Existence of Periodic Solutions in n-Dimensional Retarded Functional Differential Equations

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Abstract

The exitence of periodic orbits of n-dimensional delay systems of the form $\mathring{x}(t) = -f(x(t-p))$ is proved and applied to systems of the form

$$\mathring{x}(t) = -x(t-1)\mathbb{N}(x(t)) \ ,$$

and a cerain type of Hamiltonian system .

1. Introduction

Scalar delay equations of the form

$$\dot{x}(t) = -f(x(t-1)) \tag{1.1}$$

have been studied by many authors ; see , for example , Jones (1962) , Nussbaum (1974) , Kaplan and Yorke (1974) . These equations have important applications in population dynamics and nonlinear equations of the form

$$\dot{x}(t) = -x(t-1)N(x(t)) \tag{1.2}$$

can be transformed into equations of type (1.1) , under certain mild conditions. Moreover , it is well-known that these equations are related to nonlinear Hamiltonian systems of the form

$$\dot{x}(t) = -f(y(t))$$

$$\dot{y}(t) = f(x(t))$$

(Kaplan and Yorke , (1974) , Nussbaum and Peitgen (1984)) .

In this paper we shall generalize some of the above results to the multi-dimensional case of systems

$$\dot{x}(t) = -f(x(t-p)) \tag{1.3}$$

where $x(t) \in \mathbb{R}^n$ for each t and $f:\mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable map (with further properties to be introduced later) . We shall generalize the transformation of a system of the form (1.2) to that of (1.3) by solving the nonlinear overdetermined system of partial differntial equations

$$\nabla f = N(f) ,$$

where now $N:\mathbb{R}^n \to \mathbb{R}^n$, and $\nabla f = (\partial f_i/\partial x_j)$.

2. Fixed Point Theory

In this section we shall give a brief introduction to the fixed point index; for more details see Browder (1960) , Nussbaum (1974) , Dold (1965) and Thompson (1969) . If X is a compact , metric ANR (absolute neighbourhood retract - a space which is such that for any normal space Y and any map $f:A\to X$, where A is a closed subset of Y , f can be extended to a neighbourhood of A in Y) and $f:X\to X$ is a continuous map , then we define

$$i_X(f,X) = \sum_{i\geq 0} (-1)^i \mathrm{tr}(f_{*,i})$$

where $f_{*,i}:H_i(X) \to H_i(X)$ is the induced homology morphism (using Cech homology). $i_X(f,X)$ is also denoted by $\Lambda(f)$ and is called the *Lefschetz number* of X . It is a classical result of algebraic topology that f has a fixed point if and only if $i_X(f,X)\neq 0$ (See Spanier , 1966) . This invariant can be generalized to the case of a continuous map $f:G \to X$ where G is an open subset of X for which the set $S = \{x\in G : f(x) = x\}$ is compact or empty , and leads to an integer-valued function $i_X(f,G)$ called the *fixed point index* of f in G which satisfies the properties :

(1) If $i_X(f,G)\neq 0$ then f has a fixed point .

(2) If $S \subseteq G_1 \cup G_2$ where G_1 and G_2 are disjoint open subsets of G , then

$$i_X(f,G) = i_X(f,G_1) + i_X(f,G_2) .$$

(3) If $F:G\times[0,1] \to X$ is a homotopy such that $S' = \{(x,t)\in G\times[0,1]:F(x,t)=x\}$ is compact , then

$$i_X(F_1,G) = i_X(F_0,G) ,$$

where $F_t(x) = F(x,t)$.

(4) If $f:X \to X$ is continuous , then

$$i_X(f,X) = \Lambda(f) .$$

Now let X be a general topological space , $x_0 \in X$, and W an open neighbourhood of x_0 . Then if $f:W-\{x_0\} \to X$ is a continuous map , we say that x_0 is an *ejective point* of f if there exists an open neighbourhood U of x_0 such that there exists an integer $m>0$ such that $f^m(x)$ is defined and $f^m(x) \notin U$ for all $x \in U-\{x_0\}$.

