Reiter's properties (P_1) and (P_2) for locally compact quantum groups

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Abstract

A locally compact group G is amenable if and only if it has Reiter's property (P_p) for $p = 1$ or, equivalently, all $p \in [1, \infty)$, i.e., there is a net $(m_\alpha)_\alpha$ of non-negative norm one functions in $L^p(G)$ such that $\lim_\alpha \sup_{x \in K} \|L_{x^{-1}} m_\alpha - m_\alpha\|_p = 0$ for each compact subset $K \subset G$ ($L_{x^{-1}} m_\alpha$ stands for the left translate of m_α by x^{-1}). We extend the definitions of properties (P_1) and (P_2) from locally compact groups to locally compact quantum groups in the sense of J. Kustermans and S. Vaes. We show that a locally compact quantum group has (P_1) if and only if it is amenable and that it has (P_2) if and only if its dual quantum group is co-amenable. As a consequence, (P_2) implies (P_1).

Keywords: amenability, co-amenability, Leptin's theorem, locally compact quantum groups, operator spaces, Reiter's property (P_1), Reiter's property (P_2).

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Introduction

A locally compact group G is said to be *amenable* if there is an invariant mean on $L^\infty(G)$, i.e., a state M of the von Neumann algebra $L^\infty(G)$ such that

$$\langle L_x \phi, M \rangle = \langle \phi, M \rangle \qquad (\phi \in L^\infty(G),\, x \in G).$$

(If f is any function on G and $x \in G$, we denote by $L_x f$ the left translate of f by x, i.e., $(L_x f)(y) := f(xy)$ for $y \in G$.) Approximating M in the weak* topology of $L^\infty(G)^*$ by normal states, i.e., non-negative, norm one functions in $L^1(G)$ and then passing to convex combinations, we obtain a net $(m_\alpha)_\alpha$ of such functions in $L^1(G)$ that is asymptotically invariant in the sense that

$$\lim_\alpha \|L_{x^{-1}} m_\alpha - m_\alpha\|_1 = 0 \qquad (x \in G). \tag{1}$$

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On the other hand, whenever we have a net $(m_\alpha)_\alpha$ of non-negative norm one functions in $L^1(G)$ satisfying (1), then each of its weak* accumulation points in $L^\infty(G)^*$ is a left invariant mean, so that G is amenable.

Even though it is not obvious, the net $(m_\alpha)_\alpha$ can be chosen for amenable G in such a way that the convergence in (1) is uniform in x on each compact subset of G ([Pie, Proposition 6.12]), a condition called *Reiter's property* (P_1) in the literature. More generally, one can define *Reiter's property* (P_p) for any $p \in [1, \infty)$ ([R–St, Definition 8.3.1]), but as it turns out, the properties (P_p) are all equivalent ([R–St, Theorem 8.3.2]). In [Run 1]—see Secton 1 below—, the equivalence of amenability, (P_1), and (P_2) was used to prove Leptin's theorem ([Lep]): G is amenable if and only if $A(G)$, Eymard's Fourier algebra ([Eym]), has a bounded approximate identity.

Leptin's theorem assumes a very natural form in the language of Kac algebras (see [E–S 2]). In this language—using the terminology of [B–T]—, Leptin's theorem reads as: a locally compact group G, if viewed as a Kac algebra, is amenable if and only if its Kac algebraic dual is co-amenable. Hence, it is only natural to ask whether Leptin's theorem holds true for arbitrary Kac algebras: a Kac algebra is amenable if and only if its dual is co-amenable. In [Voi], D. V. Voiculescu showed that, indeed, the co-amenability of a Kac algebra implies the amenability of its dual. In [E–S 1], it was claimed that the converse is also true, but the proof given in [E–S 1] contains an error. Ultimately, Z.-J. Ruan was able to salvage the result at least for discrete Kac algebras ([Rua]) whereas the general case remains open.

Recently, J. Kustermans and S. Vaes introduced a surprisingly simple system of axioms for what they call *locally compact quantum groups* ([K–V 2] and [K–V 3]): those axioms cover the Kac algebras (and therefore all locally compact groups), allow for the development of a Pontryagin type duality theory, but also seem to cover all known examples of C^*-algebraic quantum groups, such as Woronowicz's $\mathrm{SU}_q(2)$ ([Wor]). For a detailed exposition on the history of locally compact quantum groups—with many references to the original literature—, we refer to the introduction of [K–V 2] and to [Vai]. Of course, the question whether amenability is dual to co-amenability— so that Leptin's theorem holds true for locally compact quantum groups— is a natural one, and—as for Kac algebras—it is only known to be true in the discrete case ([Tom]).

The problem to prove Leptin's theorem for general locally compact quantum groups appears to be formidable. R. Tomatsu's proof in the discrete case (see [Tom]) makes heavy use of the particular structure of discrete quantum groups (as does Ruan's argument in the discrete Kac algebra case) and does not appear to be adaptable to the general locally compact situation.

The present paper grew out the attempt to extend the proof of Leptin's theorem from [Run 1] to locally compact quantum groups. The problems arising with such an endeavor are numerous. How can Reiter's properties (P_1) and (P_2) be formulated? How do (P_1) and (P_2) relate to amenability and co-amenability, respectively? Finally, are (P_1) and (P_2) equivalent?

We proceed as follows. The first two sections are mostly expository. We recall the definition of Reiter's properties and reformulate them in a way that will later allow us to extend them to a quantum group setting. Then we give a brief overview of locally compact quantum groups (with references to the original literature). With these preparations, we then define property (P_1) for quantum groups and show that (P_1) and amenability are indeed equivalent; both the definition of (P_1) and the proof of the equivalence result rely heavily on the theory of operator spaces ([E–R], [Pis], and [Pau]). We then go on and define (P_2) for quantum groups, and we show that (P_2) is equivalent, not just to the amenability of the quantum group, but to the co-amenability of its dual (again, both the definition and the result are steeped in operator space theory). As a consequence, (P_2) implies (P_1) whereas the converse remains open.

1 Leptin's theorem through (P_1) and (P_2)

The original proof of Leptin's theorem, as given in [Lep], relied on Følner type conditions, for which it is difficult to see how—if at all—they can be transferred to the context of general locally compact quantum groups. In [Run 1], an alternative proof—making use of properties (P_1) and (P_2) instead—was attempted, but the argument given in [Run 1] was incomplete.

We begin this section with recalling Reiter's properties (P_p) for $p \in [1, \infty)$ ([R–St, Definition 8.3.1]):

Definition 1.1. Let G be a locally compact group, and let $p \in [1, \infty)$. We say that G has *Reiter's property* (P_p) if there is a net $(m_\alpha)_\alpha$ of non-negative norm one functions in $L^p(G)$ such that

$$\lim_\alpha \sup_{x \in K} \|L_{x^{-1}} m_\alpha - m_\alpha\|_p = 0$$

for all compact $K \subset G$.

Remarks. 1. It is not difficult to see that G has (P_p) for all $p \in [1, \infty)$ if and only if it has (P_1) ([R–St, Theorem 8.3.2]).

2. By [Pie, Proposition 6.12], (P_1) is equivalent to amenability.

We now indicate how the argument in [Run 1] can be repaired:

Proof of Leptin's theorem via properties (P_1) and (P_2). Let G be a locally compact group, and suppose that G is amenable, i.e., has Reiter's property (P_1) and thus, equivalently,

(P_2). This means that is a net $(\xi_\alpha)_{\alpha\in\mathbb{A}}$ of non-negative norm one functions in $L^2(G)$ such that

$$\lim_\alpha \sup_{x\in K} \|L_{x^{-1}}\xi_\alpha - \xi_\alpha\|_2 = 0$$

for all compact sets $K \subset G$. For $\alpha \in \mathbb{A}$, define

$$e_\alpha \colon G \to \mathbb{C}, \quad x \mapsto \langle L_{x^{-1}}\xi_\alpha, \xi_\alpha\rangle.$$

Then $(e_\alpha)_\alpha$ is a net in $A(G)$ converging to 1 uniformly on all compact subsets of G. By [G–L, Theorem B_2], this is enough for $(e_\alpha)_\alpha$ to be a bounded approximate identity for $A(G)$.

The converse implication of Leptin's theorem is easier to prove (and has long been known to extend to locally compact quantum groups; see [B–T]). □

We conclude this section with a recasting of Definition 1.1 that will enable us later to extend it from locally compact groups to quantum groups (at least for $p = 1, 2$).

Let G be a locally compact group, let $p \in [1,\infty)$, and let $g \in L^p(G)$. Then

$$L_\bullet(g) \colon G \to L^p(G), \quad x \mapsto L_{x^{-1}}g$$

is a bounded, continuous function with values in $L^p(G)$. Let $f \in \mathcal{C}_0(G)$, and define $fL_\bullet(g) \colon G \to L^p(G)$ pointwise, i.e., $(fL_\bullet(g))(x) := f(x)L_{x^{-1}}g$ for $x \in G$. Since $f \in \mathcal{C}_0(G)$, $fL_\bullet(g)$ also vanishes at infinity and thus lies in $\mathcal{C}_0(G, L^p(G)) \cong \mathcal{C}_0(G) \otimes^\lambda L^p(G)$ (following [E–R], we denote the injective Banach space tensor product by $\otimes^\lambda$).

Let $(m_\alpha)_\alpha$ be a bounded net in $L^p(G)$. Then it is straightforward to verify that

$$\lim_\alpha \sup_{x\in K} \|L_{x^{-1}}m_\alpha - m_\alpha\|_p = 0$$

holds for each compact $K \subset G$ if and only if

$$\lim_\alpha \|fL_\bullet(m_\alpha) - f \otimes m_\alpha\|_{\mathcal{C}_0(G)\otimes^\lambda L^p(G)} = 0$$

is true for all $f \in \mathcal{C}_0(G)$.

