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**Realization
and
Generalized Frequency Response
for
Nonlinear Input-Output Maps**

by

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Abstract

The purpose of this paper is to develop a realization theory and present the 'generalized frequency response' for an input-output map introduced in a previous paper. Relations with the realization of finite Volterra series will be outlined. A simple example will be given to show that the new representations have improved convergence properties.



where

$$\begin{aligned}
 ad_X^0 Y &= Y \\
 ad_X Y = ad_X^1 Y &= [X, Y] = XY - YX \quad (\text{Lie bracket}) \\
 ad_X^{m+1} Y &= [X, ad_X^m Y] \quad , \quad m \geq 1
 \end{aligned} \tag{1.2}$$

2 Nonlinear input-output maps for bilinear systems :

Consider the bilinear system

$$\begin{aligned}
 \dot{x} &= Ax + \sum_{j=1}^m u_j N_j x, \quad x(0) = x_0 \\
 y &= Cx
 \end{aligned} \tag{2.1}$$

where A, N_j, C are constant matrices of suitable dimensions. We shall present in this section a new input-output map to represent this system.

Let $u_0(t) = 1, t \geq 0$ and $N_0 = A$, then (2.1) becomes

$$\begin{aligned}
 \dot{x} &= \sum_{j=0}^m u_j N_j x, \quad x(0) = x_0 \\
 y &= Cx
 \end{aligned} \tag{2.2}$$

Consider the change of variable

$$z = e^{-N_k \int_0^t u_k(\tau) d\tau} x \tag{2.3}$$

we have $z(0) = z_0 = x_0$.

Let $U_k(t) = \int_0^t u_k(\tau) d\tau$. It is easy to prove [4] that the nonlinear input-output

map is given by

$$\begin{aligned}
y(t) &= Ce^{N_k U_k(t)} x_0 + \\
&+ \sum_{l \geq 1} \sum_{j_1 \neq k} \dots \sum_{j_l \neq k} \int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{l-1}} \\
&C e^{N_k [U_k(t) - U_k(\tau_l)]} N_{j_l} e^{N_k [U_k(\tau_l) - U_k(\tau_{l-1})]} N_{j_{l-1}} \\
&\dots N_{j_2} e^{N_k [U_k(\tau_2) - U_k(\tau_1)]} N_{j_1} e^{N_k U_k(\tau_1)} x_0 \\
&u_{j_l}(\tau_l) \dots u_{j_1}(\tau_1) d\tau_1 \dots d\tau_l
\end{aligned} \tag{2.4}$$

Returning to (2.1), the other representation [4] is given by

$$\begin{aligned}
y(t) &= C\Phi(t, 0)x_0 \\
&+ \sum_{k \geq 1} \int_0^t \int_0^{\tau_k} \dots \int_0^{\tau_2} C\Phi(t, \tau_k) A\Phi(\tau_k, \tau_{k-1}) A \\
&\dots A\Phi(\tau_2, \tau_1) A\Phi(\tau_1, 0)x_0 d\tau_1 \dots d\tau_k
\end{aligned} \tag{2.5}$$

where, $\Phi(t, \tau) = \Gamma^{-1}(t)\Gamma(\tau)$ and Γ is an $n \times n$ matrix solution of

$$\dot{\Gamma} = -\Gamma \sum_{j=1}^m u_j N_j, \quad \Gamma(0) = I \tag{2.6}$$

Example:

As a trivial example, we consider A and N commuting. Thus, the input-output map corresponding to the system

$$\begin{aligned}
\dot{x} &= Ax + uNx \\
y &= Cx
\end{aligned}$$

is given by

$$y(t) = Ce^{At + Nu(t)} x_0 \tag{2.7}$$

Therefore, the series representation introduced here (see also [4]) and the Volterra series are given by

$$y_1(t) = Ce^{NU(t)} \sum_{k \geq 0} \frac{(At)^k}{k!} x_0$$

$$y_2(t) = Ce^{At} \sum_{k \geq 0} \frac{(NU(t))^k}{k!} x_0$$

respectively. Obviously, if a truncation is to be made, y_1 will give a better estimate than y_2 for most u 's.