3. An Abstract Form for the Delay Equation

In order to apply fixed point theory to obtain periodic solutions of the delay equation

$$\dot{x}(t) = -f(x(t-p)) \quad , \quad x(t-p) = \phi(t) \ , \ t \in [0,p] \tag{3.1}$$

where $\phi \in C([0,p];\mathbb{R}^n)$, we must express the equation in an abstract form on some space . Nussbaum (1974) uses a space of continuous functions and expresses the solution of (3.1) in terms of the initial function ϕ . We shall use a different approach here which allows a direct consideration of n-dimensional equations . First note the following simple lemma .

Lemma 3.1 A necessary condition for $x(t)$ to be a periodic solution of the equation (3.1) (with period 1) is that

$$\int_{t-1}^{t} f(x(t-p))dt = 0 \tag{3.2}$$

Proof This follows directly by integrating (3.1) , namely

$$x(t) - x(t-1) = -\int_{t-1}^{t} f(x(t-p))dt = 0$$

if x is periodic . □

Since any solution of (3.1) is differentiable for $t>0$, any periodic solution $x(t)$ must satisfy

$$x(t) = x(t-1) \quad , \quad \dot{x}(t) = \dot{x}(t-1) \quad , \quad \text{for } t>1$$

and so any such solution determines an element of the space $C^1[S^1;\mathbb{R}^n]$, where S^1 is the unit circle , and conversely . The 'time' axis $[0,1]$ is mapped onto S^1 by the map $t \to e^{2\pi it}$. We shall parameterize S^1 by $\{e^{i\theta} : 0\leq\theta\leq2\pi\}$. Let X denote the Banach space $C^1[S^1;\mathbb{R}^n]\times C[S^1;\mathbb{R}^n]$. We now define the 'rotation' operator $R_p:S^1 \to S^1$ by

$$R_p(\theta) = (\theta - 2\pi p)\bmod 2\pi \quad , \quad 0\leq\theta\leq2\pi \quad . \tag{3.3}$$

Denote by K the subspace $\{(x,2\pi\dot{x}) : x\in C^1[S^1;\mathbb{R}^n] \ , \ x(0)=0 \}$ of X and define the operator F on X by

$$F(x,y) = (w,z) \tag{3.4}$$

where

$$z(\theta) = -f(x(R_p(\theta))) \ , \tag{3.5}$$

and

$$w(\theta) = \frac{1}{2.\pi}\int_0^\theta z(\theta_1)d\theta_1 \tag{3.6}$$

(Note that in the definition of K , $\dot{x}$ means $dx/d\theta$.)

More generally , consider the equation

$$\dot{x}(t) = -f(x(t-p),x(t-q)) \tag{3.7}$$

with two delays p,q . F is defined again by (3.4) where z in (3.5) is replaced by

$$z(\theta) = -f(x(R_p(\theta)),x(R_q(\theta))) \ . \tag{3.8}$$

It is even possible to consider neutral equations of the form

$$\dot{x}(t) = -f(x(t-p),x(t+q))$$

in a similar way with (3.5) replaced by

$$z(\theta) = -f(x(R_p(\theta)),x(R_{-q}(\theta))) .$$

4. Existence of Periodic Solutions

In order to prove the existence of a periodic solution of the equation (3.1) (under siutable conditions on f , to be introduced shortly) we shall use the following theorem , due to Nussbaum (1974) .

Theorem 4.1 Let G be a closed , bounded , convex infinite-dimensional subset of a Banach space X , $x_0 \epsilon G$ and $F: G - \{x_0\} \to G$ a continuous compact map . Then if x_0 is an ejective point of F and U is an open neighbourhood of x_0 such that $F(x) \neq x$ for $x \epsilon \bar{U} \backslash \{x_0\}$, then $i_G(f, G-\bar{U}) = 1$ and f has a fixed point in $G-\bar{U}$. □

The first task is to find a closed , bounded , convex (infinite-dimensional) set G such that $F: G \to G$. For this we shall make the following assumption on f:

AF1 : There exists a number A>0 such that

$$\|x\| \leqslant A \Rightarrow \|f(x)\| \leqslant A .$$

Then we have

Lemma 4.2 If f satisfies AF1 then F maps the subset

$$G = \{(x, 2\pi\dot{x}) : \|(x, 2\pi x)\| \leqslant 2A \}$$

into itself .

