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Connes-amenability of bidual and weighted semigroup algebras

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Abstract

We investigate the notion of Connes-amenability, introduced by Runde in [14], for bidual algebras and weighted semigroup algebras. We provide some simplifications to the notion of a σWC-virtual diagonal, as introduced in [10], especially in the case of the bidual of an Arens regular Banach algebra. We apply these results to discrete, weighted, weakly cancellative semigroup algebras, showing that these behave in the same way as C*-algebras with regards Connes-amenability of the bidual algebra. We also show that for each one of these cancellative semigroup algebras $l^1(S,\omega)$, we have that $l^1(S,\omega)$ is Connes-amenable (with respect to the canonical predual $c_0(S)$) if and only if $l^1(S,\omega)$ is amenable, which is in turn equivalent to S being an amenable group. This latter point was first shown by Grönbæk in [5], but we provide a unified proof. Finally, we consider the homological notion of injectivity, and show that here, weighted semigroup algebras do not behave like C*-algebras.

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1 Introduction

We first fix some notation, following [2]. For a Banach space E, we let E' be its dual space, and for $\mu \in E'$ and $x \in E$, we write $\langle \mu, x\rangle = \mu(x)$ for notational convenience. We then have the canonical map $\kappa_E : E \to E''$ defined by $\langle \kappa_E(x), \mu\rangle = \langle \mu, x\rangle$ for $\mu \in E', x \in E$. For Banach spaces E and F, we write $\mathcal{B}(E,F)$ for the Banach space of bounded linear maps between E and F. We write $\mathcal{B}(E,E) = \mathcal{B}(E)$. For $T \in \mathcal{B}(E,F)$, the *adjoint* of T is $T' \in \mathcal{B}(F',E')$, defined by $\langle T'(\mu), x\rangle = \langle \mu, T(x)\rangle$, for $\mu \in F'$ and $x \in E$.

Let $\mathcal{A}$ be a Banach algebra. A *Banach left $\mathcal{A}$-module* is a Banach space E together with a bilinear map $\mathcal{A} \times E \to E; (a,x) \mapsto a\cdot x$, such that $\|a\cdot x\| \leq \|a\|\|x\|$ and $a\cdot(b\cdot x) = ab\cdot x$ for $a,b \in \mathcal{A}$ and $x \in E$. Similarly, we have the notion of a *Banach right $\mathcal{A}$-module* and a *Banach $\mathcal{A}$-bimodule*. If E is a Banach $\mathcal{A}$-bimodule (resp. left or right module) then $\mathcal{A}'$ is a Banach $\mathcal{A}$-bimodule (resp. right or left module) with module action given by

$$\langle a\cdot\mu, x\rangle = \langle \mu, x\cdot a\rangle \qquad \langle \mu\cdot a, x\rangle = \langle \mu, a\cdot x\rangle \qquad (a \in \mathcal{A}, x \in E).$$

Notice that as $\mathcal{A}$ is certainly a bimodule over itself (with module action induced by the algebra product) we also have that $\mathcal{A}'$, $\mathcal{A}''$ etc. are Banach $\mathcal{A}$-bimodules. Given a Banach $\mathcal{A}$-bimodule E, a subspace F of E is a *submodule* if $a\cdot x, x\cdot a \in F$ for each $a \in \mathcal{A}$ and $x \in F$. For Banach $\mathcal{A}$-bimodules E and F, $T \in \mathcal{B}(E,F)$ is an *$\mathcal{A}$-bimodule homomorphism* when

$$a\cdot T(x) = T(a\cdot x) \qquad T(x)\cdot a = T(x\cdot a) \qquad (a \in \mathcal{A}, x \in E).$$

A linear map $d : \mathcal{A} \to E$ between a Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule E is a *derivation* if $d(ab) = a\cdot d(b) + d(a)\cdot b$ for $a,b \in \mathcal{A}$. For $x \in E$, we define $\delta_x : \mathcal{A} \to E$ by $\delta_x(a) = a\cdot x - x\cdot a$. Then δ_x is a derivation, called an *inner derivation*.

