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DAY-TO-DAY DYNAMICS & EQUILIBRIUM STABILITY IN A TWO-MODE TRANSPORT SYSTEM WITH RESPONSIVE BUS OPERATOR STRATEGIES

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Abstract

This paper presents a day-to-day dynamic analysis of mode choice behaviour in a transportation system. Presented results, regarding a simple two-mode system, support the conjecture that multiple equilibria can likely be observed in such systems. This condition may have a great impact on the design of transit operator strategies.

1. INTRODUCTION

Traditional, steady-state equilibrium methods allowed us for many years to study the hypothetical behaviour of transport systems under conditions of constant demand, constant travel infrastructure/costs and constant policy measures. The field of study embodying dynamic transportation systems has opened up the possibility to study the more realistic changes that occur in demands, costs and policy responses, over multiple temporal scales. While initially many advances focused on developing dynamic methods for the within-day time-scale, assuming the between-day scale to be constant, a growing body of research is developing on the converse position, namely where the between-day scale is dynamic, assuming the within-day scale constant. The present paper falls within this second body of research.

In particular, in the present paper we shall exploit the ability of such day-to-day models in representing a situation where some "parameters" of the transport system are themselves responsive, on a day-to-day scale, to the flows on the transportation system, to reflect the reactions of some 'agent' representing a responsible transport authority. The feedback measures of this agent in turn will affect the travel experiences and subsequent decision of the other agents in this system, namely the travelling public, which in turn will affect the responsive parameters in the future. In such a complex, responsive, multi-agent environment, it is natural to question whether a given strategy may be successful in stabilising the transport system (or whether it may indeed lead to greater instability), or in directing the system to towards desirable long-term states. Day-to-day dynamic process models are especially powerful in such a context, providing the opportunity for analysis of theoretical properties of system convergence to different attractors (not necessarily equilibrium or fixed-point) such as existence, uniqueness and stability.

This paper presents a day-to-day dynamic analysis of transport mode choice behaviour in a transportation system, which allows us to analyse equilibrium stability and the effects of (and on) transit operator strategies.

Quite a large number of papers have now been proposed to deal with different aspects of day-to-day dynamics in single-mode private transport networks, using both deterministic and stochastic models (e.g. Friesz *et al*, 1994; Hickman & Bernstein, 1997; Cantarella & Cascetta, 1995; Cantarella and Velonà, 2003; Hazelton & Watling, 2004; Bie & Lo, 2010; Han & Du, 2012).

All of these approaches assume that the demand for private transport is invariant to the performance and policy measures in the public transport system. Although some results in the above-quoted papers may be extended to mode choice (effectively imagining the routes to be modes), there are several distinctive features of the choice of mode that are not captured by such an analogy, and to the authors' knowledge there have yet to be any papers specifically on this topic.

Typically the design of public transport systems is based on steady state analysis, often considering the public transport mode in isolation (see the review of Guihaire & Hao, 2008). While some studies exist that consider both public and private modes, this typically retains an implicit or explicit assumption of a steady state analysis, such as in the recent study of a bi-modal transportation system by Li *et al* (2012), with no consideration of the dynamics in travellers adjusting to any new policy measures.

Furthermore, in practice it is not only travellers that may make a dynamic adjustment (of their choice, in our case mode choice), but also the transit operators may also make dynamic adjustments, and our work is distinctive in including such operators as an active agent, rather than the typical approach adopted of considering policy measures as some abstract, external force. The dynamics of the system are therefore also affected by the responsive strategies of the transit operators, and in turn the dynamics of the system also shape the long-term nature of their strategies.

Again, some limited analogies exist with the study of private transport networks. Watling (1996), developing an example originally due to Smith (1979), explored the day-to-day dynamic evolution of route choice in a network in which the traffic signals were responsive to the traffic flows. It was seen that while two out of the three equilibria were locally stable, one of the stable equilibria was much more likely to evolve from given starting conditions. Similar behaviour was observed in examples with non-monotone cost functions, such as Morlok (1979) established can arise in when bus operators strategies in which frequencies are responsive to the demands.

The overall aim of our contribution is to stress that day-to-day dynamic models are useful and often needed to support transportation supply design (e.g. Cantarella, 2010). From the motivating background described above, the first objective of the paper is to propose a class of dynamic process models for mode choice evolution over time, in which the dynamics of the transit operator demand-responsive strategies are embedded.

According to the theory of dynamical systems, bifurcations from a stable equilibrium, considered as a fixed-point attractor, may lead towards other types of attractors, such as periodic (and then a-periodic) or quasi-periodic, or towards multiple equilibria. This theoretical result supports the conjecture that multi-mode transportation systems may likely present several point equilibria, some of them being non-stable, as shown by some simple numerical example.

Our second objective, therefore, is to present a theoretical analysis of the conditions for uniqueness and stability of point and other attractors within such a model framework. Among other things, such an analysis facilitates a study of how the stability and multiplicity of equilibria is affected by on-board crowding, on-street congestion, and transit operator strategies (e.g. in terms of fleet management, fare).

Our third and final objective is to make perform and report numerical experiments concerning the bifurcation of equilibrium and transient system states with respect to assumed control measures in our transportation system. In this way, our adopted day-to-day dynamic approach allows to analysis transient and to define policies to move towards a desired equilibrium.

For simplicity, reported results in this paper refer to a single OD pair connected by two transport modes: cars and buses sharing common streets, neglecting route choice behaviour. Our analysis will be based on a simple, multi-agent, bi-modal system, with users and bus operators as agents. In our approach, the level-of-service relevant for user mode-choice behaviour is influenced by on-street congestion, monetary cost (such as bus fares, fuel cost, and the like), bus in-vehicle crowding, and service frequency (as determined by bus operator behaviour).

The proposed model is meant to be a tactical/strategic planning tool, at such a decision level a frequency-based approach is usually applied, whilst the bus timetable is designed at lower decision levels.

The paper concludes by discussing extensions of this work to more general applications, as well to stochastic process models. Our multi-agent approach, from an explicitly (day-to-day) dynamic perspective, is also open to further extension, by including not only the transit operators and travellers, but also a city authority as a third kind of active agent. Such an approach may, for example, build on the work of Friesz & Shah (2001) on disequilibrium network design, if we presume that the city authority may be represented to be acting as if 'to maximize the net present value of benefits to users of a transportation network over a fixed planning horizon'.

The paper is structured as follows: section 2 describes the formulation of the model; this model is applied to a two-mode system in section 3 where some specifications are described. In section 4 first the results of some numerical examples are reported showing that multiple, possibly non-stable, equilibria may exist in multi-mode system; then a sensitivity analysis is carried out through a bifurcation analysis with respect to bus fare as well to demand flow and user behaviour dispersion; finally some consideration about transients are also included. In section 5 major findings are recapped and some research perspectives are outlined.

2. MODEL DESCRIPTION

This section describes the general formulation of the model to represent the evolution over time of modal split between two alternatives, car and bus, connecting one Origin-Destination (OD) pair. Presented model can easily be extended to more general transportation systems.

2.1 BASIC DEFINITIONS, NOTATIONS, EQUATIONS

Let $d \geq 0$ be the demand flow of users between the one OD pair, $f_a \geq 0$ is the flow of users who choose the auto, $f_b \geq 0$ the flow of users who choose the bus. All flows are supposed measured as users per time unit.

Let $\varphi_b \geq 0$ be the frequency of bus, $\chi_a > 1$ be the car occupancy. The flow variables in terms of people and vehicles for each of the two modes are:

- Car with respect to

users:	f_a	result of the mode choice behaviour;
vehicles:	f_a / χ_a	which is derived from the mean occupancy, assumed known;

- Bus with respect to

users:	f_b	result of the mode choice behaviour;
vehicles:	φ_b	which can be defined a priori or obtained as result of interaction between the company's strategy and the strategy of public transport users (see below).

Demand conservation is expressed by:

$$f_a + f_b = d$$

The total vehicle flow f_{tot} can be written as:

$$f_{tot} = f_a / \chi_a + \varphi_b \; \varepsilon_b$$

where ε_b is the passenger car equivalence of a bus with respect to a car in terms of its relative impact on capacity.