In view of this, Definition 1.1 and the following are equivalent:

Definition 1.2. Let G be a locally compact group, and let $p \in [1,\infty)$. We say that G has *Reiter's property* (P_p) if there is a net $(m_\alpha)_\alpha$ of non-negative norm one functions in $L^p(G)$ such that

$$\lim_\alpha \|fL_\bullet(m_\alpha) - f \otimes m_\alpha\|_{\mathcal{C}_0(G)\otimes^\lambda L^p(G)} = 0$$

for all $f \in \mathcal{C}_0(G)$.

2 Locally compact quantum groups—an overview

In this section, we give a brief overview of locally compact quantum groups—as introduced by J. Kustermans and S. Vaes in [K–V 2] and [K–V 3]—with an emphasis on the von Neumann algebraic approach. For details, we refer to [K–V 2], [K–V 3], and [vDa].

As a (von Neumann algebraic) locally compact quantum group is a Hopf–von Neumann algebra with additional structure, we begin with recalling the definition of a Hopf–von Neumann algebra ($\bar{\otimes}$ denotes the W^*-tensor product):

Definition 2.1. A *Hopf–von Neumann algebra* is a pair $(\mathfrak{M}, \Gamma)$, where $\mathfrak{M}$ is a von Neumann algebra and $\Gamma : \mathfrak{M} \to \mathfrak{M}\bar{\otimes}\mathfrak{M}$ is a *co-multiplication*, i.e., a normal, unital, and injective *-homomorphism satisfying $(\mathrm{id} \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \mathrm{id}) \circ \Gamma$.

Example. For a locally compact group G, define $\Gamma_G \colon L^\infty(G) \to L^\infty(G \times G)$ by letting

$$(\Gamma_G \phi)(x, y) := \phi(xy) \qquad (\phi \in L^\infty(G),\, x, y \in G).$$

Then $(L^\infty(G), \Gamma_G)$ is a Hopf–von Neumann algebra.

Remark. Given a Hopf–von Neumann algebra $(\mathfrak{M}, \Gamma)$, one can define a product $*$ on $\mathfrak{M}_*$, the unique predual of $\mathfrak{M}$, turning it into a Banach algebra:

$$\langle f * g, x \rangle := \langle f \otimes g, \Gamma x \rangle \qquad (f, g \in \mathfrak{M}_*,\, x \in \mathfrak{M}). \tag{2}$$

If G is a locally compact group, then applying (2) to $(L^\infty(G), \Gamma_G)$ yields the usual convolution product on $L^1(G)$.

To define the additional structure that turns a Hopf–von Neumann algebra into a locally compact quantum group, we recall some basic facts about weights (see [Tak 2], for instance).

Let $\mathfrak{M}$ be a von Neumann algebra, and let $\mathfrak{M}^+$ denote its positive elements. A *weight* on $\mathfrak{M}$ is an additive map $\phi \colon \mathfrak{M}^+ \to [0, \infty]$ such that $\phi(tx) = t\phi(x)$ for $t \in [0, \infty)$ and $x \in \mathfrak{M}^+$. We let

$$\mathcal{M}_\phi^+ := \{x \in \mathfrak{M}^+ : \phi(x) < \infty\}, \qquad \mathcal{M}_\phi := \text{the linear span of } \mathcal{M}_\phi^+,$$

and

$$\mathcal{N}_\phi := \{x \in \mathfrak{M} : x^* x \in \mathcal{M}_\phi\}.$$

Then ϕ extends to a linear map on $\mathcal{M}_\phi$, and $\mathcal{N}_\phi$ is a left ideal of $\mathfrak{M}$. Using the GNS-construction ([Tak 2, p. 42]), we obtain a representation π_ϕ of $\mathfrak{M}$ on some Hilbert space $\mathfrak{H}_\phi$; we denote the canonical map from $\mathcal{N}_\phi$ into $\mathfrak{H}_\phi$ by Λ_ϕ. Moreover, we call ϕ *semifinite* if $\mathcal{M}_\phi$ is w^*-dense in $\mathfrak{M}$, *faithful* if $\phi(x) = 0$ for $x \in \mathfrak{M}^+$ implies that $x = 0$, and *normal* if $\sup_\alpha \phi(x_\alpha) = \phi(\sup_\alpha x_\alpha)$ for each bounded, increasing net $(x_\alpha)_\alpha$ in $\mathfrak{M}^+$. If ϕ is faithful and normal, then the corresponding representation π_ϕ is faithful and normal, too ([Tak 2, Proposition VII.1.4]).

Definition 2.2. A (von Neumann algebraic) *locally compact quantum group* is a Hopf–von Neumann algebra $(\mathfrak{M}, \Gamma)$ such that:

(a) there is a normal, semifinite, faithful weight ϕ on $\mathfrak{M}$—a *left Haar weight*—which is left invariant, i.e., satisfies

$$\phi((f \otimes \mathrm{id})(\Gamma x)) = \langle f, 1 \rangle \phi(x) \qquad (f \in \mathfrak{M}_*,\ x \in \mathcal{M}_\phi);$$

(b) there is a normal, semifinite, faithful weight ψ on $\mathfrak{M}$—a *right Haar weight*—which is right invariant, i.e., satisfies

$$\psi((\mathrm{id} \otimes f)(\Gamma x)) = \langle f, 1 \rangle \psi(x) \qquad (f \in \mathfrak{M}_*,\ x \in \mathcal{M}_\psi).$$

Example. Let G be a locally compact group. Then the Hopf–von Neumann algebra $(L^\infty(G), \Gamma_G)$ is a locally compact quantum group: ϕ and ψ can be chosen as left and right Haar measure, respectively.

Remarks. 1. Even though only the existence of a left and a right Haar weight, respectively, is presumed, both weights are actually unique up to a positive scalar multiple (see [K–V 2] and [K–V 3]). In order to make notation not too cumbersome, we shall thus simply write $(\mathfrak{M}, \Gamma)$ for a locally compact quantum group whose left and right Haar weight will always be denoted by ϕ and ψ, respectively.

2. As discussed in [K–V 2] and [K–V 3], locally compact quantum groups can equivalently be described in C^*-algebraic terms. The C^*-algebraic definition ([K–V 2, Definition 4.1]), however, is technically more involved, so that we shall not go into the details.

Definition 2.3. Let $(\mathfrak{M}, \Gamma)$ be a locally compact quantum group. The *multiplicative unitary* of $(\mathfrak{M}, \Gamma)$ is the unique operator $W \in \mathcal{B}(\mathfrak{H}_\phi \tilde{\otimes}_2 \mathfrak{H}_\phi)$, where $\tilde{\otimes}_2$ stands for the Hilbert space tensor product, satisfying

$$W^*(\Lambda_\phi(x) \otimes \Lambda_\phi(y)) = (\Lambda_\phi \otimes \Lambda_\phi)((\Gamma y)(x \otimes 1)) \qquad (x, y \in \mathcal{N}_\phi).$$

Example. For a locally compact group G, the multiplicative unitary W_G of $(L^\infty(G), \Gamma_G)$ is given by

$$(W_G \xi)(x, y) = \xi(x, x^{-1} y) \qquad (\xi \in L^2(G \times G),\ x, y \in G).$$

Remarks. 1. Using the left invariance of ϕ, it is easy to see that W^* is an isometry whereas it is considerably more difficult to show that W is indeed a unitary operator ([K–V 2, Theorem 3.16]).

2. The unitary W lies in $\mathfrak{M}\bar{\otimes}\mathcal{B}(\mathfrak{H}_\phi)$ and implements the co-multiplication via

$$\Gamma x = W^*(1 \otimes x)W \qquad (x \in \mathfrak{M})$$

(see the discussion following [K–V 3, Theorem 1.2]).

3. The definition of W is made via the GNS-construction arising from ϕ, so that one may want—in order to avoid confusion—rather speak of a *left* multiplicative unitary. Indeed, one can define a *right* multiplicative unitary in a similar fashion in terms of ψ: in [J–N–R], for instance, the right multiplicative unitary is used instead of the left one. It seems to be more or less a matter of taste with which of two multiplicative unitaries one prefers to work.

To emphasize the parallels between locally compact quantum groups and groups, we shall use the following notation (which was suggested by Z.-J. Ruan and is also used in [Run 2] and [J–N–R]). We use the symbol $\mathbb{G}$ for a von Neumann algebraic, locally compact quantum group $(\mathfrak{M}, \Gamma)$ and write: $L^\infty(\mathbb{G})$ for $\mathfrak{M}$, $L^1(\mathbb{G})$ for $\mathfrak{M}_*$, and $L^2(\mathbb{G})$ for $\mathfrak{H}_\phi$. If $L^\infty(\mathbb{G}) = L^\infty(G)$ for a locally compact group G and $\Gamma = \Gamma_G$, we say that $\mathbb{G}$ actually *is* a locally compact group, which is the case precisely if $L^\infty(\mathbb{G})$ is abelian (this follows from [B–S, Théorème 2.2]).