Moreover, in our representation, we have the freedom to select the stable part of the system, whereas in Volterra series representation, we are bound to take a stable in order for the series to converge on $[0, \infty]$ [9]. Of course, one of the main advantages of Volterra series representation is its 'multilinear' nature.

3 Input-Output stability of bilinear systems :

In this section we shall present sufficient conditions for the L^∞ - stability of bilinear systems. Detailed proofs can be found in [4]. We claim the following:

Theorem 1:

A sufficient condition for the system (2.2) to be L^∞ - stable is that the following hold:

(i) There exist at least one N_j ($j = 0, \dots, m$) (say N_k) having all its eigenvalues with negative real parts,

(ii) $\lim_{t \rightarrow \infty} \int_0^t [-\rho_k u_k(\tau) + \alpha_k \|P_k(\tau)\|] d\tau < \infty$

where $\alpha_k > 0$ and $\rho_k > 0$ are such that $\| e^{N_k t} \| \leq \alpha_k e^{-\rho_k t}$, $P_k(t) = \sum_{j \neq k} u_j N_j$ and $u_k \geq 0$.

Furthermore, if the limit is $-\infty$ then $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

Corollary 1:

A sufficient condition for the system (2.2) to be L^∞ -stable is that the following hold:

(i) There exist at least one N_j ($j = 0, \dots, m$) (say N_k) having all its eigenvalues with negative real parts,

(ii) $\lim_{t \rightarrow \infty} \int_0^t [-\rho_k u_k(\tau) + \alpha_k \sum_{j \neq k} |u_j(\tau)| \|N_j\|] d\tau < \infty$

where $\alpha_k > 0$ and $\rho_k > 0$ are such that $\| e^{N_k t} \| \leq \alpha_k e^{-\rho_k t}$, $P_k(t) = \sum_{j \neq k} u_j N_j$ and $u_k \geq 0$.

Furthermore, if the limit is $-\infty$ then $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

Theorem 2:

A sufficient condition for the system (2.2) to be L^∞ -stable is that the following hold:

(i) There exist a positive constant α and a non-decreasing function ρ satisfying

$\rho(0) = 0$ such that $\| \Phi(t, \tau) \| \leq \alpha e^{-[\rho(t) - \rho(\tau)]}$ for $t \geq \tau \geq 0$

(ii) $\lim_{t \rightarrow +\infty} [-\rho(t) + \alpha \|A\| t] \neq \infty$

Furthermore, if the limit is $-\infty$ then $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

Corollary 2:

A sufficient condition for the system to be L^∞ - stable is that u_j , $j = 1, \dots, m$ satisfy the following inequality:

$$\min\{\rho_1 - \alpha_1 \| E_1 + \sum_{j=1}^m u_j N_j \|, -\rho_2 + \alpha_2 \| E_2 - \sum_{j=1}^m u_j N_j \| \} \geq \alpha_1 \alpha_2 \| A \|$$

where E_1 and E_2 are stable matrices satisfying $\| e^{E_i t} \| \leq \alpha_i e^{-\rho_i t}$, $t \geq 0$ for some $\alpha_i, \rho_i > 0$.

Hint: Take $\alpha = \alpha_1 \alpha_2$ and,

$$\rho(t) = \int_0^t \max \left\{ \rho_1 - \alpha_1 \left\| E_1 + \sum_{j=1}^m u_j(\tau) N_j \right\|, -\rho_2 + \alpha_2 \left\| E_2 - \sum_{j=1}^m u_j(\tau) N_j \right\| \right\} d\tau \quad (3.1)$$

and use theorem 2.

Remark:

The sufficient condition in corollary 2 has a nice geometric interpretation in terms of the location of $\sum_{j=1}^m u_j N_j$ with respect to the balls centered at $-E_1$ and E_2 with radius $\frac{\rho_1}{\alpha_1} - \alpha_2 \| A \|$ and $\frac{\rho_2}{\alpha_2} + \alpha_1 \| A \|$ respectively.