The car generalized transportation cost, w_a, is expressed by a linear combination of two attributes:

$$w_a = vot \; t_a + cm_a$$

where:
cm_a is the monetary cost of fuel;
t_a is the travel time by car;
vot is the value of time, common to all users.

Due to congestion the travel time by car t_a depends on the total vehicles flow f_{tot}:

$$t_a = t_a(f_{tot})$$

and so the cost w_a, since it is an affine transformation of the travel time:

$w_a = w_a(f_{tot})$

Similarly, the bus cost, w_b, is expressed as a linear combination of two attributes:

$w_b = vot\ t_b + cm_b$

where:
cm_b is the monetary cost of the bus ticket (fare);
t_b is the time to travel by bus;
vot is the value of time common to all users.

Bus travel time t_b is the sum of the bus running time, of the waiting time at stop and possibly of the on-boarding crowding disutility. Walk time is not considered for simplicity's sake. Let

t_r be the bus running time, including boarding and alighting times; two cases may occur (fluctuations of boarding and/or alighting times are not considered anyway):

- *reserved lanes*, the bus running time depends on timetable only, and it is a function of the zero-flow bus speed;
- *shared lanes*, two further "bracketing" cases may occur:
  - *strict timetable*, a bus speed allowance exists to compensate the effect of congestion in order to match the (pre-fixed) timetable (a time allowance at some stops may also be considered): thus the bus running time depends on timetable, and it is a function of the lowest bus speed;
  - *flexible timetable*, the bus travels as fast as possible given the congestion in order to maximize provided capacity: the bus running time depends on the total vehicles flow f_{tot}; this case will be considered in the following;

[Most real cases are in between the two above described cases, the former mainly occurring when low frequency service is provided, the later for high frequency]

t_w is the waiting time at stop, affected by the frequency of bus φ_b, multiplied by $\beta_w > 1$ to make homogenous with running time;
t_{cr} is the disutility due to on-board bus crowding (homogenized with respect to time), which may depend on the user flow f_b.

In the following two cases will be considered:

- Hp.U1: users do not consider the crowding disutility (because e.g. they do not perceive it, or because the capacity of the bus is much larger than the flow):

$t_b = t_r(f_{tot}) + \beta_w\ t_w(\varphi_b)$

- Hp.U2: users behaviour is affected by the crowding disutility:

$t_b = t_r(f_{tot}) + \beta_w\ t_w(\varphi_b) + t_{cr}(f_b)$

Hence in the most general case the transportation cost by bus w_b is a function of the total vehicles flow f_{tot} and the user flow f_b:

$w_b = w_b(f_{tot}, f_b)$

Since for demand conservation, $f_a + f_b = d$, the flows for the two alternatives are completely defined by a single variable, for example the fraction y of users who choose the car mode with $y = f_a/d$ with $f_b = (1 - y)\ d$, thus $f_{tot} = y\ d/\chi_a + \varphi_b\ \varepsilon_b$. Hence the difference between the cost of car and of bus can be expressed by the *supply function*:

$$w(y; d) = w_a(f_{tot}) - w_b(f_{tot}, f_b) = w_a(y\ d/\chi_a + \varphi_b\ \varepsilon_b) - w_b(y\ d/\chi_a + \varphi_b\ \varepsilon_b, (1 - y)\ d) \in R$$

In addition, the bus frequency may be pre-fixed by the bus operator, or updated according to the foreseeable load. Let

N_{bus} be the maximum number of available buses,
$t_g = 2t_r$ be the time to start from and return to the end of the bus route,
Q_b be the capacity of each bus.

Hence two "bracketing" strategies are considered to define bus frequency:

- *prefixed frequency*, meeting the demand with the all buses available (Hp.S1):

$$\varphi_b = N_{bus} / t_g$$

- *daily update of frequency*, meeting the demand with the minimum number of buses needed to avoid oversaturation (Hp.S2), given the number of available buses:

$$\varphi_b = \min(N_{bus} / t_g, f_b / Q_b)$$

[Most real strategies are in between the two above described cases, the latter being a limit case, since update (if any) usually occurs over a time scale lager than one day, in this case multi-time scale models (not yet available) should be applied.]

2.2 DYNAMIC DEFINITIONS, NOTATIONS, EQUATIONS

In day-to-day dynamics, each day some users reconsider the choice of the previous day while others just repeat it. Users reconsider the previous day choice in accordance with the forecasted travel time for each mode, which is the result of the learning process and/or interaction with any information system. All the notations introduced in the previous section also apply by just adding a superscript ^{t} to each of them.

The forecasted transportation cost for mode j (= a, car, = b, bus) z^t_j on the day t may be expressed through an exponential filter, which is a convex combination of actual cost and the forecasted cost on day $t - 1$, called *cost updating equation*:

$$z^t_j = \beta\ w_j + (1 - \beta)\ z^{t-1}_j \in R \qquad\qquad k = a, b$$

where:

$\beta \in\ ]0,1]$ is the weight given to yesterday actual costs when forecasting today costs; dispersion among users is modelled through parameters of path choice function, introduced below.

The cost updating equation can model two different conditions:

- users choose according to their own forecasts for travel time, and (possibly) to information shared with other users;
- users also receive information on travel times provided by an information system.

The value of the parameter β in the former case reflects the behaviour of users, in the latter case also depends on how the information system works. In this case it can (or better should carefully) be designed if forecasted costs are provided by an ITS. Multi-user class models, including equipped and non-equipped users, will be addressed in a future paper.

Moreover, each day only some users reconsider the choice of the day before (but not necessarily change) due to habit and/or inertia to change, and their choice behaviour can be modelled through any random utility model. The other users just repeat the choice made the day before. According to most choice models the probability of choosing car or bus, for a user reconsidering the previous day choice, is a function of the difference x^t between the forecasted cost of car and of bus on day t:

$$x^t = z^t_a - z^t_b = \beta\,(w_a - w_b) + (1-\beta)\,x^{t-1}$$

or

$$x^t = z^t_a - z^t_b = \beta\,w(y; d) + (1-\beta)\,x^{t-1} \in R \qquad (1)$$

Thus, the probability p of choosing car for a user who reconsiders the previous day choice can be expressed by the *demand function* (an example in sub-section 2.3):

$$p = p(x^t; \theta) \in [0, 1]$$

where:

θ is any relevant parameter of the choice model, assumed constant over time. Its value may be affected by information availability and reliability.

Therefore the fraction y^t of users who choose the car in day t is given by the *choice updating equation*:

$$y^t = \alpha\,p + (1-\alpha)\,y^{t-1}$$

or

$$y^t = \alpha\,p(x^t; \theta) + (1-\alpha)\,y^{t-1} \in [0, 1] \qquad (2)$$

where:

$\alpha \in\ ]0,1]$ is the fraction of users reconsidering the choice of the day before; this updating parameter is assumed constant over time. The value of this parameter may be affected by information available / provided to users, and likely by information reliability as well as compliance behaviour.

Recursive equations (1) and (2) specify a time-discrete non-linear dynamic system, or a deterministic process for short, with respect to the two state variables x^t and y^t. This model can easily be generalized to any number of alternatives.

A system is dissipative, that is its evolution over time converges whichever is the starting state, if the absolute value of the determinant of the Jacobian matrix of the dynamic equations is less than one.

The Jacobian matrix, $\mathbf{J}(x, y; d, \theta)$, of the equations (1) and (2) is given by:

$$\mathbf{J}(x, y; d, \theta) = \begin{bmatrix} 1-\beta & \beta\,(\partial w(y; d)/\partial y) \\ \alpha\,(1-\beta)\,(\partial p(x; \theta)/\partial x) & \alpha\,\beta\,(\partial w(y\,; d)/\partial y)\,(\partial p(x; \theta)/\partial x) + (1-\alpha) \end{bmatrix}$$

with determinant $(1 - \alpha)$ $(1 - \beta)$ that does not depend on the point (x, y) and is a function of the updating parameters α and β only. The determinant belongs to the interval $[0,1[$, since $\alpha, \beta \in]0,1]$, thus the system is dissipative.

