ where the dimensions of N, N_1, N_2 are respectively nxn, nxm, mxn, and

$$\begin{aligned} g_0(t) &= Ce^{At}B \quad , \; g_1(t) = Ce^{At}N_1 \\ g_2(t) &= N_2e^{At}N_1 \quad , \; g_3(t) = N_2e^{At}B \\ g(t) &= Ce^{At}x_0 \quad , \; g_4(t) = N_2e^{At}x_0 \end{aligned} \qquad (3.14)$$

Equation (2.2) can therefore be rewritten as

$$\begin{aligned} y(t) = g(t) &+ (g_0*u)(t) \\ &+ \sum_{j\geq 2} (g_1*[u(g_2*[u(\cdots u(g_2*[u(g_2*[u(g_3*u)])])\cdots)])])(t) \\ &+ (g*u)(t) \\ + \sum_{j\geq 2} &(g_1*[u(g_2*[u(\cdots u(g_2*[u(g_2*[u(g_2*(g_4\cdot u))])])\cdots)])])(t) \end{aligned} \qquad (3.15)$$

where g_2 appears (j-2) times in the first summation and (j-1) times in the second.

Taking the Fourier transform of both sides of (3.15) and using its properties we obtain

$$Y(i\omega) = G(i\omega) + G_0(i\omega)\cdot U(i\omega) + \sum_{j\geq 2} \frac{1}{\sqrt{2\pi}^{\,j-1}} (G_1\cdot[U*(G_2\cdot[U*(\cdots$$

$$\cdots U*(G_2\cdot[U*(G_2\cdot[U*(G_3\ U)])])\cdots)])])(i\omega) + G(i\omega)\cdot U(i\omega)$$

$$+ \sum_{j\geq 2} \frac{1}{\sqrt{2\pi}^{\,j}} (G_1\cdot[U*(G_2\cdot[U*(\cdots$$

$$\cdots (G_2\cdot[U*(G_2\cdot[U*(G_2\cdot(G_4*U))])])\cdots)])])(i\omega) \qquad (3.16)$$

Let $x_0 = 0$ and introduce a basis $\{E_j\}_{j\geq 0}$ of $L^2[-\infty,\infty]$.

If $U(i\omega) = \sum_{j\geq 0} U_j E_j(i\omega)$ then (3.16) becomes

$$Y(i\omega) = \sum_{k_1\geq 0} U_{k_1} G_0(i\omega) E_{k_1}(i\omega)$$

$$+ \sum_{j\geq 2} \sum_{k_1\geq 0} \cdots \sum_{k_j\geq 0} U_{k_1}\cdots U_{k_j}\cdot \frac{1}{\sqrt{2\pi}^{\,j-1}} (G_1\cdot[E_{k_j}*(G_2\cdot[E_{k_{j-1}}$$

$$*(\cdots E_{k_4}*(G_2\cdot[E_{k_3}*(G_2\cdot[E_{k_2}*(G_3\cdot E_{k_1})])])\cdots)])])(i\omega) \qquad (3.17)$$

Therefore,

$$Y_1 = \sum_{j\geq 1} \sum_{k_1\geq 0} \cdots \sum_{k_j\geq 0} V_{jk_1\cdots k_j 1}\cdot U_{k_1}\cdots U_{k_j} \qquad (3.18)$$

where $$V_{1k_1 1} = \langle G_0(i\omega)E_{k_1}(i\omega)\ ,\ E_1(i\omega)\rangle \qquad (3.19)$$

and

$$V_{jk_1\cdots k_j 1} = \frac{1}{\sqrt{2\pi}^{\,j-1}} \cdot\langle (G_1\cdot[E_{k_j}*(G_2\cdot[E_{k_{j-1}}*(\cdots$$

$$\cdots E_{k_4}*(G_2\cdot[E_{k_3}*(G_2\cdot[E_{k_2}*(G_3\cdot E_{k_1})])])\cdots$$

$$\cdots)])])(i\omega), E_1(i\omega)\rangle\ , \quad \text{for } j>1 \qquad (3.20)$$

Hence, the diagram (3.5) induces the map $s_{x_0}: \ell^2 \longrightarrow \ell^2$ given by

$$s_{x_0}((U_0,U_1,\cdots))_1 = \sum_{j\geq 1} \sum_{k_1\geq 0} \cdots \sum_{k_j\geq 0} V_{jk_1\cdots k_j 1}\cdot U_{k_1}\cdots U_{k_j} \qquad (3.21)$$

4. Realization theory:

In this section we shall consider the problem of the realizability and the state space realization of an analytic map $s : \ell^2 \longrightarrow \ell^2$ which defines a ' Generalized Frequency Response'. We shall present conditions under which such s is realized by a linear or a bilinear system.