4 Realization of a nonlinear i/o map:

Consider the nonlinear input-output map given by

$$y(t) = \int_0^t \int_0^{\tau_1} \dots \int_0^{\tau_{l-1}} C e^{N[U(t)-U(\tau_1)]} A e^{N[U(\tau_1)-U(\tau_{l-1})]} A \dots A e^{N[U(\tau_{l-1})-U(\tau_1)]} A e^{NU(\tau_1)} x_0 d\tau_1 \dots d\tau_{l-1} \quad (4.1)$$

where again $U(t) = \int_0^t u(\tau) d\tau$.

Define z_1, \dots, z_{l+1} by

$$\begin{aligned}
 z_{l+1}(t) &= e^{NU(t)} x_0 \\
 z_l(t) &= \int_0^t e^{N[U(t)-(\tau)]} A z_{l+1}(\tau) d\tau \\
 &\vdots \\
 z_1(t) &= \int_0^t e^{N[U(t)-U(\tau)]} A z_2(\tau) d\tau
 \end{aligned} \tag{4.2}$$

Therefore, differentiating with respect to t , we obtain,

$$\begin{aligned}
 \dot{z} &= \bar{A}z + u\bar{N}z \\
 y &= \bar{C}z
 \end{aligned} \tag{4.3}$$

where

$$\bar{A} = \begin{pmatrix} 0 & A & 0 & \dots & 0 \\ 0 & 0 & A & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \bar{N} = \begin{pmatrix} N & 0 & 0 & 0 & 0 \\ 0 & N & 0 & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & \dots & N \end{pmatrix}$$

$$\bar{C} = (C, 0, \dots, 0), \quad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{l+1} \end{pmatrix}, \quad z(0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_0 \end{pmatrix} = z_0 \quad (4.4)$$

with dimensions $[(l+1)n] \times [(l+1)n]$, $[(l+1)n] \times [(l+1)n]$, $n \times [(l+1)n]$, $(l+1)n$ and $(l+1)n$ respectively.

Note that \bar{A} is nilpotent of degree l , i.e., $\bar{A}^l \neq 0$ and $\bar{A}^{l+1} = 0$.

If we consider now the map obtained from the $(l+1)^{\text{th}}$ term of the Volterra series, we obtain,

$$\begin{aligned} \dot{z} &= \bar{A}z + u\bar{N}z \\ y &= \bar{C}z \end{aligned} \quad (4.5)$$

where,

$$\bar{A} = \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ 0 & A & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & A \end{pmatrix}, \quad \bar{N} = \begin{pmatrix} 0 & N & 0 & \dots & 0 \\ 0 & 0 & N & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (4.6)$$

and $\bar{C} = \bar{C}$.

Therefore, we can see that

$$\begin{aligned}
 \bar{A} &= \mathcal{S}^+ \bar{A} \\
 \bar{N} &= \mathcal{S}^+ \bar{N} \\
 \bar{C} &= \bar{C}
 \end{aligned} \tag{4.7}$$

where \mathcal{S}^+ is the operator

$$\mathcal{S}^+ = \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & I \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{4.8}$$

An immediate generalization is to consider the input-output map defined by

$$\begin{aligned}
 y(t) &= e^{NU(t)} x_0 \\
 &+ \sum_{k=1}^l \int_0^t \int_0^{\tau_k} \dots \int_0^{\tau_2} C e^{N[U(t)-U(\tau_k)]} A \dots \\
 &\dots A e^{N[U(\tau_2)-U(\tau_1)]} A e^{NU(\tau_1)} x_0 d\tau_1 \dots d\tau_k
 \end{aligned} \tag{4.9}$$

If we define, as in section (4), z_1, \dots, z_{l+1} , we obtain

$$\begin{aligned}
 \dot{z} &= \bar{A}z + u\bar{N}z \\
 y &= C^*z
 \end{aligned} \tag{4.10}$$

where, this time, $\bar{C}^* = (C, \dots, C)$.