2.3 FIXED POINT STATES: DEFINITION, EXISTENCE, UNIQUENESS, STABILITY

Fixed point states of the deterministic process model (1) and (2) are obtained from condition $x^t = x^{t-1} = x^*$ and $y^t = y^{t-1} = y^*$, thus:

$$x^* = \beta\, w(y^*; d) + (1 - \beta)\, x^*$$
$$y^* = \alpha\, p(x^*; \theta) + (1 - \alpha)\, y^*$$

therefore:

$$x^* = w(y^*; d)$$
$$y^* = p(x^*; \theta)$$

or

$$y^* = p(w(y^*; d); \theta) \tag{3}$$

Model (3) may be considered a parametric fixed-point problem with respect to parameters such as d and θ. Its solutions are consistent with SUE assignment.

Sufficient conditions for fixed-point *existence* can be easily derived through Brouwer theorem applied to model (3), requiring that both the supply function and the demand function are continuous (and the network is connected). Assuming that the demand function is monotone decreasing, as for invariant probabilistic path choice functions, if the supply function is monotone strictly increasing, fixed-point *uniqueness* is guaranteed. These uniqueness conditions can be weakened under mild assumptions (see the recent comprehensive review in Cantarella et al., 2010).

Fixed-point stability can easily be analysed, without explicitly running the deterministic process model, with respect to the values of the determinant and trace. Let

$\omega(x, y) = (\partial w(y; d)/\partial y)\ (\partial p(x; \theta)/\partial x) \in R$ be the product of the derivates of the supply function $w(y; d)$ and the demand function $p = p(x; \theta)$; it depends on the point (x, y), and on all parameters, such as d and θ, but the updating ones α and β;

$\tau(\alpha, \beta, \omega) = (1 - \alpha) + (1 - \beta) + \alpha\, \beta\, \omega \in R$ be the trace of matrix **J**; it depends on the point (x, y) through variable ω;

$\delta(\alpha, \beta) = (1 - \alpha)\ (1 - \beta) \in [0,1[$ is the determinant of matrix **J**; it does not depend on the point (x, y) (see above).

General results for two-dimensional dissipative systems yield that the stability region over the trace-determinant plane, for values of the determinant in the range [0,1], is the trapezium between the two lines $\delta = \tau - 1$, $\delta = -\tau - 1$, as in figure 1, where the parabolic function $\delta = \tau^2 / 4$ is also reported.

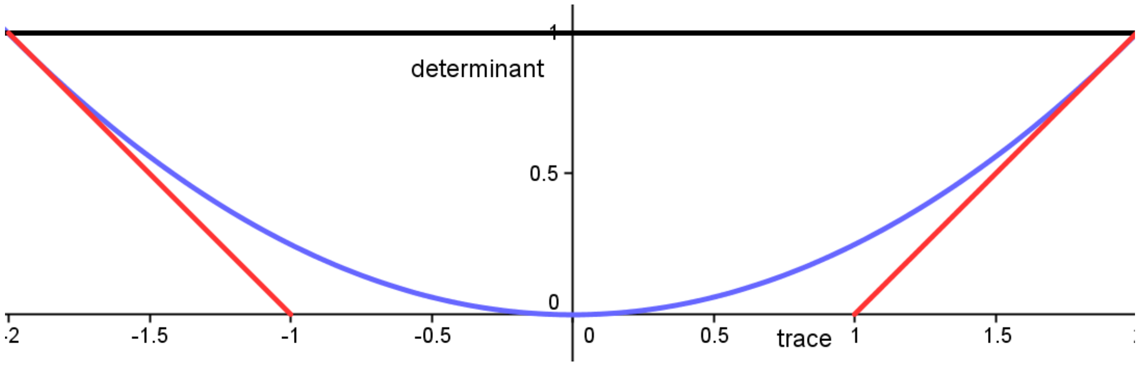


Figure 1 – Stability region over the trace-determinant plane.

Three cases may be distinguished (their order will be clear in the following).

Case ① $\delta \leq \tau^2 / 4$ and $\tau \geq 0$: the pair trace and determinant (τ, δ) is on the right side of the vertical determinant axes and below the parabola, stability is guaranteed by condition: $\delta > \tau - 1$;

Case ② $\delta \leq \tau^2 / 4$ and $\tau < 0$: the pair trace and determinant (τ, δ) is on the left side of the vertical determinant axes and below the parabola, stability is guaranteed by condition: $\delta > -\tau - 1$;

Case ③ $\delta > \tau^2 / 4$ the pair trace and determinant (τ, δ) above the parabola, stability is always guaranteed (this case is relevant for non dissipative systems with determinant greater than one.)

The above considerations only apply to two-dimensional systems, a more general approach is described below.

Applying the results of the theory of nonlinear dynamic systems, the (local) *stability* of a fixed point state (x^*, y^*) can be analysed considering the linearized model through the Jacobian matrix, $\mathbf{J}(x, y; d, \theta)$, of the dynamic equations (1) and (2):

The stability of a fixed point state (x^*, y^*) is guaranteed if both the eigenvalues λ_1^* and λ_2^* of the Jacobian calculated at the fixed point $\mathbf{J}(x^*, y^*; d, \theta)$ have modulus less than one, namely $|\lambda| < 1$. Spectral analysis of the Jacobian at a fixed-point $\mathbf{J}(x^*, y^*; d, \theta)$ allows us to check its stability without explicitly running the deterministic process model.

Eigenvalues λ_1 and λ_2 of the (2×2) matrix $\mathbf{J}$ at any point (x, y) are the solutions of the second order equation:

$$\lambda^2 - \tau\lambda + \delta = 0 \qquad \text{with } \lambda_1, \lambda_2 = (\tau \pm \sqrt{(\tau^2 - 4\delta)}) / 2$$

with $\lambda_1 + \lambda_2 = \tau$, and $\lambda_1 \lambda_2 = \delta$.

Three cases may occur depending on the values of the eigenvalues λ_1^* and λ_2^*. They may occur in a complex conjugate pair or be both real, in the latter case they have the same sign since the determinant equal to their product is non-negative.

Case ① the two eigenvalues λ_1* and λ_2* are both non-negative real,
hence the stability condition becomes λ_1 and $\lambda_2 < +1$;

Case ② the two eigenvalues λ_1* and λ_2* are both non-positive real,
hence the stability condition becomes λ_1 and $\lambda_2 > -1$;

Case ③ the two eigenvalues λ_1* and λ_2* are a complex conjugate pair,
with $| \lambda_1^* | = | \lambda_2^* | = \delta^{1/2} < 1$,
hence the stability condition always holds, and instability never occurs.

The three cases above match with the three cases already discussed with respect to trace and determinant.

The stability analysis can also be carried out in the parameter space, that is with respect to updating parameters α, β, and to the meta-parameter $\omega* = \omega(x^*, y^*;\ d, \theta)$; it is worth noting that the meta-parameter $\omega*$ may take as many values as fixed-points (x^*, y^*), and as already noted it does not depend on the updating parameters α and β. This approach is discussed below and compared with the two discussed above. Examples of all the three approaches will be discussed in sub-section 3.3.

Case ① $\tau^2 - 4\,\delta \geq 0$ and $\tau \geq 0$,
the two eigenvalues λ_1* and λ_2* are both real, since $\tau^2 - 4\,\delta \geq 0$,
have the same sign, since $\delta = \lambda_1 \lambda_2 \geq 0$,
are both non-negative, since $\tau = \lambda_1 + \lambda_2 \geq 0$;
after some algebra the stability condition yields: $0 \leq \tau \leq 1 + \delta$, or

$-((1 - \alpha) + (1 - \beta)) / (\alpha\,\beta) \leq \omega* \leq 1$;

thus, in this case instability occurs with $0 \leq 1 + \delta < \tau$ or
$1 < \omega*$.