A Banach algebra $\mathcal{A}$ is said to be *super-amenable* or *contractable* if every bounded derivation $d : \mathcal{A} \to E$, for every Banach $\mathcal{A}$-bimodule E, is inner. For example, a C*-algebra $\mathcal{A}$ is super-amenable if and only if $\mathcal{A}$ is finite-dimensional. It is conjectured that there are no infinite-dimensional, super-amenable Banach algebras.

If we restrict to derivations to E' for Banach $\mathcal{A}$-bimodules E then we arrive at the notion of *amenability*. For example, a C*-algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is nuclear; a group algebra $L^1(G)$ is amenable if and only if the locally compact group G is amenable (which is the motivating example). See [13] for further discussions of amenability and related notions.

Let E be a Banach space and F a closed subspace of E. Then we naturally, isometrically, identify F' with E'/F°, where

$$F^\circ = \{\mu \in E' : \langle \mu, x\rangle = 0 \ (x \in F)\}.$$

Definition 1.1. Let E be a Banach space and E_* be a closed subspace of E'. Let $\pi_{E_*} : E'' \to E''/E_*^\circ$ be the quotient map, and suppose that $\pi_{E_*} \circ \kappa_E$ is an isomorphism from E to E_*'. Then we say that E is a *dual Banach space* with *predual* E_*.

When $\mathcal{A}$ is a dual Banach space with predual $\mathcal{A}_*$ which is also a submodule of $\mathcal{A}'$ we say that $\mathcal{A}$ is a *dual Banach algebra.* □

For a dual Banach algebra $\mathcal{A}$ with predual $\mathcal{A}_*$, we henceforth identify $\mathcal{A}$ with $\mathcal{A}_*'$. Thus we get a weak*-topology on $\mathcal{A}$, which we denote by $\sigma(\mathcal{A}, \mathcal{A}_*)$. It is a simple exercise to show that $\mathcal{A}$ is a dual Banach algebra if and only if $\mathcal{A}$ is a dual Banach space such that the algebra product is separately $\sigma(\mathcal{A}, \mathcal{A}_*)$-continuous (see [14]). The following lemma is standard.

Lemma 1.2. *Let E and F be dual Banach spaces with preduals E_* and F_* respectively, and let $T \in \mathcal{B}(E, F)$. Then the following are equivalent:*

1. *T is $\sigma(E, E_*) - \sigma(F, F_*)$ continuous;*
2. *$T'(\kappa_{F_*}(F_*)) \subseteq \kappa_{E_*}(E_*)$;*
3. *there exists $S \in \mathcal{B}(F_*, E_*)$ such that $S' = T$.*

□

As noticed by Runde (see [14]), there are very few Banach algebras which are both dual and amenable. For von Neumann algebras, which are the motivating example of dual Banach algebras, there is a weaker notion of amenablity, called Connes-amenability, which has a natural generalisation to the case of dual Banach algebras.

Definition 1.3. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. Let E be a Banach $\mathcal{A}$-bimodule. Then E' is a *w*-Banach $\mathcal{A}$-bimodule* if, for each $\mu \in E'$, the maps

$$\mathcal{A} \to E', \quad a \mapsto \begin{cases} a \cdot \mu, \\ \mu \cdot a \end{cases}$$

are $\sigma(\mathcal{A}, \mathcal{A}_*) - \sigma(E', E)$ continuous.

Then $(\mathcal{A}, \mathcal{A}_*)$ is Connes-amenable if, for each w*-Banach $\mathcal{A}$-bimodule E', each derivation $d : \mathcal{A} \to E'$, which is $\sigma(\mathcal{A}, \mathcal{A}_*) - \sigma(E', E)$ continuous, is inner. □

Given a Banach algebra $\mathcal{A}$, we define bilinear maps $\mathcal{A}'' \times \mathcal{A}' \to \mathcal{A}'$ and $\mathcal{A}' \times \mathcal{A}'' \to \mathcal{A}'$ by

$$\langle \Phi \cdot \mu, a\rangle = \langle \Phi, \mu \cdot a\rangle \quad \langle \mu \cdot \Phi, a\rangle = \langle \Phi, a \cdot \mu\rangle \qquad (\Phi \in \mathcal{A}'', \mu \in \mathcal{A}', a \in \mathcal{A}).$$