Proof Since

$$x = \int_0^\theta \dot{x} \, d\theta$$

for $(x, 2\pi\dot{x}) \epsilon G$ we have

$$\|x\| \leqslant 2\pi \|\dot{x}\| .$$

Hence

$$2 \|x\| \leqslant 2\pi \|\dot{x}\| + \|x\| = \|(x, 2\pi\dot{x})\| \leqslant 2A ,$$

and so

$$\|f(x)\| \leqslant A ,$$

by AF1 . Now ,

$$F((x,2\pi\dot{x}))(\theta) = \left(-\frac{1}{2.\pi}\int_0^\theta f(x(R_p(\theta))), -f(x(R_p(\theta)))\right)$$

and since R_p is just a rotation it is clear that

$$\|F((x,2\pi\dot{x}))\| \leqslant 2\|f(x(R_p(.)))\|$$

$$\leqslant 2A . \quad \square$$

Next we show that $F:G \to G$ is a continuous compact map under the assumption

AF2 : f is continuously differentiable and

$$\|x\| \leqslant A \Rightarrow \|\nabla f(x)\| \leqslant B$$

for some $B>0$.

Lemma 4.3 If f satisfies AF1 and AF2 , then $F:G \to G$ is a continuous compact map.

Proof If $C \subseteq G$ is a bounded set then , for $(x,2\pi\dot{x}) \epsilon C$ we have

$$\frac{d}{d\theta}F(x,2\pi\dot{x}) = \left(-\frac{1}{2.\pi} f(x(R_p(\theta))) , - f(x(R_p(\theta)))\dot{x}\frac{dR_p}{d\theta}\right)$$

and so by AF2 the set $(d/d\theta)C = \{d\zeta/d\theta : \zeta \epsilon C\}$ is bounded . Hence the lemma follows from the Arzela-Ascoli theorem . $\square$

It remains , therefore , to show that 0 is an ejective point of F . To do this we restrict attention to the subspace G' of G consisting of elements $(x,2\pi\dot{x})$ such that x satisfies the condition

C: $x(\pi) = 0$, $x|_{(0,\pi)} > 0$, $x|_{(\pi,2\pi)} < 0$

$x|_{[0,\pi]}$ is symmetric about $t=\pi/2$

$x|_{[\pi,2\pi]}$ is symmetric about $t=3\pi/2$

x is monotonically increasing on $[0,\pi/2] \cup [3\pi/2,2\pi]$ and monotonically decreasing on $[\pi/2,3\pi/2]$.

Furthermore , suppose that f satifies the condition

AF3 : $\begin{cases} f(x_1,\dots,x_n) > 0 & \text{if } x_1>0,\dots,x_n>0 \ , \\ f(x_1,\dots,x_n) < 0 & \text{if } x_1<0,\dots,x_n<0 \ . \end{cases}$

Then it is easy to check that the restriction F' of F to G' maps G' into G' , provided

$$p = 1/4 \ . \tag{4.1}$$

In order to show that 0 is an ejective point of G' we require a final condition on f , namely ,

AF4 : For each i , we have

$$\frac{\partial f}{\partial x_i}(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n) > 2\pi$$

for any $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\epsilon\mathbb{R}^{n-1}$.

such that $x_j \geqslant 0$ for all j or $x_j \leqslant 0$ for all j .

Then we have

Theorem 4.4 Under the conditions AF1-AF4 , equation (3.1) has a periodic solution (of period 1) if p satisfies (4.1) .