Case ② $\tau^2 - 4\,\delta \geq 0$ and $\tau < 0$,
the two eigenvalues λ_1* and λ_2* are both real, since $\tau^2 - 4\,\delta \geq 0$,
have the same sign, since $\delta = \lambda_1 \lambda_2 \geq 0$,
are both negative, since $\tau = \lambda_1 + \lambda_2 < 0$,
after some algebra the stability condition yields: $-1 - \delta \leq \tau < 0$, or

$-2((1 - \alpha) + (1 - \beta)) / \alpha\,\beta - 1 \leq \omega* < -((1 - \alpha) + (1 - \beta)) / \alpha\,\beta$;

thus, in this case instability occurs with $\tau < -1 - \delta \leq 0$ or
$\omega* < -(1 + 2((1 - \alpha) + (1 - \beta)) / \alpha\,\beta)$

Case ③ $\tau^2 - 4\,\delta < 0$,
the two eigenvalues λ_1* and λ_2* are a complex conjugate pair,
with $| \lambda_1^* | = | \lambda_2^* | = \delta^{1/2} < 1$, hence, as already noted,
the stability condition always holds, and instability never occurs.

The above considerations allow us to map the stability conditions from the space (τ, δ) into the parameter space (α, β, ω^*). According to the above analysis the stability of a fixed-point state (x^*, y^*) may be assessed in three equivalent ways:

- by computing the Jacobian matrix at the fixed-point $\mathbf{J}(x^*, y^*; d, \theta)$, then its trace, τ, and determinant, δ, and mapping them into the stability region; this approach can only be applied to any system with two state variables only;

- by computing the Jacobian matrix at the fixed-point $\mathbf{J}(x^*, y^*; d, \theta)$, then its eigenvalues λ_1^* and λ_2^*, and mapping them into the unitary circle on the complex plane; this approach can be applied to any system with any number of state variables;

- by computing function $\omega = (\partial w(y;\ d)/\partial y)\ (\partial p(x;\ \theta)/\partial x)$ at the fixed-point ω^*, and checking whether it is within the range $]\ -(1 + 2((1\ -\ \alpha) + (1\ -\ \beta)) / \alpha\ \beta),\ 1\ [$; this approach can be extended to a transportation system with any number of state variables. It is worth noting that this approach is based on the value of one parameter only, instead of the two eigenvalues needed to apply the previous approach.

3. SOME SPECIFICATIONS

This section describes some specifications of the model (1) – (2), which stem from three scenarios obtained by combining the assumptions about user behaviour (Hp.U) and those relating to bus operator (Hp.S), as in the table 1, to check the effect of bus operator reacting to actual loads, and of crowding on user mode choice behaviour.

Table 1 - Specifications

	User disutility	Operator strategy
A)	Hp.U1 – no crowding disutility	Hp.S2 – minimum bus number
B)	Hp.U2 – crowding disutility	Hp.S2 – minimum bus number
C)	Hp.U2 – crowding disutility	Hp.S1 – all available buses

3.1 MODEL SPECIFICATIONS

The total user demand flow is expressed in the following with respect to total capacity supplied by car:

$d = \kappa\ Q_a\ \chi_a$

where:
$Q_a > 0$ is the capacity available for car;
κ is the level of demand, $\kappa = 1$ gives the maximum demand flow that can be served by car only without over saturation.

The mode choice behaviour is modelled by a binomial Logit model:

$p(x; \theta) = 1 / (1 + \exp(x/\theta))$

where:

$\theta > 0$ is the dispersion parameter, proportional to the standard deviation of perceived utility: $\theta = (\sqrt{6}/\pi)\, \sigma \cong 0.780\, \sigma$, it is measured consistently with costs.

The car travel time t_a is calculated using a (BPR-like) power function:

$t_a = t_o\, (1 + \mu_a\, (f_{tot}\, /Q_a)^{\nu_a})$

where:

$t_o > 0$ is the zero-flow time;
$\mu_a > 0$ and $\nu_a > 1$ are parameters.

The car travel time function may be further elaborated taking into account that the total vehicle flow is given by (sub-section 2.1):

$f_{tot} = (y\, d\, /\, \chi_a) + \varphi_b\, \varepsilon_b$

thus

$t_a = t_o\, (1 + \mu_a\, (y\, d\, /(\chi_a\, Q_a) + \varphi_b\, \varepsilon_b\, /\, Q_a\,)^{\nu_a}) \;\Rightarrow$
$t_a = t_o\, (1 + \mu_a\, (y\, \kappa + \varphi_b\, \varepsilon_b\, /\, Q_a\,)^{\nu_a})$

As already said, bus time travel is the sum of three terms:

$t_b = t_r + \beta_w\, t_w + t_{cr}$

- the running time is given by:

$t_r = (1 + \eta)\, t_a$

where:
$\eta \in [0, 1[$ models boarding and alighting extra-times as well as the lower speed of bus with respect to car;
- the waiting time is given by:

$t_w = 0.5\, /\, \varphi_b$

- the crowding disutility (specifications B) and C) only) is given by:

$t_{cr} = t_r\, (\mu_b\, (\, f_b\, /\, (Q_b\, \varphi_b))^{\nu_b})$

where:
$\mu_b > 0$ and $\nu_b > 1$ are parameters.

It should be noted that in order to compute car or bus travel time it is necessary to define the bus frequency φ_b.

- According to Hp.S1 (all available buses) – scenario C).

$\varphi_b = N_{bus}\, /\, (2\, (1 + \eta)\, t_a)$

Thus, for any given y, car travel time can be obtained by solving the following system of two nonlinear equations:

$t_a^* = t_o\,(1 + \mu_a\,((y\,d\,/\chi_a + \varphi_b^*\,\varepsilon_b)\,/\,Q_a\,)^{\nu_a})$
$\varphi_b^* = N_{bus}\,/\,(2(1 + \eta)\,t_a^*)$

To simplify notation, let

$t_a(\varphi_b; y) = t_o\,(1 + \mu_a\,((y\,d\,/\chi_a + \varphi_b\,\varepsilon_b)\,/\,Q_a\,)^{\nu_a})$
$\varphi_b(t_a) = N_{bus}\,/\,(2(1 + \eta)\,t_a)$

Then, the above system of two non-linear equations is equivalent to the (parametric) fixed point problem:

$t_a^* = t_a(\varphi_b(t_a^*); y)$ with $t_a \geq t_o > 0$ (4)

For any given y, existence of at least a solution of the above fixed-point problem is assured by Brouwer's theorem, since the car travel time t_a is upper bounded by $t_a(\varphi_b; 1)$ for any finite bus frequency φ_b, and the two functions $t_a(\cdot)$ and $\varphi_b(\cdot)$ are continuous (note $t_a \geq t_o > 0$). Moreover uniqueness is assured since $t_a(\cdot)$ is strictly increasing with respect to φ_b and $\varphi_b(\cdot)$ is strictly decreasing with respect to $t_a \geq t_o > 0$.
Let $t_a^*(y)$ be the function between the value of fraction of car users, y, and the corresponding car travel time, t_a^*. In the following function $t_a^*(y)$ and thus related functions such as the supply function $w(y; d, \omega_a)$ is assumed continuous as occurred in several experimental tests, not reported for brevity's sake. Still, this is an issue worth of further research efforts.

- According to Hp.S2 (minimum bus number) – scenarios A and B.

  $\varphi_b = \min(N_{bus}\,/\,(2\,(1 + \eta)\,t_a)\,, (1 - y)\,d\,/Q_b)$

  - If the result is $\varphi_b = (1 - y)\,d\,/Q_b$, for any given y car and bus travel times can directly be obtained;
  - If the result is $\varphi_b = N_{bus}\,/\,(2(1 + \eta)\,t_a)$, the above procedure should be followed.

Some examples of cost functions $w_a(y)$, $w_b(y)$, $w(y)$, are described below, consistent with above introduced travel time specifications, and equations in sub-section 2.1:

$w_a = vot\;t_a + cm_a$, $w_b = vot\;t_b + cm_b$, $w = w_a - w_b$

Numerical results discussed in the following are obtained with values of parameters reported in table 2 below, and vot = 0.10 € / min.