We then define two bilinear maps $\Box, \Diamond : \mathcal{A}'' \times \mathcal{A}'' \to \mathcal{A}''$ by

$$\langle \Phi \Box \Psi, \mu \rangle = \langle \Phi, \Psi \cdot \mu \rangle \quad \langle \Phi \Diamond \Psi, \mu \rangle = \langle \Psi, \mu \cdot \Phi \rangle \qquad (\Phi, \Psi \in \mathcal{A}'', \mu \in \mathcal{A}').$$

We can check that $\Box$ and $\Diamond$ are actually algebra products, called the *first* and *second Arens products* respectively. Then $\kappa_A : \mathcal{A} \to \mathcal{A}''$ is a homomorphism with respect to either Arens product. When $\Box = \Diamond$, we say that $\mathcal{A}$ is *Arens regular*. In particular, when $\mathcal{A}$ is Arens regular, we may check that $\mathcal{A}''$ is a dual Banach algebra with predual $\mathcal{A}'$.

Theorem 1.4. *Let $\mathcal{A}$ be an Arens regular Banach algebra. When $\mathcal{A}$ is amenable, $\mathcal{A}''$ is Connes-amenable. If $\kappa_{\mathcal{A}}(\mathcal{A})$ is an ideal in $\mathcal{A}''$ and $\mathcal{A}''$ is Connes-amenable, then $\mathcal{A}$ is amenable.*

Let $\mathcal{A}$ be a C-algebra. Then $\mathcal{A}$ is Arens regular, and $\mathcal{A}''$ is Connes-amenable if and only if $\mathcal{A}$ is amenable.*

Proof. The first statements are [14, Corollary 4.3] and [14, Theorem 4.4]. The statement about C*-algebras is detailed in [13, Chapter 6]. $\Box$

Another class of Connes-amenable dual Banach algebras is given by Runde in [11], where it is shown that $M(G)$, the measure algebra of a locally compact group G, is amenable if and only if G is amenable.

The organisation of this paper is as follows. Firstly, we study intrinsic characterisations of amenability, recalling a result of Runde from [10]. We then simplify these conditions in the case of Arens regular Banach algebras. We recall the notion of an *injective* module, and quickly note how Connes-amenability can be phrased in this language. The final section of the paper then applies these ideas to weighted semigroup algebras. We finish with some open questions.

2 Characterisations of amenability

Let E and F be Banach spaces, and form the algebraic tensor product $E \otimes F$. We can norm $E \otimes F$ with the *projective tensor norm*, defined as

$$\|u\|_\pi = \inf \left\{ \sum_{k=1}^{n} \|x_k\| \|y_k\| : u = \sum_{k=1}^{n} x_k \otimes y_k \right\} \qquad (u \in E \otimes F).$$

Then the completion of $(E \otimes F, \|\cdot\|_\pi)$ is $E \widehat{\otimes} F$, the *projective tensor product* of E and F.

Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A} \widehat{\otimes} \mathcal{A}$ is a Banach $\mathcal{A}$-bimodule for the module actions given by

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \qquad (a \in \mathcal{A}, b \otimes c \in \mathcal{A} \widehat{\otimes} \mathcal{A}).$$

Define $\Delta_{\mathcal{A}} : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ by $\Delta_{\mathcal{A}}(a \otimes b) = ab$. Then $\Delta_{\mathcal{A}}$ is an $\mathcal{A}$-bimodule homomorphism.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*

1. *$\mathcal{A}$ is amenable;*
2. *$\mathcal{A}$ has a* virtual diagonal*, which is a functional $M \in (\mathcal{A} \widehat{\otimes} \mathcal{A})''$ such that $a \cdot M = M \cdot a$ and $\Delta''_{\mathcal{A}}(M) \cdot a = \kappa_{\mathcal{A}}(a)$ for each $a \in \mathcal{A}$.*

$\Box$

Runde introduced, in [10], the following notion in order to prove a version of the above theorem for Connes-amenability.