Proof G' is closed , bounded , convex and infinite-dimensional and so , by theorem 4.1 it is sufficient to show that 0 is an ejective point of F . If $0 \neq (x,2\pi\dot{x})\epsilon G'$ then , since x is monotonically increasing on $[0,\pi/2]\cup[3\pi/2,2\pi]$ there exists $i\epsilon\{1,\dots,n\}$ and $\alpha>0$ such that $|x_i(\theta)| \geqslant |v(\theta)|$, $t\epsilon[0,2\pi]$, where $v = \alpha\sin\theta$. Consider the operation of F on $(v,2\pi\dot{v})$. Since p=1/4 , F first rotates v counterclockwise by $\pi/2$, then applies -f and finally integrates the resulting function (and multiplies by $1/2\pi$) . If α is sufficiently small , then $|f(x_1,\dots,x_{i-1},\alpha v,x_{i+1},\dots,x_n)| > 2\pi\alpha|v|$ by AF4 and AF3 now shows that $F(v,2\pi\dot{v})=(\beta v,2\pi\beta\dot{v})$ where $\beta>1$. Hence 0 is an ejective point in G' and the result follows. □

Remark 4.1 In Nussbaum (1974) and Kaplan and Yorke (1974) , where scalar equations are considered with a delay of p=1 , the condition AF4 is replaced by

$$\frac{\partial f}{\partial x}(x) > \pi/2 .$$

The difference in the growth constants is entirely due to the difference in the delays . If theorem 4.4 is reworked for a delay of 1 then in condition AF4 , 2π is replaced by $\pi/2$.

Remark 4.2 The above proof does not require a delicate consideration of the linearized eigenvalue equation of the delay system used in Nussbaum (1974) .

5. Example

Several authors (see Nussbaum , 1974) have considered the scalar equation

$$\dot{x}(t) = -\alpha x(t-1)N(x(t))$$

for some nonlinear function N . Here we shall generalize this to an n-dimensional equation

$$\dot{x}(t) = -\alpha N(x(t))x(t-p) \qquad (5.1)$$

where N is a matrix-valued nonlinear function with properties to be specified later . In order to transform such an equation into the form (3.1) we must consider the following overdetermined system of nonlinear partial differential equations :

$$\nabla f(y) = N(f(y)) , \; f(0)=0 \qquad (5.2)$$

for the function $f:\mathbb{R}^n \to \mathbb{R}^n$, where

$$\nabla f = \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{pmatrix}$$

is the gradient matrix of f . We have the following necessary compatibility conditions for the solubility of (5.2) .

Lemma 5.1 If the equations (5.2) have a solution $f \in C^2$, then

$$\sum_p \frac{\partial N_{ij}}{\partial z_p}(z)N_{pk}(z) = \sum_p \frac{\partial N_{ik}}{\partial z_p}(z)N_{pj}(z) \qquad (5.3)$$

for any point $z \in$ range(f) and for all i,j,k .

Proof Clearly we have

$$\frac{\partial^2 f_i}{\partial y_k \partial y_j} = \frac{\partial^2 f_i}{\partial y_j \partial y_k}$$

and so , from (5.2) ,

$$\sum_p \frac{\partial N_{ij}}{\partial z_p}(f(y)) \frac{\partial f_p}{\partial y_k}(y) = \sum_p \frac{\partial N_{ik}}{\partial z_p}(f(y)) \frac{\partial f_p}{\partial y_j}(y) . \tag{5.4}$$

However , again by (5.2) we have

$$\frac{\partial f_p}{\partial y_k}(y) = N_{pk}(f(y)) . \quad \square$$

In order to find sufficient conditions for the existence of a (nontrivial) solution of (5.2) , suppose that f(y) is a solution of this equation which satisfies (5.4) . Then , if $\eta : I \subseteq \mathbb{R} \to \mathbb{R}^n$ is a path in $\mathbb{R}^n$, we have

$$\dot{f}(\eta(t)) = \nabla f \, \dot{\eta}(t)$$

and so

$$f(y) = \int_0^y N(f(\eta))d\eta$$

where the integral is taken along the path η . However , by (5.4) , the integral is independent of path and so f must satisfy the integral equation

$$f(y) = \int_0^y N(f(y))dy \tag{5.5}$$

for any path joining 0 to y .

The following theorem is proved in a similar way to the proof of theorem 4.4.

Theorem 5.2 Suppose first that N is such that the set G of functions $f \in C^2$ which satisfy f(0) = 0 , AF3 and (5.4) is nonempty , closed , bounded , convex and infinite-dimensional . Next assume that $N(z)$ is strictly positive in some neighbourhood of 0 . Then (5.5) has a solution for small enough $\|y\|$. □

(We consider the operator $\mathcal{N}: f(y) \to \int_0^y N(f(y))dy$ on G , which is well-defined by (5.4) and use the stated properties to show that $\mathcal{N}$ is compact and that the zero function is an ejective point of $\mathcal{N}$.)