Table 2 - Values of parameters

Car	t_o	15 min	Bus	η	0.2
	χ_a	1.2		ε_b	3 car / bus
				β_w	2
	μ_a	0.33		μ_b	0.25
	ν_a	4		ν_b	3
	Q_a	1000 veihc / h		Q_b	100 seats
	cm_a	2.50 €		cm_b	{0.50, 0.90, 0.95, 1.50} €
				N_{bus}	3 bus

As it can be shown by numerical examples, not reported here, in most cases cost function are not monotone strictly increasing sufficient conditions for fixed-point uniqueness based on cost function monotonicity (cfr sub-section 2.3) are not guaranteed, then multiple fixed points may occur, possibly depending on demand level.

Generally it is not easy to find algebraically w'(y) since it is defined through the solution of fixed-point (4); thereby deducing an algebraic condition for w'(y) < 0 is still an open research issue.

4. UNIQUENESS, STABILITY, BIFURCATIONS

This section discusses uniqueness and stability of fixed-points for the three scenarios introduced earlier, and reports results of a bifurcations analysis.

4.1 STABILITY AND UNIQUENESS

This section compares, for an instance of scenario A) only, the different approaches to assessing the stability of a fixed-point, described in sub-section 2.3. Results of stability are matched with some considerations about uniqueness. All results are reported in figures 5 and 6 for low and high values of updating parameters α and β, respectively.

As noted in paragraph 2.3 fixed point states can be determined by solving the equation $y^* = p(w(y^*;\ d);\ \theta)$ which, according to the specification made above for the model of mode choice, becomes:

$$y^* = 1\ /\ (1 + \exp(w(y^*)\ /\ \theta)) \tag{5}$$

The one-dimensional fixed-point problem (5) may be solved by looking for the zeroes of the following function:

$$g(y) = y - 1/(1+\exp(w(y)/\theta)) \qquad \text{with } y \in [0, 1] \tag{6}$$

It should be noted that solution of the fixed point problem (4) is needed to compute function $g(y)$. In the first column of figures 2 and 3, function $g(y)$ is plotted against y showing that three fixed-points exist, but, as expected, they do not depend on the values of updating parameters α and β, as shown by a comparison between figures 5 and 6.

In the second column the three pairs of eigenvalues, λ_1^* and λ_2^*, corresponding to the three fixed points are shown with respect to the unitary circle, pointing out that:

- the middle fixed-point is not stable (white dot) since one eigenvalue is out the circle,
- the other two fixed-points are stable (black dots), since both eigenvalues are inside the circle.

The non-stable fixed-point is also the border between starting states leading to either stable fixed-point, as shown in the left column. It is worth recalling that the system is dissipative, thus the evolution over time always converges to some kind of attractor.

In the third column the three values of meta-parameter ω^* are matched with the stability range introduced in sub-section 2.3. It should be noted that meta-parameter ω^* does not depend on updating parameters, as expected. In scenario A), both the supply function $w(y)$ and the demand function $p(w)$ are monotone decreasing, thus the meta-parameter $\omega = (\partial w(y;\ d)/\partial y)\ (\partial p(x\ ;\theta)/\partial x)$ always gets values greater than zero, therefore only the non-negative part [0, 1[of the stability range is depicted (upper bound 1 is marked by a black triangle). It is worth noting that in this case the stability of a fixed-point is not affected at all by the updating parameters. Results are consistent with those in second column, the value of $\omega*$ being greater than 1 only for the middle fixed-point.

Finally in the forth column the value of determinant, δ, and trace, τ, are matched with the stability region in figure 1 (the stability region is not coloured to increase readability). The value of determinant only depends on updating parameters.

The above results stress that the static (equilibrium) analysis can evidence the number of fixed-points, but only through a dynamic their stability can be assessed.

4.2 BIFURCATIONS

In this section a sensitivity analysis is carried out through a bifurcation analysis with respect to bus fare as well to demand flow and to user behaviour dispersion, for each of three scenarios introduced above. In a **dissipative** system a variation of a parameter may move a stable fixed-point towards non-stability according three bifurcations:

- **pitchfork**, leading to multiple fixed-point states, when one positive real eigenvalue becomes greater than one, cfr case ① in sub-section 2.3;
- **flip**, leading to a periodic attractor (and possibly to an a-periodic one), when one negative real eigenvalue becomes less than one, cfr case ② in sub-section 2.3;
- **Neimark**, leading to quasi-periodic attractor, when the modulus of a pair of complex conjugate eigenvalues becomes greater than one, cfr case ③ in sub-section 2.3; as observed in sub-section 2.3.

Neimark bifurcations may never be observed in a bi-modal system (sub-section 2.3). If the two eigenvalues occur in a conjugate pair, their modulus is equal to the square root of their product, equal to the determinant, always less than one (as already noted).

In the proposed two-mode system pitchfork bifurcations are observed, as described below in figures 4, 5, and 6, for scenarios A), B), C) respectively. Four different values (rows) of bus fare, cm_b, and two different values (columns) of demand flow, d, are considered. This approach allows anticipating the effects of a change of bus fare or an increase of demand.

Each bifurcation diagram shows stable (black dots) and non-stable (white dots) equilibria against the dispersion parameter, θ, in the range [0, 6] likely values consistent with costs. It is worth noting that the lower the dispersion parameter, the less dispersed the user mode choice behaviour is. As this parameter goes to infinity, the dispersion of choice behaviour increases leading towards equal probabilities for both modes, a unique and stable fixed-point. When three fixed-points exist the middle one is always non-stable and separates the starting states leading to either stable fixed-point.

Figure 4 shows results for scenario A), the bus operator try to use the minimum number of bus neglecting the crowding effect. Multiple fixed points occur for value of the dispersion parameter in the range [0, 4] for low demand, or in the range [0, 2] for high demand. For low values of the dispersion parameter, the two stable fixed-points corresponds to all users on car or many of them on bus, but as the dispersion parameter increases the fixed-point “many-on-bus” moves towards an equal distribution between the two modes. This pattern is the unique fixed-point for high values of the dispersion parameter. As it may be easily anticipated, the lower the bus fare, the greater is the range of values of dispersion parameters allowing for the “many-on-bus” fixed-point. Hence, if the “all-on-car” fixed-point is to be avoided for social concern, the dispersion parameter should be kept low (for instance through advertising) and other measures should be implemented to move the modal split toward the “many-on-bus” fixed-point.

Figure 5 shows results for scenario B), the bus operator try to use the minimum number of bus but user behaviour is affected by the crowding effect. Multiple fixed points occur for value of the dispersion parameter greater than around 4 for low demand, or than around 2 for high demand. The two stable fixed-points correspond to all users on car or to an equal distribution between the two modes. For low values of the dispersion parameter, the “all-on-car” fixed-point is unique (and stable). Quite paradoxically, if the “all-on-car” fixed-point is to be avoided for social concern, the dispersion parameter should be kept high, so that an equal distribution between the two modes can be obtained.

Figure 6 shows results for scenario C), the bus operator always to use all the available buses, and user behaviour is affected by the crowding effect. A unique (and stable) fixed-point occurs for any value of the dispersion parameter, going towards an equal distribution between the two modes as the dispersion parameter increases. For low values of demand and bus fare modal split is in favour of bus.

All the above results show that the bus operator fleet management strategy and the bus fare (at least for low values of the dispersion parameter) affect the fixed-point pattern, but they may also affect uniqueness and stability. These considerations should carefully be taken into account when assessing any intervention in a bi-modal transportation system. Above results also point out that updating the number of buses and increasing the bus fare may have an effect much greater than expected, moving the system towards a completely different “all-on-car” fixed-point.

4.3 TRANSIENT

As well established (see for instance Cantarella and Velonà, 2003), the values of updating parameters not only affect fixed-point stability, but also the length of transient, that is the time required for the system reaching a fixed-point.

Figure 7 shows the transients for some instances of scenario A), with low (top) and high (bottom) values of demand; the updating parameters are equal for simplicity’s sake, but they need not to be. Different stable fixed-points are reached from different starting states; the non-stable starting state may only be obtained if the starting state is exactly equal to it.