Similar result for the finite Volterra series can be obtained and $\bar{\bar{C}}^* = \bar{C}^*$.

Remark:

There is clearly a one to one map between this representation and the Volterra series representation, and since we know how to solve a minimal realization from the Volterra series, we can deduce a minimal realization for this representation. From the above discussion and the remark we have proved:

Theorem 3:

Given an i/o map of the form (4.9) there exists a minimal bilinear realization.

5 Generalized frequency response:

In [2] (see also [3]) we have introduced the notion of the 'generalized frequency response' as follows:

Consider a system S given in terms of an input-output map

$$S : R^n \times L_w^2[0, \infty] \longrightarrow L_w^2[0, \infty] \quad (5.1)$$

defined by

$$y(t) = S(x_0, u(\cdot))(t) \quad (5.2)$$

where u and y are respectively the input and the output of the system and x_0 is the initial state in some given state-space realization. For each fixed initial state x_0 , we have a map

$$S_{x_0} = S(x_0, \cdot) : L_w^2[0, \infty] \longrightarrow L_w^2[0, \infty] \quad (5.3)$$

For simplicity, we have assumed scalar input and scalar output.

Let \mathcal{I} denotes the natural isomorphism between $L_w^2[0, \infty]$ and ℓ^2 , then the 'generalized frequency response' for S is the induced map from ℓ^2 to ℓ^2 such that the diagram

$$\begin{array}{ccc}
 L_w^2[0, \infty] & \xrightarrow{S_{x_0}} & L_w^2[0, \infty] \\
 \mathcal{I} \downarrow & & \downarrow \mathcal{I} \\
 \ell^2 & \xrightarrow{s_{x_0}} & \ell^2
 \end{array}$$

commutes.

In this section, we shall derive the 'generalized frequency response' associated with the input-output map (4.9). First we shall Consider the bilinear system

$$\begin{aligned}
 \dot{x} &= Ax + uNx, & x(0) &= x_0 \\
 y &= Cx
 \end{aligned} \tag{5.4}$$

We have,

$$x(t) = e^{NU(t)}x_0 + \int_0^t e^{N[U(t)-U(\tau)]}Ax(\tau)d\tau \tag{5.5}$$

from which we get

$$y(t) = e^{NU(t)}x_0 +$$

$$\begin{aligned}
& + \sum_{k \geq 1} \int_0^t \int_0^{\tau_k} \dots \int_0^{\tau_2} C e^{N[U(t)-U(\tau_k)]} A \dots \\
& \dots A e^{N[U(\tau_2)-U(\tau_1)]} A e^{NU(\tau_1)} x_0 d\tau_1 \dots d\tau_k \quad (5.6)
\end{aligned}$$

We claim the following:

Theorem 4:

A sufficient condition for the nonlinear operator h defined by (5.6) to map $L_w^2[0, \infty] \cap L^\infty[0, \infty]$ into itself is that the following hold:

- (i) N has all its eigenvalues with negative real parts,
- (ii) there exists a function η such that $\int_0^t [-\rho u(\tau) + \alpha \|A\|] d\tau \leq \eta(t) < \infty$ with $\lim_{t \rightarrow \infty} \eta(t) = -\infty$.

where $\alpha > 0$, $\rho > 0$ are such that $\|e^{Nt}\| \leq \alpha e^{-\rho t}$, $t \geq 0$.

Proof:

(5.5) yields

$$\|x(t)\| \leq \alpha e^{-\rho U(t)} \|x_0\| + \alpha \int_0^t e^{-\rho[U(t)-U(\tau)]} \|A\| \|x(\tau)\| d\tau \quad (5.7)$$