Given a locally compact quantum group $\mathbb{G}$ with multiplicative unitary W, we set

$$\mathcal{C}_0(\mathbb{G}) := \overline{\{(\mathrm{id} \otimes \omega)(W) : \omega \in \mathcal{B}(L^2(\mathbb{G}))_*\}}^{\|\cdot\|}$$

It is relatively easy to see that $\mathcal{C}_0(\mathbb{G})$ is a closed subalgebra of $\mathcal{B}(L^2(\mathbb{G}))$, but—which is much harder to show—it is even a C^*-subalgebra. Restricting Γ to $\mathcal{C}_0(\mathbb{G})$ then yields a reduced C^*-algebraic quantum group in the sense of [K–V 2, Definition 4.1] (see [K–V 3, Proposition 1.6]). If $\mathbb{G}$ is a locally compact group G, then $\mathcal{C}_0(G)$ just has the usual meaning: the continuous function on G vanishing at infinity. Consequently, we write $M(\mathbb{G})$ for $\mathcal{C}_0(\mathbb{G})^*$. Like $L^1(\mathbb{G})$, the dual space $M(\mathbb{G})$ has a canonical product induced by Γ, turning it into a Banach algebra ([K–V 2, p. 913]) containing $L^1(\mathbb{G})$ as a closed ideal ([K–V 2, p. 914]).

Given a locally compact quantum group $\mathbb{G}$ with multiplicative unitary W, the *left regular representation* of $\mathbb{G}$ is the map

$$\lambda_2 \colon L^1(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G})), \quad f \mapsto (f \otimes \mathrm{id})(W). \tag{3}$$

Since $W \in L^\infty(\mathbb{G})\bar{\otimes}\mathcal{B}(L^2(\mathbb{G}))$, it is clear that λ_2 is well defined, and it is easy to see that λ_2 is a contractive algebra homomorphism.

Example. For a locally compact group G, we have

$$(\lambda_2(f)\xi)(y) = \int_G f(x)\xi(x^{-1}y)\,dx \qquad (f \in L^1(G),\, \xi \in L^2(G))$$

for almost all $y \in G$, i.e., λ_2 according to (3) is just the usual left regular representation of $L^1(G)$ on $L^2(G)$.

Locally compact quantum groups allow for the development of a duality theory that extends Pontryagin duality for locally compact abelian groups.

For a locally compact quantum group $\mathbb{G}$, set

$$L^\infty(\hat{\mathbb{G}}) := \overline{\lambda_2(L^2(\mathbb{G}))}^{\sigma\text{-strongly}^*};$$

it can be shown that $L^\infty(\hat{\mathbb{G}})$ is a von Neumann algebra. Let σ denote the flip map on $L^2(\mathbb{G})\tilde{\otimes}_2 L^2(\mathbb{G})$, i.e., $\sigma(\xi \otimes \eta) = \eta \otimes \xi$ for $\xi, \eta \in L^2(\mathbb{G})$. Set $\hat{W} := \sigma W^* \sigma$. Then

$$\hat{\Gamma} \colon L^\infty(\hat{\mathbb{G}}) \to L^\infty(\hat{\mathbb{G}})\bar{\otimes}L^\infty(\hat{\mathbb{G}}), \quad x \mapsto \hat{W}^*(1 \otimes x)\hat{W}$$

is a co-multiplication. One can also define a left Haar weight $\hat{\phi}$ and a right Haar weight $\hat{\psi}$ for $(L^\infty(\hat{\mathbb{G}}), \hat{\Gamma})$ turning it into a locally compact quantum group again, the *dual quantum group* of $\mathbb{G}$, which we denote by $\hat{\mathbb{G}}$, and whose multiplicative unitary is $\hat{W}$ as defined above. Finally, a Pontryagin duality theorem holds, i.e., $\hat{\hat{\mathbb{G}}} = \mathbb{G}$. For the details of this duality, we refer again to [K–V 2] and [K–V 3].

Example. Let G be a locally compact group. Since $L^\infty(\hat{G})$ is the σ-strong* closure of $\lambda_2(L^1(G))$, it equals $\mathrm{VN}(G)$, the group von Neumann algebra of G. Further, the co-multiplication $\hat{\Gamma}_G \colon \mathrm{VN}(G) \to \mathrm{VN}(G)\bar{\otimes}\,\mathrm{VN}(G)$ is given by

$$\hat{\Gamma}_G(\lambda(x)) = \lambda(x) \otimes \lambda(x) \qquad (x \in G).$$

Consequently, the product $*$ according to (2) on $\mathrm{VN}(G)_*$ is the usual pointwise product on $A(G)$, so that $L^1(\hat{G}) = A(G)$. The Plancherel weight on $\mathrm{VN}(G)$ ([Tak 2, Definition VII.3.2]) is both a left and a right Haar weight for $(\mathrm{VN}(G), \hat{\Gamma}_G)$. Finally note that $\mathcal{C}_0(\hat{G})$ is the reduced group C^*-algebra of G, so that $M(\hat{G})$ is the reduced Fourier–Stieltjes algebra from [Eym].

3 (P_1) for locally compact quantum groups

With an eye on Definition 1.2, we shall, in this section, formulate a version of property (P_1) for locally compact quantum groups. To this end, we require the framework of operator space theory, as laid out in the monographs [E–R], [Pau], and [Pis]. We shall mostly follow [E–R] in our choice of notation; in particular, for two operator spaces E and F, we denote the completely bounded operators from E to F by $\mathcal{CB}(E, F)$, we write $\|\cdot\|_{\mathrm{cb}}$ for the cb-norm, and we use $\check{\otimes}$ for the injective tensor product *of operator spaces*. (Note that, if $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras, then $\mathfrak{A}\check{\otimes}\mathfrak{B}$ is just the spatial tensor product of C^*-algebras.)

We begin with an elementary lemma:

Lemma 3.1. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces, and let $A, B \in \mathcal{B}(\mathfrak{K})\check{\otimes}\mathcal{K}(\mathfrak{H})$, where $\mathcal{K}(\mathfrak{H})$ denotes the compact operators on $\mathfrak{H}$. Then the map*

$$\mathcal{B}(\mathfrak{H}) \to \mathcal{B}(\mathfrak{K})\check{\otimes}\mathcal{K}(\mathfrak{H}), \quad x \mapsto A(1 \otimes x)B \tag{4}$$

is completely bounded and belongs to the cb*-norm closure of the finite rank operators in $\mathcal{CB}(\mathcal{B}(\mathfrak{H}), \mathcal{B}(\mathfrak{K})\check{\otimes}\mathcal{K}(\mathfrak{H}))$.*

Proof. The complete boundedness of (4) is clear.

To see that (4) is a norm limit of finite rank operators in $\mathcal{CB}(\mathcal{B}(\mathfrak{H}), \mathcal{B}(\mathfrak{K})\check{\otimes}\mathcal{K}(\mathfrak{H}))$, first note that it is enough to suppose that $A = S \otimes K$ and $B = T \otimes L$ with $S, T \in \mathcal{B}(\mathfrak{K})$ and $K, L \in \mathcal{K}(\mathfrak{H})$. Let $(K_n)_{n=1}^{\infty}$ and $(L_n)_{n=1}^{\infty}$ be finite rank operators on $\mathfrak{H}$ such that $K = \lim_{n\to\infty} K_n$ and $L = \lim_{n\to\infty} L_n$ in the norm topology of $\mathcal{B}(\mathfrak{H})$. For each $n \in \mathbb{N}$, the operator

$$\mathcal{B}(\mathfrak{H}) \to \mathcal{B}(\mathfrak{K})\check{\otimes}\mathcal{K}(\mathfrak{H}), \quad x \mapsto (S \otimes K_n)(1 \otimes x)(T \otimes L_n)$$

has finite rank, and it is immediate that these operators converge to (4) in $\|\cdot\|_{\mathrm{cb}}$. $\square$

Let $\mathbb{G}$ be a locally compact quantum group, and let $g \in L^1(\mathbb{G})$. We define

$$(\Gamma|g) : L^{\infty}(\mathbb{G}) \to L^{\infty}(\mathbb{G}), \quad x \mapsto (\mathrm{id} \otimes g)(\Gamma x).$$

It is immediate that $(\Gamma|g)$ is a weak*-weak* continuous, completely bounded map.

For our next result—which will enable us to formulate property (P_1) for locally compact quantum groups—, we use the following conventions:

- if $\mathfrak{A}$ is any algebra, and a and b are any elements of $\mathfrak{A}$, then $M_{a,b}$ denotes the two-sided multiplication map on $\mathfrak{A}$ given by $M_{a,b}x := axb$ for $x \in \mathfrak{A}$;
- for any C^*-algebra $\mathfrak{A}$, its multiplier algebra ([Tak 1, Definition III.6.22]) is denoted by $\mathcal{M}(\mathfrak{A})$;
- if $\mathfrak{M}$ is a von Neumann algebra on a Hilbert space $\mathfrak{H}$ and ξ and η are vectors in $\mathfrak{H}$, we write $\omega_{\xi,\eta}$ for the vector functional given by $\langle \omega_{\xi,\eta}, x \rangle = \langle x\xi, \eta \rangle$ for $x \in \mathfrak{M}$.

We also recall that, if E and F are operator spaces, then the closure of the finite rank operators in $\mathcal{CB}(E, F)$ can be canonically identified with $F\check{\otimes}E^*$ ([E–R, Proposition 8.1.2]).

Proposition 3.2. *Let $\mathbb{G}$ be a locally compact quantum group, let $g \in L^1(\mathbb{G})$, and let $a, b \in \mathcal{C}_0(\mathbb{G})$. Then $M_{a,b} \circ (\Gamma|g)$ is a completely bounded operator from $L^{\infty}(\mathbb{G})$ to $\mathcal{C}_0(\mathbb{G})$ that lies in the* cb*-norm closure of the finite rank operators in $\mathcal{CB}(L^{\infty}(\mathbb{G}), \mathcal{C}_0(\mathbb{G}))$ and can be identified with an element of $\mathcal{C}_0(\mathbb{G})\check{\otimes}L^1(\mathbb{G})$.*

Proof. Since $L^\infty(\mathbb{G})$ on $L^2(\mathbb{G})$ is in standard form ([Tak 2, Definition IX.1.13]), there are $\xi, \eta \in L^2(\mathbb{G})$ such that $g = \omega_{\xi,\eta}$ (this follows from [Tak 2, Lemma IX.1.6]). Choose $K, L \in \mathcal{K}(L^2(\mathbb{G}))$ such that $L\xi = \xi$ and $K^*\eta = \eta$ (clearly, rank one operators will do).