4.A: Linear system :

Theorem 2 :

A necessary and sufficient condition for a sequence of numbers $\{G_{lj}\}_{l,j\geq 0}$ to be the 'Generalized Frequency Response ' of a linear system with zero initial condition (with respect to a given basis $\{E_k\}_{k\geq 0}$ of $L^2[-\infty,\infty]$) is that there exists a strictly proper rational function $G(i\omega)$ such that

$$\sum_{l\geq 0} G_{lj} \cdot E_l(i\omega) = G(i\omega) \cdot E_j(i\omega) \qquad (4.1)$$

for all $j\geq 0$. $G(i\omega)$ is then the Fourier transform of the impulse response of the linear system.

Proof :

immediate and shall be ommited.

4.B Bilinear systems :

Let e_k , $k\geq 0$ be the Laguerre functions defined by

$$e_k(t) = e^{-t/2} \sum_{m=0}^{k} \frac{(-1)^m}{m!} \binom{k}{m} t^m$$

They constitute a complete orthonormal basis for $L^2[0,\infty]$.

Consider E_k the Fourier transform of e_k , we have

$$E_k(i\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\frac{1}{2} + i\omega} \left[\frac{-\frac{1}{2} + i\omega}{\frac{1}{2} + i\omega}\right]^k$$

$\{E_k\}_{k\geq 0}$ is therefore a complete basis for $L^2[-\infty,\infty]$.

Remark 1 : The coefficients in the expansion of $F(i\omega)$ with respect to the basis $\{E_k\}_{k\geq 0}$ are the coefficients in the Taylor expansion of $\overline{F}$

$$\overline{F}(z) = \sqrt{2\pi} \frac{1}{1-z} F(\frac{1}{2} \frac{1+z}{1-z})$$

at z=0. So $F_k = \frac{\sqrt{2\pi}}{k!}\left[\frac{1}{1-z}\,F(\frac{1}{2}\,\frac{1+z}{1-z})\right]^{(k)}_{z=0}$

where $F(i\omega) = \sum_{k\geq 0} F_k\cdot E_k(i\omega)$.

Theorem 3 :

A necessary and sufficient condition for a sequence of numbers $\{V_{jk_1\cdots k_j l}\}_{j\geq 1, k_1,\cdots,k_j, l\geq 0}$ to be the ' Generalized Frequency Response ' of a bilinear system with zero initial condition (with respect to the given basis $\{E_k\}_{k\geq 0}$ of $L^2[-\infty,\infty]$) is that there exist four matrices $G_0(i\omega)$, $G_1(i\omega)$, $G_2(i\omega)$, $G_3(i\omega)$ with dimensions respectively 1x1, 1xm, mxm, mx1 of strictly proper rational functions such that

(i) $\sum_{l\geq 0} V_{1k_1 l}\, E_l(i\omega) = G_0(i\omega)\cdot E_{k_1}(i\omega)$ (4.2)

(ii) $$\sum_{l\geq 0} V_{jk_1\cdots k_j l}E_l(i\omega) = \frac{1}{k_j!\cdots k_2!}\, D_j^{k_j}\cdots D_2^{k_2}\cdot \cdot[G_1(i\omega)\cdot G_2(i\omega-z_j)\cdot G_2(i\omega-z_j-z_{j-1})\cdots \cdots G_2(i\omega-z_j-\cdots-z_3)\cdot G_3(i\omega-z_j-\cdots-z_2)\cdot \cdot E_{k_1}(i\omega-z_j-\cdots-z_2)\cdot(-\tfrac{1}{2}+z_2)^{k_2}\cdots(-\tfrac{1}{2}+z_j)^{k_j}]\Big|_{z_2=\cdots=z_j=-\frac{1}{2}}$$

for all $k_1,\cdots,k_j\geq 0$, $j>1$ (4.3)

where $D_j^{k_j} = \partial^{k_j}/\partial z_j^{k_j}$.

Proof :

(i) (4.2) can readily be obtained from (3.9) by premultiplying by $E_l(i\omega)$ and summing over $l\geq 0$. Whereas (3.14) yields $G_0(i\omega) = C(i\omega-A)^{-1}B$.