Using Gronwall's lemma, we obtain

$$e^{\rho U(t)} \|x(t)\| \leq \alpha \|x_0\| e^{\|A\|t} \quad (5.8)$$

Therefore,

$$\begin{aligned}
|y(t)| & \leq \alpha \|C\| \|x_0\| e^{\int_0^t [-\rho u(\tau) + \alpha \|A\|] d\tau} \\
& \leq \alpha \|C\| \|x_0\| e^{\eta(t)} < \infty
\end{aligned}$$

hence,

$$\begin{aligned}\|y(t)\|_w^2 &= \int_0^\infty |y(t)|^2 w(t) dt \\ &\leq \alpha^2 \|C\|^2 \|x_0\|^2 \int_0^\infty w(t) e^{2\eta(t)} dt\end{aligned}\quad (5.9)$$

Using the Cauchy-Bouniakovsky inequality we obtain,

$$\|y\|_w^2 \leq \alpha^2 \|C\|^2 \|x_0\|^2 \left\{ \int_0^\infty w^2(t) dt \cdot \int_0^\infty e^{2\eta(t)} dt \right\}^{\frac{1}{2}} < \infty \quad (5.10)$$

Theorem 5:

The 'generalized frequency response' associated with the input-output map (4.9) is the map $s : \ell^2 \rightarrow \ell^2$ defined by $\bar{y} = s(\bar{u})$ where

$$\begin{aligned}y_k &= Cv_k^0 x_0 \\ &+ \sum_{m=1}^l \sum_{j_1 \geq 0} \dots \sum_{j_{m+1} \geq 0} \beta_{j_1 \dots j_{m+1}}^{m,k} Cv_{j_{m+1}}^0 v_{j_m} \dots v_{j_1} x_0\end{aligned}\quad (5.11)$$

and $v_k^0 = v_k^0(\bar{u})$ and $v_k = v_k(\bar{u})$ are given by (5.20) and (5.21).

Proof :

Consider v_0 and v defined by

$$\begin{aligned}v_0(t) &= e^{NU(t)} \\ v(t) &= e^{-NU(t)} A e^{NU(t)}\end{aligned}\quad (5.12)$$

We have

$$v_0(t) = \sum_{m \geq 0} N^m \frac{1}{m!} U^m(t) \quad (5.13)$$

and the Baker-Campbell-Hausdorff formula (1.1) yields

$$v(t) = \sum_{m \geq 0} (-1)^m \text{ad}_N^m A \frac{1}{m!} U^m(t) \quad (5.14)$$

but

$$\frac{1}{m!} U^m(t) = \int_0^t \int_0^{\tau_m} \dots \int_0^{\tau_2} u(\tau_1) \dots u(\tau_m) d\tau_1 \dots d\tau_m \quad (5.15)$$

Thus

$$v_0(t) = \sum_{m \geq 0} N^m \int_0^t \int_0^{\tau_m} \dots \int_0^{\tau_2} u(\tau_1) \dots u(\tau_m) d\tau_1 \dots d\tau_m \quad (5.16)$$

and

$$v(t) = \sum_{m \geq 0} (-1)^m \text{ad}_N^m A \int_0^t \int_0^{\tau_m} \dots \int_0^{\tau_2} u(\tau_1) \dots u(\tau_m) d\tau_1 \dots d\tau_m \quad (5.17)$$

Assume u , v , and v_0 are in $L_w^2[0, \infty]$, so we can write $u = \sum_{k \geq 0} u_k e_k$,

$v = \sum_{k \geq 0} v_k e_k$, and $v^0 = \sum_{k \geq 0} v_k^0 e_k$

Now assume that the basis $\{e_j\}$ satisfies the assumption

(A) the functions

$$\alpha_{j_1 \dots j_m}^m : t \longrightarrow \int_0^t \int_0^{\tau_m} \dots \int_0^{\tau_2} e_{j_1}(\tau_1) \dots e_{j_m}(\tau_m) d\tau_1 \dots d\tau_m$$

and

$$\beta_{j_1 \dots j_{m+1}}^m : t \longrightarrow e_{j_{m+1}}(t) \int_0^t \int_0^{\tau_m} \dots \int_0^{\tau_2} e_{j_1}(\tau_1) \dots e_{j_m}(\tau_m) d\tau_1 \dots d\tau_m$$

belong to $L_w^2[0, \infty]$ for all indices.