The larger the values of the updating parameters, the shorter the transient; on the other hand for larger values the fixed point is more likely not stable. This effect is more relevant of high values of demand.

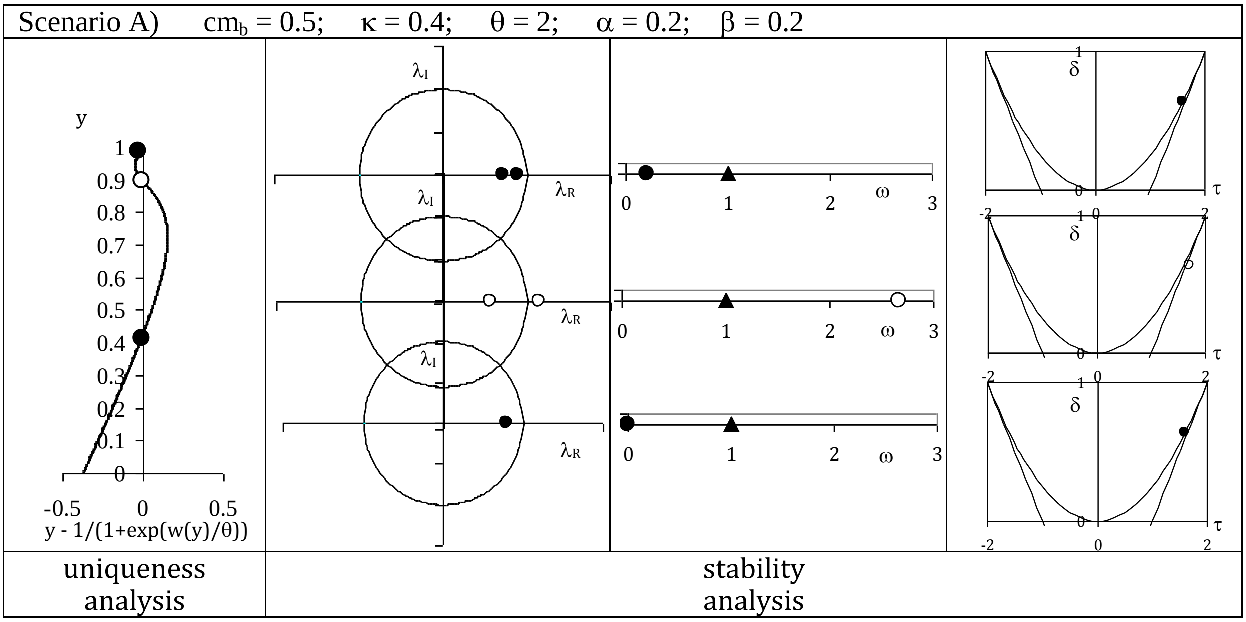


Figure 2 - Fixed-point uniqueness and stability for scenario A) – low values of α and β (black dots refer to stable fixed-points, white to non-stable).

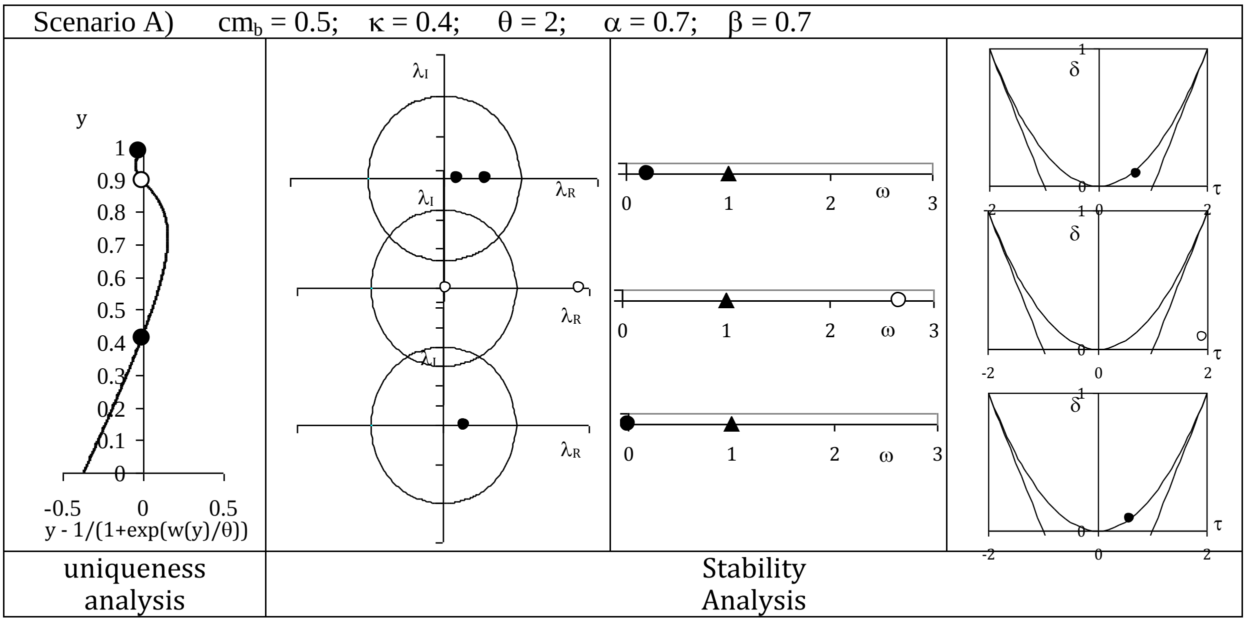


Figure 3 - Fixed-point uniqueness and stability for scenario A) – high values of α and β (black dots refer to stable fixed-points, white to non-stable).