Definition 2.2. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$, and let E be a Banach $\mathcal{A}$-bimodule. Then $x \in \sigma WC(E)$ if and only if the maps $\mathcal{A} \to E$,

$$a \mapsto \begin{cases} a \cdot x, \\ x \cdot a \end{cases}$$

are $\sigma(\mathcal{A}, \mathcal{A}_*) - \sigma(E, E')$ continuous. □

It is clear that $\sigma WC(E)$ is a closed submodule of E. The $\mathcal{A}$-bimodule homomorphism $\Delta_{\mathcal{A}}$ has adjoint $\Delta'_{\mathcal{A}} : \mathcal{A}' \to (\mathcal{A}\widehat{\otimes}\mathcal{A})'$. In [10, Corollary 4.6] it is shown that $\Delta'_{\mathcal{A}}(\mathcal{A}_*) \subseteq \sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})')$. Consequently, we can view $\Delta'_{\mathcal{A}}$ as a map $\mathcal{A}_* \to \sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})')$, and hence view $\Delta''_{\mathcal{A}}$ as a map $\sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})')' \to \mathcal{A}'_* = \mathcal{A}$, denoted by $\tilde{\Delta}_{\mathcal{A}}$.

Theorem 2.3. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. Then the following are equivalent:*

1. *$\mathcal{A}$ is Connes-amenable;*

2. *$\mathcal{A}$ has a σWC-*virtual diagonal*, which is $M \in \sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})')'$ such that $a \cdot M = M \cdot a$ and $a\tilde{\Delta}_{\mathcal{A}}(M) = a$ for each $a \in \mathcal{A}$.*

Proof. This is [10, Theorem 4.8]. □

In particular, we see that a Connes-amenable Banach algebra is unital (which can of course be shown in an elementary fashion, as in [14, Proposition 4.1]).

3 Connes-amenability for biduals of algebras

Recall Gantmacher's theorem, which states that a bounded linear map $T : E \to F$ between Banach spaces E and F is *weakly-compact* if and only if $T''(E'') \subseteq \kappa_F(F)$. We write $\mathcal{W}(E, F)$ for the collection of weakly-compact operators in $\mathcal{B}(E, F)$.

Lemma 3.1. *Let E be a dual Banach space with predual E_*, let F be a Banach space, and let $T \in \mathcal{B}(E, F')$. Then the following are equivalent, and in particular each imply that T is weakly-compact:*

1. *T is $\sigma(E, E_*) - \sigma(F', F'')$ continuous;*

2. *$T'(F'') \subseteq \kappa_{E_*}(E_*)$;*

3. *there exists $S \in \mathcal{W}(F, E_*)$ such that $S' = T$.*

Proof. That (1) and (2) are equivalent is standard (compare with Lemma 1.2).

Suppose that (2) holds, so that we may define $S \in \mathcal{B}(F, E_*)$ by $\kappa_{E_*} \circ S = T' \circ \kappa_F$. Then, for $x \in E$ and $y \in F$, we have

$$\langle x, S(y) \rangle = \langle T'(\kappa_F(y)), x \rangle = \langle T(x), y \rangle,$$

so that $S' = T$. Then $S''(F'') = T'(F'') \subseteq \kappa_{E_*}(E_*)$, so that S is weakly-compact, by Gantmacher's Theorem, so that (3) holds.

Conversely, if (3) holds, as S is weakly-compact, we have $\kappa_{E*}(E_*) \supseteq S''(F'') = T'(F'')$, so that (2) holds. □

It is standard that for Banach spaces E and F, we have $(E\widehat{\otimes}F)' = \mathcal{B}(F, E')$ with duality defined by

$$\langle T, x \otimes y\rangle = \langle T(y), x\rangle \qquad (T \in \mathcal{B}(F, E'), x \otimes y \in E\widehat{\otimes}F).$$

Then we see, for $a, b, c \in \mathcal{A}$ and $T \in (\mathcal{A}\widehat{\otimes}\mathcal{A})' = \mathcal{B}(\mathcal{A}, \mathcal{A}')$, that $\langle a \cdot T, b \otimes c\rangle = \langle T(ca), b\rangle$ and that $\langle T \cdot a, b \otimes c\rangle = \langle T(c), ab\rangle = \langle T(c) \cdot a, b\rangle$ so that

$$(a \cdot T)(c) = T(ca), \quad (T \cdot a)(c) = T(c) \cdot a \qquad (a, c \in \mathcal{A}, T : \mathcal{A} \to \mathcal{A}'). \tag{1}$$

Notice that we could also have defined $(E\widehat{\otimes}F)'$ to be $\mathcal{B}(E, F')$. This would induce a different bimodule structure on $\mathcal{B}(\mathcal{A}, \mathcal{A}')$, and we shall see in Section 4 that our chosen convention seems more natural for the task at hand.