Let $W \in \mathcal{B}(L^2(\mathbb{G})\tilde{\otimes}_2 L^2(\mathbb{G}))$ be the multiplicative unitary of $\mathbb{G}$. By [K–V 2, Proposition 3.21 and pp. 913–914]—with the appropriate identifications made—, we have $W \in \mathcal{M}(\mathcal{C}_0(\mathbb{G})\check{\otimes}\mathcal{K}(L^2(\mathbb{G})))$, so that $(a \otimes K)W^*, W(b \otimes L) \in \mathcal{C}_0(\mathbb{G})\check{\otimes}\mathcal{K}(L^2(\mathbb{G}))$. From Lemma 3.1, we conclude that

$$L^\infty(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G}))\check{\otimes}\mathcal{K}(L^2(\mathbb{G})), \quad x \mapsto (a \otimes K)W^*(1 \otimes x)W(b \otimes L) \tag{5}$$

is a norm limit of finite rank operators in $\mathcal{CB}(L^\infty(\mathbb{G}), \mathcal{B}(L^2(\mathbb{G}))\check{\otimes}\mathcal{K}(L^2(\mathbb{G})))$. It is straightforward that (5) actually lies in $\mathcal{CB}(L^\infty(\mathbb{G}), \mathcal{C}_0(\mathbb{G})\check{\otimes}\mathcal{K}(L^2(\mathbb{G})))$.

Let $\xi', \eta' \in L^2(\mathbb{G})$, and note that, for $x \in L^\infty(\mathbb{G})$, we have

$$\begin{aligned}
\langle((M_{a,b} \circ (\Gamma|g))x)\xi', \eta'\rangle &= \langle a(\mathrm{id} \otimes g)(W^*(1 \otimes x)W)b\xi', \eta'\rangle \\
&= \langle(a \otimes 1)W^*(1 \otimes x)W(b \otimes 1)(\xi' \otimes \xi), \eta' \otimes \eta\rangle \\
&= \langle(a \otimes 1)W^*(1 \otimes x)W(b \otimes 1)(\xi' \otimes L\xi), \eta' \otimes K^*\eta\rangle \\
&= \langle(a \otimes K)W^*(1 \otimes x)W(b \otimes L)(\xi' \otimes \xi), \eta' \otimes \eta\rangle \\
&= \langle(\mathrm{id} \otimes g)((a \otimes K)W^*(1 \otimes x)W(b \otimes L))\xi', \eta'\rangle.
\end{aligned}$$

Since $\xi', \eta' \in L^2(\mathbb{G})$ were arbitrary, this means that

$$(M_{a,b} \circ (\Gamma|g))x = (\mathrm{id} \otimes g)((a \otimes K)W^*(1 \otimes x)W(b \otimes L)) \qquad (x \in L^\infty(\mathbb{G})),$$

i.e., $M_{a,b} \circ (\Gamma|g)$ is the composition of (5) with the Tomiyama slice map $\mathrm{id} \otimes g$ and thus is a norm limit of finite rank operators in $\mathcal{CB}(L^\infty(\mathbb{G}), \mathcal{C}_0(\mathbb{G}))$.

By [E–R, Proposition 8.1.2], $M_{a,b} \circ (\Gamma|g)$ can be canonically identified with an element of $\mathcal{C}_0(\mathbb{G})\check{\otimes}L^\infty(\mathbb{G})^*$. In order to prove that $M_{a,b} \circ (\Gamma|g)$ actually lies in $\mathcal{C}_0(\mathbb{G})\check{\otimes}L^1(\mathbb{G})$, we show that $(M_{a,b} \circ (\Gamma|g))^* \colon M(\mathbb{G}) \to L^\infty(\mathbb{G})^*$ attains its values in $L^1(\mathbb{G})$. For $\mu \in M(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$, we have

$$\begin{aligned}
\langle(M_{a,b} \circ (\Gamma|g))^*\mu, x\rangle &= \langle(M_{a,b} \circ (\Gamma|g))x, \mu\rangle \\
&= \langle(a \otimes 1)(\Gamma x)(b \otimes 1), \mu \otimes g\rangle \\
&= \langle\Gamma x, b\mu a \otimes g\rangle \\
&= \langle b\mu a * g, x\rangle,
\end{aligned}$$

so that

$$(M_{a,b} \circ (\Gamma|g))^*\mu = b\mu a * g. \tag{6}$$

(We denote the canonical module actions of a C^*-algebra on its dual by juxtaposition.) Since $L^1(\mathbb{G})$ is an ideal in $M(\mathbb{G})$, it follows from (6) that $(M_{a,b} \circ (\Gamma|g))^*M(\mathbb{G}) \subset L^1(\mathbb{G})$, so that $M_{a,b} \circ (\Gamma|g)$ is canonically represented by an element of $\mathcal{C}_0(\mathbb{G})\check{\otimes}L^1(\mathbb{G})$. $\square$

Let G be a locally compact group, let $g \in L^1(G)$, and let $a, b \in \mathcal{C}_0(G)$. Then $M_{a,b} \circ (\Gamma|g) \in \mathcal{C}_0(G)\check{\otimes}L^1(G) = \mathcal{C}_0(G) \otimes^\lambda L^1(G)$ is nothing but $abL_\bullet(g)$ in the notation of Section 1, as a routine verification shows.

With Definition 1.2 in mind, we can thus extend property (P_1) from locally compact groups to locally compact quantum groups:

Definition 3.3. A locally compact quantum group $\mathbb{G}$ is said to have *Reiter's property* (P_1) if there is a net $(m_\alpha)_\alpha$ of states in $L^1(\mathbb{G})$ such that

$$\lim_\alpha \|M_{a,b} \circ (\Gamma|m_\alpha) - ab \otimes m_\alpha\|_{\mathcal{C}_0(\mathbb{G})\check{\otimes}L^1(\mathbb{G})} = 0$$

for all $a, b \in \mathcal{C}_0(\mathbb{G})$.

4 Amenability and (P_1)

Recall the definition of an amenable, locally compact quantum group:

Definition 4.1. A locally compact quantum group $\mathbb{G}$ is called *amenable* if it has a *left invariant mean*, i.e., a state M on $L^\infty(\mathbb{G})$ such that

$$\langle (f \otimes \mathrm{id})(\Gamma x), M \rangle = \langle f, 1 \rangle \langle x, M \rangle \qquad (f \in L^1(\mathbb{G}),\, x \in L^\infty(\mathbb{G})). \tag{7}$$

Remarks. 1. Our use of the term amenable is the same as in [B–T], but there is no general consensus in the literature about terminology: an amenable, locally compact quantum group according to Definition 4.1 is called *Voiculescu amenable* in [Rua] and *weakly amenable* in [D–Q–V].

2. There is an element of asymmetry in Definition 4.1: a state M on $L^\infty(\mathbb{G})$ is called a *right invariant mean* if

$$\langle (\mathrm{id} \otimes f)(\Gamma x), M \rangle = \langle f, 1 \rangle \langle x, M \rangle \qquad (f \in L^1(\mathbb{G}),\, x \in L^\infty(\mathbb{G})) \tag{8}$$

holds and an *invariant mean* if both (7) and (8) are satisfied. So, $\mathbb{G}$ is amenable if and only if there is a *left* invariant mean on $L^\infty(\mathbb{G})$. However, by [D–Q–V, Proposition 3], the amenability of $\mathbb{G}$ already implies the existence of an invariant mean.

3. The standard approximation argument (see [E–S 1], for instance) immediately yields that $\mathbb{G}$ is amenable if and only if there is a net $(m_\alpha)_\alpha$ of states in $L^1(\mathbb{G})$ such that

$$\lim_\alpha \|f * m_\alpha - \langle f, 1 \rangle m_\alpha\| = 0 \qquad (f \in L^1(\mathbb{G})). \tag{9}$$

4. If G is a locally compact group, then a state M as in Definition 4.1 is topologically left invariant in the sense of [Pie, Definition 4.3]. By [Pie, Theorem 4.19], this means that G is amenable in the sense of Definition 4.1 if and only if it is amenable in the classical sense.

It is easy to see that (P_1) implies amenability:

Proposition 4.2. *Let $\mathbb{G}$ be a locally compact quantum group with Reiter's property (P_1). Then $\mathbb{G}$ is amenable.*

Proof. Let $(m_\alpha)_\alpha$ be a net as in Definition 3.3, and let $f \in L^1(\mathbb{G})$. By Cohen's factorization theorem ([Dal, Corollary 2.9.26]), there are $a, b \in \mathcal{C}_0(\mathbb{G})$ and $g \in L^1(\mathbb{G})$ such that $f = bga$.

For any Banach space E, we denote its closed unit ball by $\mathrm{Ball}(E)$.