Corollary 5.3 Under the conditions of theorem 5.2 , if , in addition , N is an analytic function , then f is analytic and has an analytic extension to some maximal region in $\mathbb{R}^n$. □

If the conditions of corollary 5.3 hold , let f be a solution of (5.2) and define

$$x = f(y) .$$

Then

$$\begin{aligned} \dot{x}(t) &= \nabla f(y).\dot{y}(t) \\ &= -\alpha N(x(t))x(t-p) \\ &= -\alpha N(f(y(t)))f(y(t-p)) \end{aligned}$$

if x(t) satifies equation (5.1) . Hence , if $y(t) \in U$ for all t we have

$$\dot{y}(t) = -\alpha f(y(t-p)) \tag{5.6}$$

and a periodic solution of (5.6) satisfying $y(t) \in U$ is also a periodic solution of (5.1) .

Consider , for example , the system

$$\dot{x}(t) = -\alpha \begin{pmatrix} 1 + x_1(t) & x_2(t) \\ x_2(t) & 1 + x_1(t) \end{pmatrix} x(t-p) \quad . \tag{5.7}$$

Then condition (5.3) is satisfied by

$$N(z) = \begin{pmatrix} 1 + z_1 & z_2 \\ z_2 & 1 + z_1 \end{pmatrix}$$

and the conditions in (5.4) become

$$\frac{\partial f_1}{y_2} = \frac{\partial f_2}{y_1} \tag{5.8}$$

$$\frac{\partial f_1}{\partial y_1} = \frac{\partial f_2}{\partial y_2} \tag{5.9}$$

The equations (5.2) reduce to

$$\begin{aligned} \frac{\partial f_1}{\partial y_1}(y_1, y_2) &= 1 + f_1(y_1, y_2) \\ \frac{\partial f_2}{\partial y_1}(y_1, y_2) &= f_2(y_1, y_2) \end{aligned} \tag{5.10}$$

with $f(0,0)=0$. Solving these equations gives

$$f_1 = -1 + e^{y_1}\cosh(y_2) \tag{5.11}$$

$$f_2 = e^{y_1}\sinh(y_2) \ . \tag{5.12}$$

Note that f_1 does not satisfy AF3 everywhere , but it does so on the region

$$|y_1| > |y_2| \ .$$

An easy extension of theorem 4.4 shows that the result remains valid in this case . Condition AF4 now shows that if $\alpha > 2\pi$ then equation (5.7) has a periodic solution .

As a second application we recall that Kaplan and Yorke (1974) used the two-dimensional Hamiltonian system

$$\begin{aligned} \dot{x} &= -f(y) \\ \dot{y} &= f(x) \end{aligned} \tag{5.13}$$

to prove the existence of periodic orbits in a scalar delay system of the form (3.1) . To do this they use a Lyapunov argument to prove the existence of a family of periodic solutions of (5.13) . This argument no longer works for 2n-dimensional systems of the form (5.13) . However , we can use theorem 4.4 to reverse the argument and show the existence of periodic solutions of (5.13) from the existence of periodic solutions of (3.1) . In fact , if f satisfies the conditions of theorem 4.4 and moreover , f is odd , i.e.

$$f(-x) = -f(x) \quad , \tag{5.14}$$

then we put

$$y(t) = x(t-p)$$

in (3.1) and obtain

$$\dot{x}(t) = -f(y(t))$$

$$\dot{y}(t) = -f(x(t-2p)) \; .$$

With p=1/4 , using (5.14) and the fact that

$$x(t-2p) = -x(t) \; ,$$

the result follows .

6. Conclusions

In this paper we have generalized the existence theory of periodic solutions of scalar FDE's to the vector case . This has been achieved by the use of the fixed point index . Using a transformation which depends on the solution of the system

$$\nabla f = N(f)$$

we have given a simple example which generalizes the well-known one-dimensional case .

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