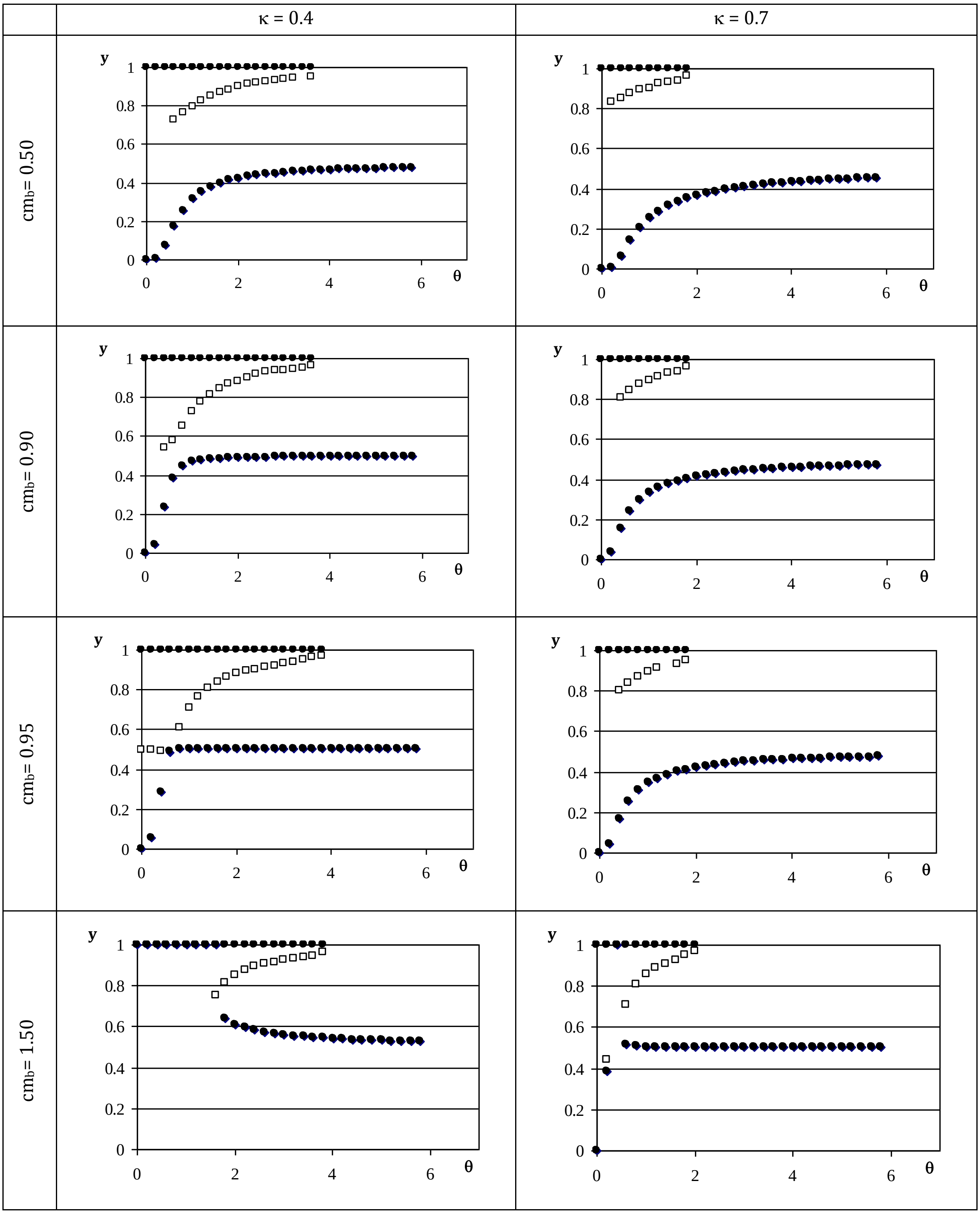


Figure 4 –Fixed-point bifurcations for scenario A)
(black dots refer to stable fixed-points, white to non-stable).

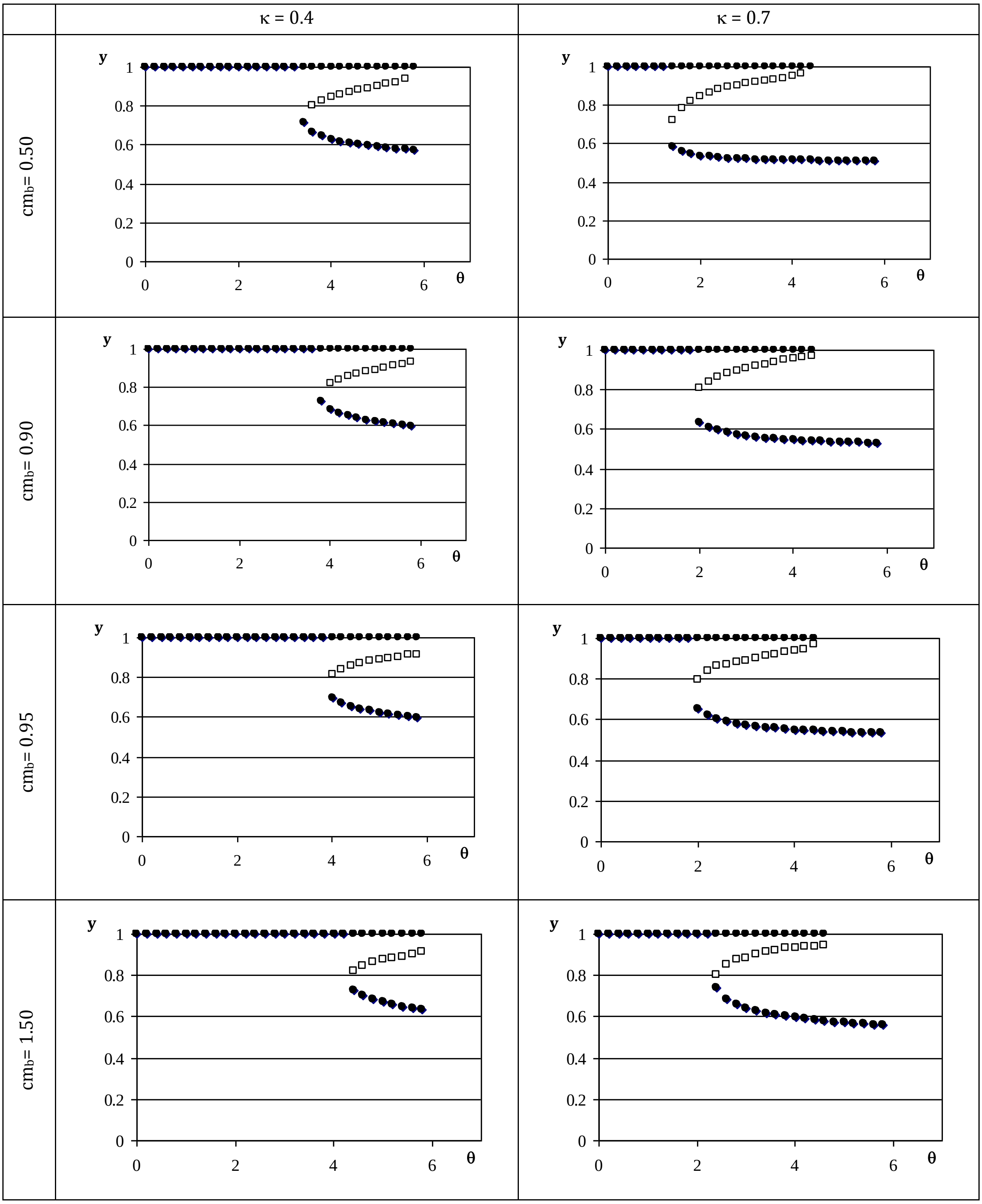


Figure 5 –Fixed-point bifurcations for scenario B)
(black dots refer to stable fixed-points, white to non-stable).