We then have:

$$\begin{aligned}
\|f * m_\alpha - \langle 1, f\rangle m_\alpha\| &= \sup\{|\langle f \otimes m_\alpha, \Gamma x - 1 \otimes x\rangle| : x \in \mathrm{Ball}(L^\infty(\mathbb{G}))\} \\
&= \sup\{|\langle bga \otimes m_\alpha, \Gamma x - 1 \otimes x\rangle| : x \in \mathrm{Ball}(L^\infty(\mathbb{G}))\} \\
&= \sup\{|\langle g \otimes m_\alpha, (a \otimes 1)(\Gamma x)(b \otimes 1) - ab \otimes x\rangle| : x \in \mathrm{Ball}(L^\infty(\mathbb{G}))\} \\
&= \sup\{|\langle g, (M_{a,b} \circ (\Gamma | m_\alpha))x - \langle m_\alpha, x\rangle ag\rangle| : x \in \mathrm{Ball}(L^\infty(\mathbb{G}))\} \\
&\leq \|g\| \sup\{\|(M_{a,b} \circ (\Gamma | m_\alpha))x - \langle m_\alpha, x\rangle ab\| : x \in \mathrm{Ball}(L^\infty(\mathbb{G}))\} \\
&\leq \|g\| \|M_{a,b} \circ (\Gamma | m_\alpha) - ab \otimes m_\alpha\|_{\mathcal{C}_0(\mathbb{G}) \check{\otimes} L^1(\mathbb{G})} \\
&\overset{\alpha}{\longrightarrow} 0.
\end{aligned}$$

It is clear that any weak* accumulation point of $(m_\alpha)_\alpha$ in $L^\infty(\mathbb{G})^*$ is a left invariant mean. $\square$

For the converse of Proposition 4.2, we require a few preparations.

Let E be an operator space; deviating from [E–R], but for the sake of notational clarity, we denote, for $n \in \mathbb{N}$, the n-th matrix level of E by $\mathbb{M}_n(E)$. A *matricial subset* of E is a sequence $\boldsymbol{S} = (S_n)_{n=1}^\infty$ with $S_n \subset \mathbb{M}_n(E)$ for $n \in \mathbb{N}$. We use the usual set theoretic symbols for matricial points and subsets termwise, e.g., if $\boldsymbol{S} = (S_n)_{n=1}^\infty$ and $\boldsymbol{T} = (T_n)_{n=1}^\infty$ are matricial subsets of E, then $\boldsymbol{S} \cup \boldsymbol{T}$ is defined as $(S_n \cup T_n)_{n=1}^\infty$.

Given two operator spaces E and F, $n \in \mathbb{N}$, and a linear map $T \colon E \to F$, we write (again, not following [E–R]) $T^{(n)} \colon \mathbb{M}_n(E) \to \mathbb{M}_n(F)$ for the n-th amplification of T.

Definition 4.3. Let E and F be operator spaces, let $(T_\alpha)_\alpha$ be a net in $\mathcal{CB}(E, F)$, let $T \in \mathcal{CB}(E, F)$, and let $\boldsymbol{S} = (S_n)_{n=1}^\infty$ be a matricial subset of E. We say that $(T_\alpha)_\alpha$ converges to T *completely uniformly on $\boldsymbol{S}$* if

$$\lim_\alpha \sup_{n \in \mathbb{N}} \sup \left\{ \left\| T_\alpha^{(n)} x - T^{(n)} x \right\|_n : x \in S_n \right\} \to 0.$$

Lemma 4.4. *Let $E_1, \ldots, E_m$, E, and F be operator spaces, and let $S_j \in \mathcal{CB}(E_j, E)$ for $j = 1, \ldots, m$ lie in the* cb*-norm closure of the finite rank operators. Let*

$$\boldsymbol{K}_j := \left(S_j^{(n)}(\mathrm{Ball}(\mathbb{M}_n(E_j))) \right)_{n=1}^\infty \qquad (j = 1, \ldots, m),$$

and set $\boldsymbol{K} := \boldsymbol{K}_1 \cup \cdots \cup \boldsymbol{K}_m$. Then every norm bounded net $(T_\alpha)_\alpha$ in $\mathcal{CB}(E, F)$ that converges to $T \in \mathcal{CB}(E, F)$ pointwise on E converges to T completely uniformly on $\boldsymbol{K}$.

Proof. Suppose without loss of generality that $m = 1$.

The completely uniform convergence of $(T_\alpha)_\alpha$ to T on $\boldsymbol{K}_1$ amounts to $\|T_\alpha S_1 - TS_1\|_{\mathrm{cb}} \to 0$. Since $(T_\alpha)_\alpha$ is norm bounded in $\mathcal{CB}(E, F)$ and since S_1 is a norm limit of finite rank operators in $\mathcal{CB}(E_1, E)$, there is no loss of generality to suppose that S_1 *is* a finite rank operator.

Let E_0 be a finite-dimensional subspace of E with $S_1 E_1 \subset E_0$. Since $\dim E_0 < \infty$, the identity map $\mathrm{id}_{E_0} : E_0 \to \max E_0$ is completely bounded ([Pau, Theorem 14.3(ii)]). Hence, we have the (Banach space) isomorphisms

$$\mathcal{CB}(E_0, F) \cong \mathcal{CB}(\max E_0, F) \cong \mathcal{B}(E_0, F),$$

where the last isomorphism holds by the definition of $\max E$ and is, in fact, isometric ([E–R, (3.3.9)]). Since the unit ball of E_0 is compact, and since $(T_\alpha)_\alpha$ is norm bounded in $\mathcal{B}(E_0, F)$, we conclude that $T_\alpha|_{E_0} \to T|_{E_0}$ in the norm topology of $\mathcal{B}(E_0, F)$ and thus of $\mathcal{CB}(E_0, F)$. Finally, note that

$$\|T_\alpha S_1 - TS_1\|_{\mathrm{cb}} \leq \frac{1}{\max\{\|S_1\|_{\mathrm{cb}}, 1\}} \|T_\alpha|_{E_0} - T|_{E_0}\|_{\mathrm{cb}} \to 0,$$

which completes the proof. $\square$

Remark. Let E and F be Banach spaces, let $(T_\alpha)_\alpha$ be a norm bounded net in $\mathcal{B}(E, F)$, and let $T \in \mathcal{B}(E, F)$ be such that $T_\alpha \to T$ pointwise on E. Then it is elementary (and was used in the proof of Lemma 4.4) that $T_\alpha \to T$ uniformly on all compact subsets of E. Lemma 4.4 is a fairly crude attempt to adapt this fact to an operator space setting. One major obstacle to establishing a more satsifactory operator space variant is the apparent difficulty of finding a proper notion of compactness suited for operator spaces (see [Web] and [Yew]).

We can now prove the first main result of this paper:

Theorem 4.5. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following are equivalent:*

(i) $\mathbb{G}$ *is amenable;*

(ii) $\mathbb{G}$ *has Reiter's property* (P_1)*.*

Proof. As (ii) $\Longrightarrow$ (i) is Proposition 4.2, all we need to prove is (i) $\Longrightarrow$ (ii).

Let $a_1, b_1, \ldots, a_\nu, b_\nu \in \mathcal{C}_0(\mathbb{G})$, and let $\epsilon > 0$. We need to show that there is a state $m \in L^1(\mathbb{G})$ such that

$$\|M_{a_j,b_j} \circ (\Gamma|m) - a_j b_j \otimes m\|_{\mathcal{C}_0(\mathbb{G})\check{\otimes}L^1(\mathbb{G})} < \epsilon \qquad (j = 1, \ldots, \nu). \tag{10}$$

Since $\mathbb{G}$ is amenable, there is a net $(m_\alpha)_{\alpha\in\mathbb{A}}$ of states in $L^1(\mathbb{G})$ such that (9) holds. For $\alpha \in \mathbb{A}$, define

$$T_\alpha \colon L^1(\mathbb{G}) \to L^1(\mathbb{G}), \quad f \mapsto f * m_\alpha - \langle f, 1\rangle m_\alpha.$$

The net $(T_\alpha)_\alpha$ lies in $\mathcal{CB}(L^1(\mathbb{G}))$, is norm bounded, and converges to 0 pointwise on $L^1(\mathbb{G})$ by (9).

Let $m_0 \in L^1(\mathbb{G})$ be an arbitrary state. For $j = 1, \ldots, \nu$, let the matricial subset $\boldsymbol{K}_j = (K_{j,n})_{n=1}^\infty$ of $L^1(\mathbb{G})$ be defined through

$$K_{j,n} := \{[b_j \mu_{k,l} a_j * m_0] : [\mu_{k,l}] \in \operatorname{Ball}(\mathbb{M}_n(M(\mathbb{G})))\} \qquad (n \in \mathbb{N}).$$

For $j = 1, \ldots, \nu$, let $S_j \in \mathcal{CB}(M(\mathbb{G}), L^1(\mathbb{G}))$ be defined as $(M_{a_j,b_j} \circ (\Gamma|m_0))^*$. By Proposition 3.2, this means that S_j belongs to the norm closure of the finite rank operators in $\mathcal{CB}(M(\mathbb{G}), L^1(\mathbb{G}))$. A simple calculation shows that

$$S_j \mu = b_j \mu a_j * m_0 \qquad (j = 1, \ldots, \nu,\ \mu \in M(\mathbb{G})),$$

so that

$$K_{j,n} = S_j^{(n)}(\operatorname{Ball}(\mathbb{M}_n(M(\mathbb{G})))) \qquad (j = 1, \ldots, \nu,\ n \in \mathbb{N}).$$

Invoking Lemma 4.4—with $\boldsymbol{K}_1, \ldots, \boldsymbol{K}_\nu$ as just defined—as well as (9), we obtain $\alpha_\epsilon \in \mathbb{A}$ such that

$$\sup_{n\in\mathbb{N}} \sup\left\{\left\|T_{\alpha_\epsilon}^{(n)} f\right\| : f \in K_{1,n} \cup \cdots \cup K_{\nu,n}\right\} < \frac{\epsilon}{2}$$

as well as

$$\|m_0 * m_{\alpha_\epsilon} - m_{\alpha_\epsilon}\| < \frac{1}{\max\{\|a_1 b_1\|, \ldots, \|a_\nu b_\nu\|, 1\}} \frac{\epsilon}{2}.$$

Set $m := m_0 * m_{\alpha_\epsilon}$.