(ii) From (3.20) we obtain

$$\sum_{l\geq 0} V_{jk_1\cdots k_j l}E_l(i\omega) = \frac{1}{\sqrt{2\pi}^{\,j-1}}\,(G_1\cdot[E_{k_j}*(G_2\cdot[E_{k_{j-1}}*(\cdots \cdots E_{k_4}*(G_2\cdot[E_{k_3}*(G_2\cdot[E_{k_2}*(G_3\cdot E_{k_1})])])\cdots)])])(i\omega)$$

Starting from the inner square bracket and proceeding outward, we obtain, after a change of variable $i\Omega = z$ and the use of residue theory,

$$-\{E_{k_2} * (G_3 \cdot E_{k_1})\}(i\omega) = \frac{\sqrt{2\pi}}{k_2!} D_2^{k_2}[G_3(i\omega - z_2)\cdot E_{k_1}(i\omega - z_2) \cdot (-\tfrac{1}{2} + z_2)^{k_2}]\Big|_{z_2=-\frac{1}{2}}$$

$$-\{E_{k_3} * \{G_2 \cdot [E_{k_2} * (G_3 \cdot E_{k_1})]\}\}(i\omega) = \int_{-\infty}^{\infty} E_{k_3}(i\Omega) G_2(i\omega - i\Omega) \frac{\sqrt{2\pi}}{k_2!} \cdot D_2^{k_2}\Big[G_3(i\omega - i\Omega - z_2)\cdot E_{k_1}(i\omega - i\Omega - z_2)\cdot(-\tfrac{1}{2} + z_2)^{k_2}\Big]_{z_2=-\frac{1}{2}} d\Omega$$

$$= \frac{\sqrt{2\pi}}{k_2!}(-i)\int_{-i\infty}^{i\infty} E_{k_3}(z_3) G_2(i\omega - z_3)\cdot D_2^{k_2}\Big[G_3(i\omega - z_3 - z_2)\cdot E_{k_1}(i\omega - z_3 - z_2)\cdot(-\tfrac{1}{2} + z_2)^{k_2}\Big]_{z_2=-\frac{1}{2}} dz_3$$

$$= \frac{(\sqrt{2\pi})^2}{k_3! k_2!} D_3^{k_3}\Big[G_2(i\omega - z_3)\cdot D_2^{k_2}\Big[G_3(i\omega - z_3 - z_2)\cdot E_{k_1}(i\omega - z_3 - z_2)\cdot(-\tfrac{1}{2} + z_2)^{k_2}\Big]_{z_2=-\frac{1}{2}} \cdot (-\tfrac{1}{2} + z_3)^{k_3}\Big]_{z_3=-\frac{1}{2}}$$

$$-\{E_{k_j} * (G_2 \cdot [E_{k_{j-1}} * (\cdots E_{k_3} * G_2 \cdot [E_{k_2} * (G_3 \cdot E_{k_1})]\cdots)])\}(i\omega) = \frac{\sqrt{2\pi}^{\,j-1}}{k_j!\cdots k_2!} D_j^{k_j}\Big[G_2(i\omega - z_j)\cdot D_{j-1}^{k_{j-1}}\Big[G_2(i\omega - z_j - z_{j-1})\cdots$$
$$\cdots D_3^{k_3}\Big[G_2(i\omega - z_j - \cdots - z_3)\cdot D_2^{k_2}\Big[G_3(i\omega - z_j - \cdots - z_2)\cdot E_{k_1}(i\omega - z_j - \cdots - z_2)\cdot(-\tfrac{1}{2} + z_2)^{k_2}\Big](-\tfrac{1}{2} + z_3)^{k_3}\Big]$$
$$\cdots(-\tfrac{1}{2} + z_{j-1})^{k_{j-1}}\Big]\cdot(-\tfrac{1}{2} + z_j)^{k_j}\Big]_{z_1=\cdots=z_j=-\frac{1}{2}}$$

Hence, (4.3).

From (3.14) we get

$$G_0(i\omega) = C(i\omega I - A)^{-1}B \quad , \quad G_1(i\omega) = C(i\omega I - A)^{-1}N_1$$
$$G_2(i\omega) = N_2(i\omega I - A)^{-1}N_1 \quad , \quad G_3(i\omega) = N_2(i\omega I - A)^{-1}B \qquad (4.6)$$