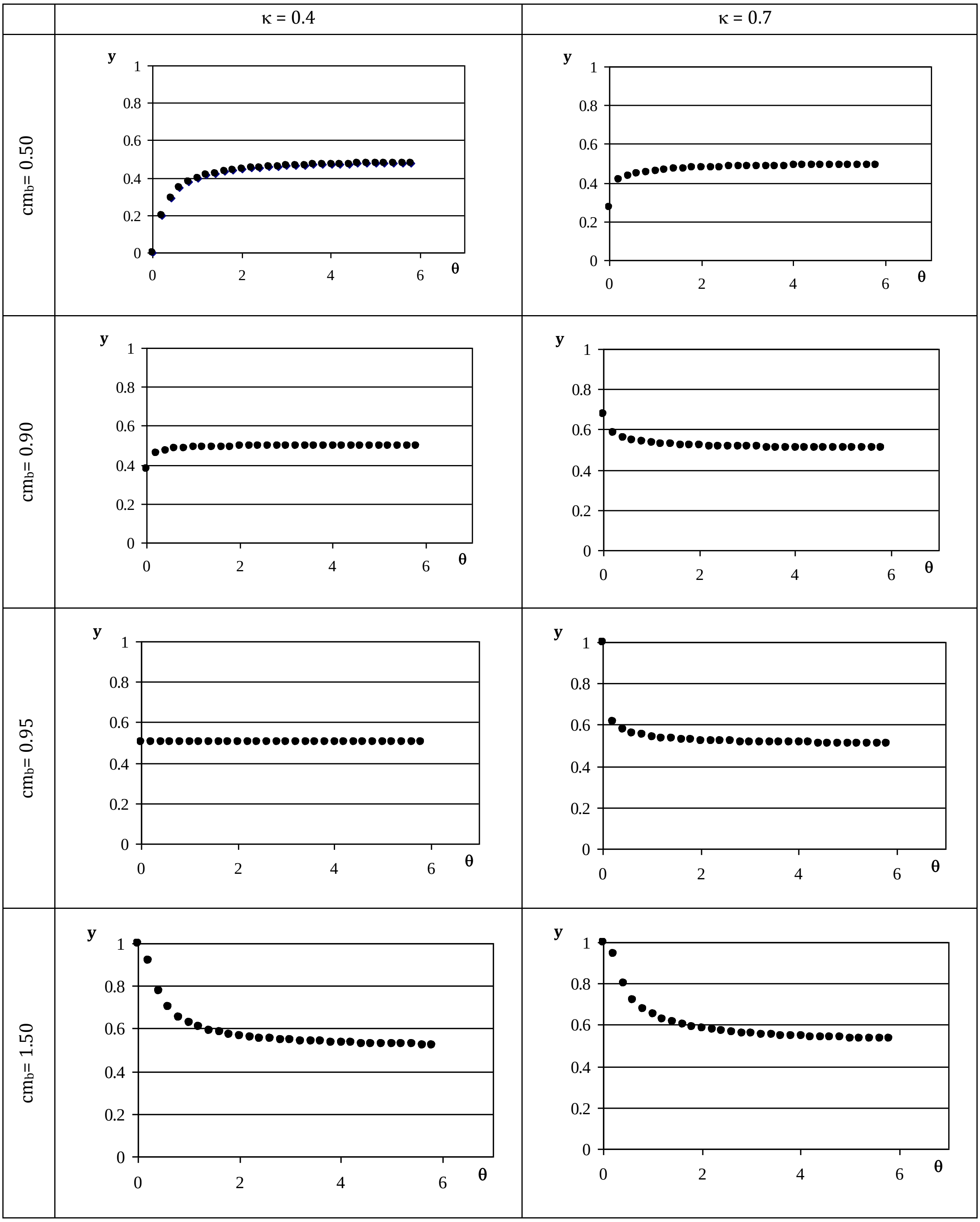
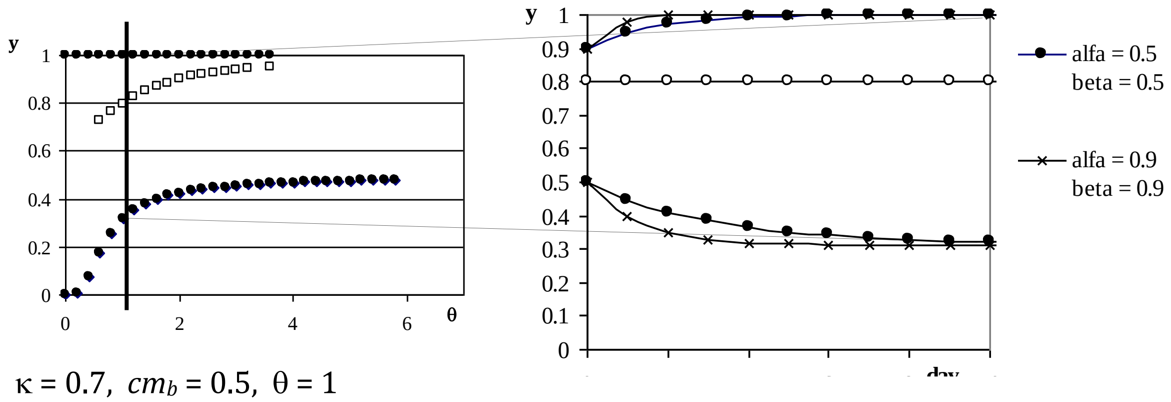


Figure 6 –Fixed-point bifurcations for scenario C)
(black dots refer to stable fixed-points, white to non-stable).

Scenario A

$\kappa = 0.4$, $cm_b = 0.5$, $\theta = 1$



$\kappa = 0.7$, $cm_b = 0.5$, $\theta = 1$

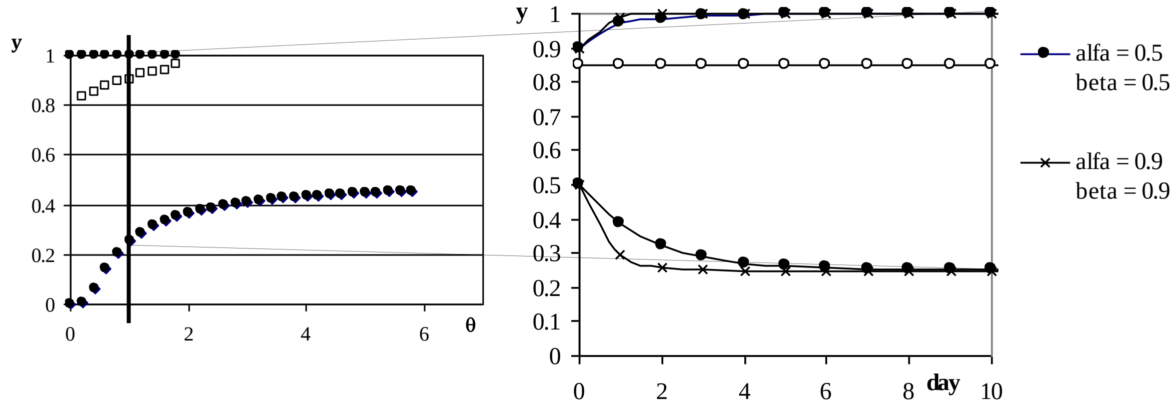


Figure 7 –Length of transients for scenario A).

5. CONCLUSIONS

In this article we have applied a dynamic process model to a two-mode system for analysis of equilibrium states as fixed-points. It has been shown that multiple stable fixed-points may exist resulting from the interaction between bus operator strategies and user behaviour, as well as demand. The modelling of an transport authority is an interesting extension of the presented approach.

Several issues seem worth of further research effort, and will be addressed in future papers, such as the extension to multi-mode systems, and to VoT distributed among users, as well as to multi-user and/or multi-scale dynamic models, including stochastic process models. Including route choice behaviour is another relevant issue, as well as the analysis of other bus operator strategies.

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BIBLIOGRAPHY

Bie J. & Lo H.K. (2010). Stability and attraction domains of traffic equilibria in a day-to-day dynamical system formulation. Transportation Research Part B 44, 90–107.

Cantarella G.E. (2010). Signal setting with dynamic process assignment. In New Developments in Transport Planning: Advances in Dynamic Traffic Assignment. Immers L.H., Tampere C.M.J., Viti F. (eds), 2009. Edward Elgar, Cheltenham, UK - Northampton, MA, USA.

Cantarella G.E & Cascetta E. (1995). Dynamic Process and Equilibrium in Transportation Networks: Towards a Unifying Theory. Transportation Science, 29, 305-329.

Cantarella G. E., Velonà P. (2003). Stability analysis of equilibrium patterns in a transportation network. In Proceedings of AET Conference 2003, 8-10 October 2003, Strasbourg, France.

Cantarella G.E., Gentile G. and Velonà P. (2010). Uniqueness of Stochastic User Equilibrium. In Proceedings of 5th IMA conference on Mathematics in Transportation. (London, UK, April 2010).

Friesz, T.L., Berstein, D., Mehta, N.J., Tobin, R.L. & Ganjalizadeh, S. (1994). Day-to-day dynamic network disequilibria and idealized traveler information systems. Operations Research 42(6), 1120–1136.

Friesz T.L. & Shah S. (2001). An overview of nontraditional formulations of static and dynamic equilibrium network design. Transportation Research Part B 35, 5–21.

Guihaire V. & Hao J.-K. (2008). Transit network design and scheduling: A global review. Transportation Research Part A 42, 1251–1273

Han L. & Du L. (2012). On a link-based day-to-day traffic assignment model. Transportation Research Part B 46, 72–84.

Hazelton M.L. & Watling D.P. (2004). Computation of equilibrium distributions of Markov traffic assignment models. Transportation Science 38 (3), 331-342.

Hickman M.D. & Bernstein D.H. (1997). Transit Service and Path Choice Models in Stochastic and Time-Dependent Networks. Transportation Science 31(2): 129-146.

Li Z.-C., Lam W.H.K. & Wong S.C. (2012). Modeling intermodal equilibrium for bimodal transportation system design problems in a linear monocentric city. Transportation Research Part B 46, 30–49.

Morlok E.K. (1979). Short run supply functions with decreasing user costs. Transportation Research Part B 13, 183-187.

Smith M.J. (1979). Traffic control and route choice: A simple example. Transportation Research Part B 13, 289-294.

Watling D.P. (1996). Asymmetric problems and stochastic process models of traffic assignment. Transportation Research Part B 30, 339-357.