Proposition 3.2. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. For $T \in \mathcal{B}(\mathcal{A}, \mathcal{A}') = (\mathcal{A}\widehat{\otimes}\mathcal{A})'$, define maps $\phi_r, \phi_l : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}'$ by*

$$\phi_r(a \otimes b) = T'\kappa_{\mathcal{A}}(a) \cdot b, \quad \phi_l(a \otimes b) = a \cdot T(b) \qquad (a \otimes b \in \mathcal{A}\widehat{\otimes}\mathcal{A}).$$

Then $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$ if and only if ϕ_r and ϕ_l are weakly-compact and have ranges contained in $\kappa_{\mathcal{A}_*}(\mathcal{A}_*)$.

Proof. For $T \in \mathcal{B}(\mathcal{A}, \mathcal{A}') = (\mathcal{A}\widehat{\otimes}\mathcal{A})'$, define $R_T, L_T : \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})'$ by $R_T(a) = a \cdot T$ and $L_T = T \cdot a$, for $a \in \mathcal{A}$. By definition, $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$ if and only if R_T and L_T are $\sigma(\mathcal{A}, \mathcal{A}_*) - \sigma(\mathcal{B}(\mathcal{A}, \mathcal{A}'), (\mathcal{A}\widehat{\otimes}\mathcal{A})'')$ continuous. By Lemma 3.1, this is if and only if there exist $\varphi_r, \varphi_l \in \mathcal{W}(\mathcal{A}\widehat{\otimes}\mathcal{A}, \mathcal{A}_*)$ such that $\varphi_r' = R_T$ and $\varphi_l' = L_T$.

For $a \otimes b \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ and $c \in \mathcal{A}$, we see that

$$\begin{aligned}
\langle c, \varphi_r(a \otimes b)\rangle &= \langle R_T(c), a \otimes b\rangle = \langle c \cdot T, a \otimes b\rangle = \langle T(bc), a\rangle \\
&= \langle T'\kappa_{\mathcal{A}}(a), bc\rangle = \langle T'\kappa_{\mathcal{A}}(a) \cdot b, c\rangle = \langle \phi_r(a \otimes b), c\rangle, \\
\langle c, \varphi_l(a \otimes b)\rangle &= \langle L_T(c), a \otimes b\rangle = \langle T \cdot c, a \otimes b\rangle = \langle T(b), ca\rangle \\
&= \langle a \cdot T(b), c\rangle = \langle \phi_l(a \otimes b), c\rangle.
\end{aligned}$$

Thus $\kappa_{\mathcal{A}_*} \circ \varphi_r = \phi_r$ and $\kappa_{\mathcal{A}_*} \circ \varphi_l = \phi_l$. Consequently, we see that $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$ if and only if ϕ_r and ϕ_l are weakly-compact and take values in $\kappa_{\mathcal{A}_*}(\mathcal{A}_*)$. ☐

The following definition is [10, Definition 4.1].

Definition 3.3. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. An element $x \in E$ is *weakly almost periodic* if the maps

$$\mathcal{A} \to E, \quad a \mapsto \begin{cases} a \cdot x, \\ x \cdot a \end{cases}$$

are weakly-compact. The collection of weakly almost periodic elements in E is denoted by $\mathrm{WAP}(E)$. ☐

Lemma 3.4. *Let $\mathcal{A}$ be a Banach algebra, and let $T \in \mathcal{B}(\mathcal{A}, \mathcal{A}') = (\mathcal{A}\widehat{\otimes}\mathcal{A})'$. Let $\phi_r, \phi_l : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}'$ be as above. Then $T \in \mathrm{WAP}(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$ if and only if ϕ_r and ϕ_l are weakly-compact.*