To see that (10) holds, first note that

$$\begin{aligned}
\|M_{a_j,b_j} \circ (\Gamma|m) - a_j b_j \otimes m\| &\leq \|M_{a_j,b_j} \circ (\Gamma|m) - a_j b_j \otimes m_{\alpha_\epsilon}\| + \|a_j b_j \otimes m_{\alpha_\epsilon} - a_j b_j \otimes m\| \\
&= \|M_{a_j,b_j} \circ (\Gamma|m) - a_j b_j \otimes m_{\alpha_\epsilon}\| + \|a_j b_j\| \|m - m_{\alpha_\epsilon}\| \\
&< \|M_{a_j,b_j} \circ (\Gamma|m) - a_j b_j \otimes m_{\alpha_\epsilon}\| + \frac{\epsilon}{2} \qquad (j = 1, \ldots, \nu).
\end{aligned}$$

In order to establish (10), it is thus enough to show that

$$\|M_{a_j,b_j} \circ (\Gamma|m) - a_j b_j \otimes m_{\alpha_\epsilon}\| < \frac{\epsilon}{2} \qquad (j = 1, \ldots, \nu). \tag{11}$$

With $j \in \{1, \ldots, \nu\}$ fixed, note that

$$\begin{aligned}
&\|M_{a_j,b_j} \circ (\Gamma|m) - a_j b_j \otimes m_{\alpha_\epsilon}\| \\
&= \sup_{n\in\mathbb{N}} \sup\{\|[(M_{a_j,b_j} \circ (\Gamma|m)) x_{k,l} - a_j b_j \langle m_{\alpha_\epsilon}, x_{k,l}\rangle]\|_n : [x_{k,l}] \in \operatorname{Ball}(\mathbb{M}_n(L^\infty(\mathbb{G})))\}.
\end{aligned} \tag{12}$$

Let $\langle\langle\cdot,\cdot\rangle\rangle$ denote the matrix duality of [E–R]. By [E–R, (3.2.4)], the second supremum of the right hand side of (12) is then computed as

$$\sup\{\|\langle\langle[(M_{a_j,b_j}\circ(\Gamma|m))x_{k,l}-\langle m_{\alpha_\epsilon},x_{k,l}\rangle a_jb_j],[\mu_{\kappa,\lambda}]\rangle\rangle\|_{n^2}:\\ [x_{k,l}]\in\mathrm{Ball}(\mathbb{M}_n(L^\infty(\mathbb{G}))),\ [\mu_{\kappa,\lambda}]\in\mathrm{Ball}(\mathbb{M}_n(M(\mathbb{G})))\}. \quad (13)$$

For $x\in L^\infty(\mathbb{G})$ and $\mu\in M(\mathbb{G})$, we have

$$\begin{aligned}
&\langle(M_{a_j,b_j}\circ(\Gamma|m))x-\langle m_{\alpha_\epsilon},x\rangle a_jb_j,\mu\rangle\\
&=\langle\Gamma x,b_j\mu a_j\otimes m\rangle-\langle 1\otimes x,b_j\mu a_j\otimes m_{\alpha_\epsilon}\rangle\\
&=\langle\Gamma x,b_j\mu a_j\otimes(m_0*m_{\alpha_\epsilon})\rangle-\langle 1\otimes x,b_j\mu a_j\otimes m_{\alpha_\epsilon}\rangle\\
&=\langle(\mathrm{id}\otimes\Gamma)(\Gamma x),b_j\mu a_j\otimes m_0\otimes m_{\alpha_\epsilon}\rangle-\langle 1\otimes x,b_j\mu a_j\otimes m_{\alpha_\epsilon}\rangle\\
&=\langle(\Gamma\otimes\mathrm{id})(\Gamma x),b_j\mu a_j\otimes m_0\otimes m_{\alpha_\epsilon}\rangle-\langle 1\otimes x,b_j\mu a_j\otimes m_{\alpha_\epsilon}\rangle\\
&=\langle(\Gamma\otimes\mathrm{id})(\Gamma x)-\Gamma 1\otimes x,b_j\mu a_j\otimes m_0\otimes m_{\alpha_\epsilon}\rangle\\
&=\langle\Gamma x-1\otimes x,(b_j\mu a_j*m_0)\otimes m_{\alpha_\epsilon}\rangle\\
&=\langle(b_j\mu a_j*m_0)*m_{\alpha_\epsilon}-\langle b_j\mu a_j*m_0,1\rangle m_{\alpha,\epsilon},x\rangle.
\end{aligned}$$

Again from [E–R, (3.2.4)], we therefore conclude that (13) equals

$$\sup\{\|[(b_j\mu_{\kappa,\lambda}a_j*m_0)*m_{\alpha_\epsilon}-\langle b_j\mu_{\kappa,\lambda}a_j*m_0,1\rangle m_{\alpha_\epsilon}]\|_n:[\mu_{\kappa,\lambda}]\in\mathrm{Ball}(\mathbb{M}_n(M(\mathbb{G})))\}.$$

We thus have

$$\begin{aligned}
&\|M_{a_j,b_j}\circ(\Gamma|m)-a_jb_j\otimes m_{\alpha_\epsilon}\|\\
&=\sup_{n\in\mathbb{N}}\\
&\qquad\sup\{\|[(b_j\mu_{\kappa,\lambda}a_j*m_0)*m_{\alpha_\epsilon}-\langle b_j\mu_{\kappa,\lambda}a_j*m_0,1\rangle m_{\alpha_\epsilon}]\|_n:[\mu_{\kappa,\lambda}]\in\mathrm{Ball}(\mathbb{M}_n(M(\mathbb{G})))\}\\
&\le\sup_{n\in\mathbb{N}}\sup\{\|[f_{\kappa,\lambda}*m_{\alpha_\epsilon}-\langle f_{\kappa,\lambda},1\rangle m_{\alpha_\epsilon}]\|_n:[f_{\kappa,\lambda}]\in K_{j,n}\}\\
&=\sup_{n\in\mathbb{N}}\sup\{\|[T_{\alpha_\epsilon}f_{\kappa,\lambda}]\|_n:[f_{\kappa,\lambda}]\in K_{j,n}\}\\
&<\frac{\epsilon}{2},\qquad\text{by the choice of }\alpha_\epsilon,
\end{aligned}$$

for $j=1,\ldots,\nu$, i.e., (11) holds. $\square$

5 (P_2) and co-amenability

We finally turn to defining Reiter's property (P_2) for locally compact quantum groups.

Following [E–R], we denote the column and row operator space over a Hilbert space $\mathfrak{H}$ by $\mathfrak{H}_c$ and $\mathfrak{H}_r$, respectively. Given $T\in\mathcal{B}(\mathfrak{H})\bar{\otimes}\mathcal{B}(\mathfrak{H})$ and $\xi\in\mathfrak{H}$, we have a linear map

$$(T|\xi):\overline{\mathfrak{H}}\to\mathcal{B}(\mathfrak{H}),\quad \eta\mapsto(\mathrm{id}\otimes\omega_{\xi,\eta})(T),$$

where $\overline{\mathfrak{H}}$ denotes the complex conjugate Hilbert space of $\mathfrak{H}$. From the definition of row Hilbert space, it is routine to verify that $(T|\xi) \in \mathcal{CB}\left(\overline{\mathfrak{H}}_r, \mathcal{B}(\mathfrak{H})\right)$. By [E–R, p. 59], we can canonically identity $\overline{\mathfrak{H}}_r$ with $\mathfrak{H}_c^*$, and thus view $(T|\xi)$ as an operator in $\mathcal{CB}(\mathfrak{H}_c^*, \mathcal{B}(\mathfrak{H}))$.

We have the following L^2-analog of Proposition 3.2:

Proposition 5.1. *Let $\mathbb{G}$ be a locally compact quantum group with multiplicative unitary W, let $\xi \in L^2(\mathbb{G})$, and let $a, b \in \mathcal{C}_0(\mathbb{G})$. Then $M_{a,b} \circ (W|\xi)$ is a completely bounded operator from $L^2(\mathbb{G})_c^*$ to $\mathcal{C}_0(\mathbb{G})$ that lies in the* cb*-norm closure of the finite rank operators in $\mathcal{CB}(L^2(\mathbb{G})_c^*, \mathcal{C}_0(\mathbb{G}))$ and can be identified with an element of $\mathcal{C}_0(\mathbb{G})\check{\otimes}L^2(\mathbb{G})_c$.*

Proof. Choose $L \in \mathcal{K}(L^2(\mathbb{G}))$ with $L\xi = \xi$, so that $(a \otimes 1)W(b \otimes L) \in \mathcal{C}_0(\mathbb{G})\check{\otimes}\mathcal{K}(L^2(\mathbb{G}))$. By the definition of $L^2(\mathbb{G})_c$, the linear map

$$T_\xi \colon \mathcal{K}(L^2(\mathbb{G})) \to L^2(\mathbb{G})_c, \quad K \mapsto K\xi$$

is completely bounded, so that

$$(\mathrm{id} \otimes T_\xi)((a \otimes 1)W(b \otimes L)) \in \mathcal{C}_0(\mathbb{G})\check{\otimes}L^2(\mathbb{G})_c.$$