Then, under assumption (A), we can write

$$\alpha_{j_1 \dots j_m}^m(t) = \sum_{k \geq 0} \alpha_{j_1 \dots j_m}^{m,k} e_k(t) \quad (5.18)$$

and

$$\beta_{j_1 \dots j_{m+1}}^m(t) = \sum_{k \geq 0} \beta_{j_1 \dots j_{m+1}}^{m,k} e_k(t) \quad (5.19)$$

Therefore,

$$v_k^0 = \sum_{m \geq 0} \sum_{j_1 \geq 0} \dots \sum_{j_m \geq 0} N^m \alpha_{j_1 \dots j_m}^{m,k} u_{j_1} \dots u_{j_m} = v_k^0(\bar{u}) \quad (5.20)$$

$$v_k = \sum_{m \geq 0} \sum_{j_1 \geq 0} \dots \sum_{j_m \geq 0} (-1)^m \text{ad}_N^m A \alpha_{j_1 \dots j_m}^{m,k} u_{j_1} \dots u_{j_m} = v_k(\bar{u}) \quad (5.21)$$

Thus,

$$y_k = C v_k^0 x_0 + \sum_{m=1}^l \sum_{j_{m+1} \geq 0} \dots \sum_{j_1 \geq 0} \beta_{j_1 \dots j_{m+1}}^{m,k} C v_{j_{m+1}}^0 v_{j_m} \dots v_{j_1} x_0 \quad (5.22)$$

where $y_k = \langle y, e_k \rangle_w$. Hence, we have proved the theorem.

6 Conclusion:

In this paper we have presented a realization theory for an input-output map introduced in [4], outlining the link with the realization of finite Volterra series. We have defined the ‘generalized frequency response’ [2](see also [3]) for this input-output map. A simple example has been included to show how this representation compares with the Volterra series representation.

References

- [1] C. Bruni, G. DiPillo and G. Koch, 'On the Mathematical Models Of Bilinear Systems ', *Ricerche di Automatica*, 2 pp. 11-26, 1972.
- [2] S. P. Banks and B. Chanane, 'A generalized frequency response for nonlinear systems ', *I.M.A. J. Of Math. Cont. & Inf.*, 5, 147-166, 1988.
- [3] S. P. Banks and B. Chanane, 'On frequency response for nonlinear systems ', to appear in the Conference Proceedings of the Fifth I.M.A. Conference on Control Theory, 1988.
- [4] B. Chanane and S. P. Banks, 'Nonlinear Input-Output Maps For Bilinear Systems and Stability ', submitted to *I.M.A. J. Of Math. Cont. & Inf.*, 1988
- [5] C. Lesiak and A. J. Krener, 'The Existence And Uniqueness Of Volterra Series For Nonlinear Systems ', *I.E.E.E. Trans. Aut. Cont.*, AC-23, pp. 1090-1095, 1978.
- [6] W. J. Rugh. **Nonlinear System Theory: The Volterra/Wiener Approach** Johns Hopkins Press, Baltimore, MD, 1981.
- [7] P. D'Alessandro, A. Isodori and Ruberti, 'Realization And Structure Theory Of Bilinear Systems ', *S.I.A.M. J. Control.* 12 pp 517-535, 1974.
- [8] S. P. Banks, **Mathematical Theories Of Nonlinear Systems** Prentice-Hall. 1988

- [9] R. W. Brockett, 'Convergence Of Volterra Series On Infinite Intervals And Bilinear Approximations', in 'Nonlinear Systems And Applications' edited by V. Lakshmikantham, Academic Press, 1977.
- [10] M. Fliess and F. Lamnabhi-Lagarrigue, 'Volterra Series and Optimal Control', in Algebraic and Geometric Methods in Nonlinear Control Theory, D. Reidel Publishing Company, (Eds.) M. Fliess and M. Hazewinkel, pp 371-387, 1986.
- [11] Bourbaki, Groupes et Algebre de Lie, Chap 2 et Chap 3, Hermann, Paris, 1972.

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