Proof. Let $R_T, L_T : \mathcal{A} \to \mathcal{B}(\mathcal{A}, \mathcal{A}')$ be as in the above proof. By definition, $T \in \mathrm{WAP}(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$ if and only if L_T and R_T are weakly-compact. We can verify that

$$\phi_r' \circ \kappa_{\mathcal{A}} = R_T, \ \phi_l' \circ \kappa_{\mathcal{A}} = L_T, \ R_T' \circ \kappa_{\mathcal{A}\widehat{\otimes}\mathcal{A}} = \phi_r, \ L_T' \circ \kappa_{\mathcal{A}\widehat{\otimes}\mathcal{A}} = \phi_l,$$

which completes the proof. ☐

Corollary 3.5. *Let $\mathcal{A}$ be a unital, dual Banach algebra with predual $\mathcal{A}_*$, and let $T \in \mathcal{B}(\mathcal{A}, \mathcal{A}') = (\mathcal{A}\widehat{\otimes}\mathcal{A})'$. The following are equivalent, and, in particular, each imply that T is weakly-compact:*

1. *$T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$;*
2. *$T(\mathcal{A}) \subseteq \kappa_{\mathcal{A}_*}(\mathcal{A}_*)$, $T'(\kappa_{\mathcal{A}}(\mathcal{A})) \subseteq \kappa_{\mathcal{A}_*}(\mathcal{A}_*)$, and $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$;*
3. *$T(\mathcal{A}) \subseteq \kappa_{\mathcal{A}_*}(\mathcal{A}_*)$, $T'(\kappa_{\mathcal{A}}(\mathcal{A})) \subseteq \kappa_{\mathcal{A}_*}(\mathcal{A}_*)$, and $T \in \mathrm{WAP}(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$.*

Proof. Let $e_{\mathcal{A}}$ be the unit of $\mathcal{A}$, so that for $a \in \mathcal{A}$, we have $T(a) = \phi_l(e_{\mathcal{A}} \otimes a)$ and $T'\kappa_{\mathcal{A}}(a) = \phi_r(a \otimes e_{\mathcal{A}})$, which shows that (1) implies (2); clearly (2) implies (1).

As $\mathcal{A}_*$ is an $\mathcal{A}$-bimodule, (2) and (3) are equivalent by an application of Lemma 3.4 and Proposition 3.2. □

Theorem 3.6. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. Then $\mathcal{A}$ is Connes-amenable if and only if $\mathcal{A}$ is unital and there exists $M \in (\mathcal{A}\widehat{\otimes}\mathcal{A})''$ such that:*

1. *$\langle M, a \cdot T - T \cdot a\rangle = 0$ for $a \in \mathcal{A}$ and $T \in \sigma WC(\mathcal{W}(\mathcal{A}, \mathcal{A}'))$;*
2. *$\kappa_{\mathcal{A}_*}' \Delta_{\mathcal{A}}''(M) = e_{\mathcal{A}}$, where $e_{\mathcal{A}}$ is the unit of $\mathcal{A}$.*

Proof. As $\sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})')'$ is a quotient of $(\mathcal{A}\widehat{\otimes}\mathcal{A})''$, this is just a re-statement of Theorem 2.3. □

When $\mathcal{A}$ is an Arens regular Banach algebra, $\mathcal{A}''$ is a dual Banach algebra with canonical predual $\mathcal{A}'$. In this case, we can make some significant simplifications in the characterisation of when $\mathcal{A}''$ is Connes-amenable.

For a Banach algebra $\mathcal{A}$, we define the map $\kappa_{\mathcal{A}} \otimes \kappa_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}''\widehat{\otimes}\mathcal{A}''$ by

$$(\kappa_{\mathcal{A}} \otimes \kappa_{\mathcal{A}})(a \otimes b) = \kappa_{\mathcal{A}}(a) \otimes \kappa_{\mathcal{A}}(b) \qquad (a \otimes b \in \mathcal{A}\widehat{\otimes}\mathcal{A}).$$

We turn $\mathcal{A}''\widehat{\otimes}\mathcal{A}''$ into a Banach $\mathcal{A}$-bimodule in the canonical way. Then $\kappa_{\mathcal{A}} \otimes \kappa_{\mathcal{A}}$ is an $\mathcal{A}$-bimodule homomorphism. The following is a simple verification.