Embedding $\mathcal{C}_0(\mathbb{G})\check{\otimes}L^2(\mathbb{G})_c$ canonically into $\mathcal{CB}(L^2(\mathbb{G})_c^*, \mathcal{C}_0(\mathbb{G}))$, we see that

$$\begin{aligned}
(\mathrm{id} \otimes T_\xi)((a \otimes 1)W(b \otimes L))\eta &= (\mathrm{id} \otimes \omega_{\xi,\eta})((a \otimes 1)W(b \otimes L)) \\
&= M_{a,b} \circ (\mathrm{id} \otimes \omega_{L\xi,\eta})(W) \\
&= M_{a,b} \circ (\mathrm{id} \otimes \omega_{\xi,\eta})(W) \\
&= (M_{a,b} \circ (W|\xi))\eta \qquad (\eta \in \mathfrak{H}),
\end{aligned}$$

which completes the proof. $\square$

Let G be a locally compact group, let $a, b \in \mathcal{C}_0(G)$, and let $\xi \in L^2(G)$. Then it is easily checked that $M_{a,b} \circ (W|\xi) = abL_\bullet(\xi)$. With this in mind, we define:

Definition 5.2. Let $\mathbb{G}$ be a locally compact quantum group with multiplicative unitary W. We say that $\mathbb{G}$ has *Reiter's property* (P_2) if there is a net $(\xi_\alpha)_\alpha$ of unit vectors in $L^2(\mathbb{G})$ such that

$$\lim_\alpha \|M_{a,b} \circ (W|\xi_\alpha) - ab \otimes \xi_\alpha\|_{\mathcal{C}_0(\mathbb{G})\check{\otimes}L^2(\mathbb{G})_c} = 0$$

for all $a, b \in \mathcal{C}_0(\mathbb{G})$.

Remarks. 1. Let G be a locally compact group with (P_2) in the sense of Definition 1.2, and let $(\xi_\alpha)_\alpha$ be a net in $L^2(G)$ as required by that definition; then $(\xi_\alpha)_\alpha$ clearly satisfies Definition 5.2. On the other hand, if G has property (P_2) in the sense of Definition 5.2 and if $(\xi_\alpha)_\alpha$ is a corresponding net of unit vectors in $L^2(G)$, then $(|\xi_\alpha|)_\alpha$, where the modulus is taken pointwise almost everywhere, satisfies Definition 1.1.

2. Since $L^\infty(\mathbb{G})$ is in standard form on $L^2(\mathbb{G})$ ([Tak 2, Definition IX.1.13]), there is a canonical self-dual cone $L^2(\mathbb{G})_+$ in $L^2(\mathbb{G})$ that provides a notion of positivity in $L^2(\mathbb{G})$. We could thus have required the net $(\xi_\alpha)_\alpha$ in Definition 5.2 to be from $L^2(\mathbb{G})_+$. The reason why we haven't done this is Theorem 5.4 below: we do not know if it remains true with this additional requirement. (Unlike in the group case, we cannot conclude that, if a net as in Definition 5.2 exists, then it can always be found in $L^2(\mathbb{G})_+$; see also the remark immediately after the proof of Theorem 5.4.)

For our second main result, recall the definition of a co-amenable, locally compact quantum group:

Definition 5.3. A locally compact quantum group $\mathbb{G}$ is called *co-amenable* if the Banach algebra $L^1(\mathbb{G})$ has a bounded approximate identity.

Remarks. 1. There are several descriptions of co-amenability equivalent to Definition 5.3: see [B–T, Theorem 3.1]. In particular, $\mathbb{G}$ is co-amenable if and only if there is a net $(\xi_\alpha)_\alpha$ of unit vectors in $L^2(\mathbb{G})$ such that

$$\|W(\xi_\alpha \otimes \eta) - \xi_\alpha \otimes \eta\| \to 0 \qquad (\eta \in L^2(\mathbb{G})).$$

2. If $\hat{\mathbb{G}}$ is co-amenable ([B–T, Theorem 3.2]), then $\mathbb{G}$ is amenable whereas the converse is unknown unless $\mathbb{G}$ is discrete ([Tom]) or a group ([Lep]).

Theorem 5.4. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following are equivalent:*

(i) *$\mathbb{G}$ has (P_2);*

(ii) *$\hat{\mathbb{G}}$ is co-amenable.*

Proof. (i) $\Longrightarrow$ (ii): Let $(\xi_\alpha)_\alpha$ be a net as required by Definition 5.2. We claim that

$$\|\hat{W}(\xi_\alpha \otimes \eta) - \xi_\alpha \otimes \eta\| \to 0 \qquad (\eta \in L^2(\mathbb{G}))$$

or rather—equivalently by the definition of $\hat{W}$—

$$\|W(\eta \otimes \xi_\alpha) - \eta \otimes \xi_\alpha\| \to 0 \qquad (\eta \in L^2(\mathbb{G})). \tag{14}$$

Let $\eta \in L^2(\mathbb{G})$ be a unit vector, and use Cohen's factorization theorem ([Dal, Corollary 2.9.26]) to obtain $a \in \mathcal{C}_0(\mathbb{G})$ and $\zeta \in L^2(\mathbb{G})$ such that $\eta = a\zeta$. By Definition 5.2,

$$\|M_{a^*,a} \circ (W|\xi_\alpha) - a^*a \otimes \xi_\alpha\|_{\mathcal{C}_0(\mathbb{G})\check{\otimes} L^2(\mathbb{G})_c} \to 0$$

holds. By the definition of column Hilbert space, the map

$$T_\zeta \colon \mathcal{C}_0(\mathbb{G}) \to L^2(\mathbb{G})_c, \quad b \mapsto b\zeta$$

lies in $\mathcal{CB}(\mathcal{C}_0(\mathbb{G}), L^2(\mathbb{G})_c)$ with $\|T_\zeta\|_{\mathrm{cb}} \leq \|\zeta\|$. Since $L^2(\mathbb{G})_c \check{\otimes} L^2(\mathbb{G})_c = (L^2(\mathbb{G}) \tilde{\otimes}_2 L^2(\mathbb{G}))_c$ ([E–R, Proposition 9.3.5]), it follows that

$$\begin{aligned}
&\|(a^* \otimes 1)W(\eta \otimes \xi_\alpha) - a^*\eta \otimes \xi_\alpha\|_{L^2(\mathbb{G})\tilde{\otimes}_2 L^2(\mathbb{G})} \\
&= \|(a^* \otimes 1)W(\eta \otimes \xi_\alpha) - a^*\eta \otimes \xi_\alpha\|_{L^2(\mathbb{G})_c \check{\otimes} L^2(\mathbb{G})_c} \\
&= \|(T_\zeta \otimes \mathrm{id})(M_{a^*,a} \circ (W|\xi_\alpha) - a^*a \otimes \xi_\alpha)\|_{L^2(\mathbb{G})_c \check{\otimes} L^2(\mathbb{G})_c} \\
&\leq \|\zeta\| \|M_{a^*,a} \circ (W|\xi_\alpha) - a^*a \otimes \xi_\alpha\|_{\mathcal{C}_0(\mathbb{G}) \check{\otimes} L^2(\mathbb{G})_c} \\
&\to 0
\end{aligned}$$

and thus

$$\begin{aligned}
1 &= \lim_\alpha \langle \eta \otimes \xi_\alpha, \eta \otimes \xi_\alpha \rangle \\
&= \lim_\alpha \langle a\zeta \otimes \xi_\alpha, \eta \otimes \xi_\alpha \rangle \\
&= \lim_\alpha \langle \zeta \otimes \xi_\alpha, a^*\eta \otimes \xi_\alpha \rangle \\
&= \lim_\alpha \langle \zeta \otimes \xi_\alpha, (a^* \otimes 1)W(\eta \otimes \xi_\alpha) \rangle \\
&= \lim_\alpha \langle a\zeta \otimes \xi_\alpha, W(\eta \otimes \xi_\alpha) \rangle \\
&= \lim_\alpha \langle \eta \otimes \xi_\alpha, W(\eta \otimes \xi_\alpha) \rangle,
\end{aligned}$$

which means that (14) holds.

(ii) $\Longrightarrow$ (i): Let $a_1, b_1, \ldots, a_\nu, b_\nu \in \mathcal{C}_0(\mathbb{G})$, and let $\epsilon > 0$. It is enough to show that there is a vector $\xi \in \mathrm{Ball}(L^2(\mathbb{G}))$ with $\|\xi\| \geq 1 - \epsilon$ such that

$$\|M_{a_j,b_j} \circ (W|\xi) - a_j b_j \otimes \xi\|_{\mathcal{C}_0(\mathbb{G}) \check{\otimes} L^2(\mathbb{G})_c} < \epsilon \qquad (j = 1, \ldots, \nu). \tag{15}$$

Since $\hat{\mathbb{G}}$ is co-amenable, there is a net $(\xi_\alpha)_{\alpha \in \mathbb{A}}$ of unit vectors in $L^2(\mathbb{G})$ such that (14) holds; it follows easily from (14) that

$$\|\lambda_2(f)\xi_\alpha - \langle f, 1 \rangle \xi_\alpha\| \to 0 \qquad (f \in L^1(\mathbb{G})). \tag{16}$$

For $\alpha \in \mathbb{A}$, define

$$T_\alpha \colon L^1(\mathbb{G}) \to L^2(\mathbb{G}), \quad f \mapsto \lambda_2(f)\xi_\alpha - \langle f, 1 \rangle \xi_\alpha.$$

The net $(T_\alpha)_\alpha$ lies in $\mathcal{CB}(L^1(\mathbb{G}), L^2(\mathbb{G})_c)$, is norm bounded, and converges to 0 pointwise on $L^1(\mathbb{G})$ by (16).

Let $m_0 \in L^1(\mathbb{G})$ be an arbitrary state, and define, for $j = 1, \ldots, \nu$, matricial subsets $\boldsymbol{K}_j = (K_{j,n})_{n=1}^\infty$ of $L^1(\mathbb{G})$ just as in the proof of Theorem 4.5. By Lemma 4.4 and (16), there is $\alpha_\epsilon \in \mathbb{A}$ such that

$$\sup_{n \in \mathbb{N}} \sup \left\{ \left\| T_{\alpha_\epsilon}^{(n)} f \right\| : f \in K_{1,n} \cup \cdots \cup K_{\nu,n} \right\} < \frac{\epsilon}{2}$$

as well as

$$\|\lambda_2(m_0)\xi_{\alpha_\epsilon} - \xi_{\alpha_\epsilon}\| < \frac{1}{\max\{\|a_1b_1\|, \ldots, \|a_\nu b_\nu\|, 1\}} \frac{\epsilon}{2}. \tag{17}$$

Set $\xi := \lambda_2(m_0)\xi_{\alpha_\epsilon}$. It is clear that $\|\xi\| \leq 1$, and by (17), we also have $\|\xi\| > 1 - \frac{\epsilon}{2} > 1 - \epsilon$.

To prove (15) holds, first note that

$$\begin{aligned}\|M_{a_j,b_j} \circ (W|\xi) - a_jb_j \otimes \xi\| &\leq \|M_{a_j,b_j} \circ (W|\xi) - a_jb_j \otimes \xi_{\alpha_\epsilon}\| + \|a_jb_j \otimes \xi_{\alpha_\epsilon} - a_jb_j \otimes \xi\| \\ &= \|M_{a_j,b_j} \circ (W|\xi) - a_jb_j \otimes \xi_{\alpha_\epsilon}\| + \|a_jb_j\|\|\xi - \xi_{\alpha_\epsilon}\| \\ &< \|M_{a_j,b_j} \circ (W|\xi) - a_jb_j \otimes \xi_{\alpha_\epsilon}\| + \frac{\epsilon}{2} \qquad (j = 1, \ldots, \nu),\end{aligned}$$

so that it is sufficient to show that

$$\|M_{a_j,b_j} \circ (W|\xi) - a_jb_j \otimes \xi_{\alpha_\epsilon}\| < \frac{\epsilon}{2} \qquad (j = 1, \ldots, \nu). \tag{18}$$

With $j \in \{1, \ldots, \nu\}$ fixed, observe that

$$\begin{aligned}&\|M_{a_j,b_j} \circ (W|\xi) - a_jb_j \otimes \xi_{\alpha_\epsilon}\| \\ &\quad = \sup_{n\in\mathbb{N}} \sup\{\|[(M_{a_j,b_j} \circ (W|\xi))\eta_{k,l} - \langle \xi_{\alpha_\epsilon}, \eta_{k,l}\rangle a_jb_j]\|_n : [\eta_{k,l}] \in \mathrm{Ball}(\mathbb{M}_n(L^2(\mathbb{G})_c^*))\}\end{aligned} \tag{19}$$

and that the second supremum of the right hand side of (19) is

$$\begin{aligned}&\sup\{\|\langle\langle[(M_{a_j,b_j} \circ (W|\xi))\eta_{k,l} - \langle \xi_{\alpha_\epsilon}, \eta_{k,l}\rangle a_jb_j], [\mu_{\kappa,\lambda}]\rangle\rangle\|_{n^2} : \\ &\qquad [\eta_{k,l}] \in \mathrm{Ball}(\mathbb{M}_n(L^2(\mathbb{G})_c^*)),\ [\mu_{\kappa,\lambda}] \in \mathrm{Ball}(\mathbb{M}_n(M(\mathbb{G})))\}.\end{aligned} \tag{20}$$

Then note that, for $\eta \in L^2(\mathbb{G})^*$ and $\mu \in M(\mathbb{G})$, we have

$$\begin{aligned}&\langle (M_{a_j,b_j} \circ (W|\xi))\eta - \langle \xi_{\alpha_\epsilon}, \eta\rangle a_jb_j, \mu\rangle \\ &= \langle \lambda_2(b_j\mu a_j)\xi - \langle b_j\mu a_j, 1\rangle \xi_{\alpha_\epsilon}, \eta\rangle \\ &= \langle \lambda_2(b_j\mu a_j * m_0)\xi_{\alpha_\epsilon} - \langle b_j\mu a_j * m_0, 1\rangle \xi_{\alpha_\epsilon}, \eta\rangle,\end{aligned}$$

so that (20) can also be computed as

$$\sup\{\|[\lambda_2(b_j\mu_{\kappa,\lambda}a_j * m_0)\xi_{\alpha_\epsilon} - \langle b_j\mu_{\kappa,\lambda}a_j * m_0, 1\rangle \xi_{\alpha_\epsilon}]\|_n : [\mu_{\kappa,\lambda}] \in \mathrm{Ball}(\mathbb{M}_n(M(\mathbb{G})))\}.$$

We conclude that

$$\begin{aligned}&\|M_{a_j,b_j} \circ (W|\xi) - a_jb_j \otimes \xi_{\alpha_\epsilon}\| \\ &= \sup_{n\in\mathbb{N}} \sup\{\|[\lambda_2(b_j\mu_{\kappa,\lambda}a_j * m_0)\xi_{\alpha_\epsilon} - \langle b_j\mu_{\kappa,\lambda}a_j * m_0, 1\rangle \xi_{\alpha_\epsilon}]\|_n : [\mu_{\kappa,\lambda}] \in \mathrm{Ball}(\mathbb{M}_n(M(\mathbb{G})))\} \\ &\leq \sup_{n\in\mathbb{N}} \sup\{\|[\lambda_2(f_{\kappa,\lambda})\xi_{\alpha_\epsilon} - \langle f_{\kappa,\lambda}, 1\rangle \xi_{\alpha_\epsilon}]\|_n : [f_{\kappa,\lambda}] \in K_{j,n}\} \\ &= \sup_{n\in\mathbb{N}} \sup\{\|[T_{\alpha_\epsilon} f_{\kappa,\lambda}]\|_n : [f_{\kappa,\lambda}] \in K_{j,n}\} \\ &< \frac{\epsilon}{2}\end{aligned}$$

for $j = 1, \ldots, \nu$, so that (18) holds. $\square$

Remark. In the proof of (ii) $\Longrightarrow$ (i), we could have chosen the net $(\xi_\alpha)_\alpha$ satisfying (14) from $L^2(\hat{\mathbb{G}})_+$. This, however, does not mean that the resulting net satisfying Definition 5.2 belongs to $L^2(\mathbb{G})_+$. First of all, even though $L^2(\mathbb{G}) = L^2(\hat{\mathbb{G}})$ holds by the definition of $\hat{\mathbb{G}}$, there is no need for $L^2(\mathbb{G})_+$ and $L^2(\hat{\mathbb{G}})_+$ to coincide (or even be related). Furthermore, even *if* we could pick a net $(\xi_\alpha)_\alpha$ from $L^2(\mathbb{G})_+$ such that (14) holds, then it is not clear that the resulting net for Definition 5.2 would lie in $L^2(\mathbb{G})_+$ as well.

Combining Theorem 4.5 and 5.4 and [B–T, Theorem 3.2], we obtain:

Corollary 5.5. *Let $\mathbb{G}$ be a locally compact quantum group with (P_2). Then $\mathbb{G}$ has (P_1).*

Remarks.

1. We believe that (P_1) and (P_2) are, in fact, equivalent, which—in view of Theorems 4.5 and 5.4—would immediately yield Leptin's theorem for locally compact quantum groups. For a locally compact group G, the implication from (P_1) to (P_2) is a straightforward consequence of the elementary inequality

$$\|f - g\|_2^2 \leq \|f^2 - g^2\|_1 \qquad (f, g \in L^2(G)_+). \tag{21}$$

There is a non-commutative variant of (21) for von Neumann algebras in standard form ([Tak 2, Theorem IX.1.2(iv)]), however, in order to get from (P_1) to (P_2) in the group case, we have to apply (21) to L^2-valued, continuous functions on G. We thus believe that, in order to derive (P_2) from (P_1) in a general quantum group context, an operator valued version of [Tak 2, Theorem IX.1.2(iv)] is necessary, e.g., in the framework of C^*-valued weights (see [K–V 1] and [Kus, Section 1], for instance).

2. We have not dealt with property (P_p) for locally compact quantum groups for any $p \in [1, \infty)$ other than 1 or 2. For any von Neumann algebra $\mathfrak{M}$ and $p \in (1, \infty)$, there is a unique so-called non-commutative L^p-space $L^p(\mathfrak{M})$ (see [Haa], [Izu], and [Ter] for various constructions). For a locally compact quantum group $\mathbb{G}$, one might thus define $L^p(\mathbb{G})$ as $L^p(L^\infty(\mathbb{G}))$. However, it seems to be unclear, at least for now, how $L^1(\mathbb{G})$ could be made to act on $L^p(\mathbb{G})$ in a satisfactory manner that would enable us to even define (P_p) for arbitrary p. For a locally compact group G, B. E. Forrest, H. H. Lee, and E. Samei recently equipped $L^p(\hat{G})$ with an $L^1(\hat{G})$-, i.e., $A(G)$-, module structure ([F–L–S]), and a related, but not entirely identical construction was carried out by the first named author in [Daw]. Both in [F–L–S] and [Daw], the non-commutative L^p-spaces are obtained through complex interpolation, following [Izu]. It remains to be seen whether the constructions from [F–L–S] or [Daw] can be extended to general locally compact quantum groups and whether they can be used to define, in a meaningful way, property (P_p) for locally compact quantum groups for arbitrary p.

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