Lemma 3.7. *Let $\mathcal{A}$ be a Banach algebra. The map*

$$\iota_{\mathcal{A}} : \mathcal{B}(\mathcal{A}, \mathcal{A}') \to \mathcal{B}(\mathcal{A}'', \mathcal{A}'''); \ T \mapsto T'',$$

is an $\mathcal{A}$-bimodule homomorphism which is an isometry onto its range. Furthermore, we have that $(\kappa_{\mathcal{A}} \otimes \kappa_{\mathcal{A}})' \circ \iota_{\mathcal{A}} = I_{\mathcal{B}(\mathcal{A},\mathcal{A}')}$. Define $\rho_{\mathcal{A}} : \mathcal{A}''\widehat{\otimes}\mathcal{A}'' \to (\mathcal{A}\widehat{\otimes}\mathcal{A})''$ by

$$\langle \rho_{\mathcal{A}}(\tau), T\rangle = \langle T'', \tau\rangle \qquad (\tau \in \mathcal{A}''\widehat{\otimes}\mathcal{A}'', T \in \mathcal{B}(\mathcal{A}, \mathcal{A}') = (\mathcal{A}\widehat{\otimes}\mathcal{A})').$$

Then $\rho_{\mathcal{A}}$ is a norm-decreasing $\mathcal{A}$-bimodule homomorphism which satisfies $\rho_{\mathcal{A}} \circ (\kappa_{\mathcal{A}} \otimes \kappa_{\mathcal{A}}) = \kappa_{\mathcal{A}\widehat{\otimes}\mathcal{A}}$. □

For a Banach algebra $\mathcal{A}$, it is clear that $\mathcal{W}(\mathcal{A}, \mathcal{A}')$ is a sub-$\mathcal{A}$-bimodule of $\mathcal{B}(\mathcal{A}, \mathcal{A}') = (\mathcal{A}\widehat{\otimes}\mathcal{A})'$.

Theorem 3.8. *Let $\mathcal{A}$ be an Arens regular Banach algebra such that $\mathcal{A}''$ is unital, and let $T \in \mathcal{B}(\mathcal{A}'', \mathcal{A}''') = (\mathcal{A}''\widehat{\otimes}\mathcal{A}'')'$. Then the following are equivalent:*

1. *$T \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}'''))$, where we treat $\mathcal{B}(\mathcal{A}'', \mathcal{A}''')$ as an $\mathcal{A}''$-bimodule;*
2. *$T = S''$ for some $S \in \mathrm{WAP}(\mathcal{W}(\mathcal{A}, \mathcal{A}'))$, where now we treat $\mathcal{W}(\mathcal{A}, \mathcal{A}')$ as an $\mathcal{A}$-bimodule.*

Proof. We apply Corollary 3.5 to $\mathcal{A}''$, so that (1) is equivalent to T being weakly-compact, $T(\mathcal{A}'') \subseteq \kappa_{\mathcal{A}'}(\mathcal{A}')$, $T'(\kappa_{\mathcal{A}''}(\mathcal{A}'')) \subseteq \kappa_{\mathcal{A}'}(\mathcal{A}')$, and $T \in \mathrm{WAP}(\mathcal{B}(\mathcal{A}'', \mathcal{A}'''))$. Thus, if (1) holds, then there exists $T_0 \in \mathcal{W}(\mathcal{A}'', \mathcal{A}')$ such that $T = \kappa_{\mathcal{A}'} \circ T_0$, and there exists $T_1 \in \mathcal{W}(\mathcal{A}'', \mathcal{A}')$ such that $T' \circ \kappa_{\mathcal{A}''} = \kappa_{\mathcal{A}'} \circ T_1$. Let $S = T_0 \circ \kappa_{\mathcal{A}} \in \mathcal{W}(\mathcal{A}, \mathcal{A}')$. Then, for $a \in \mathcal{A}$ and $\Psi \in \mathcal{A}''$, we have

$$\begin{aligned}\langle S'(\Psi), a\rangle &= \langle \Psi, T_0(\kappa_{\mathcal{A}}(a))\rangle = \langle T(\kappa_{\mathcal{A}}(a)), \Psi\rangle = \langle T'(\kappa_{\mathcal{A}''}(\Psi)), \kappa_{\mathcal{A}}(a)\rangle \\ &= \langle \kappa_{\mathcal{A}}(a), T_1(\Psi)\rangle = \langle T_1(\Psi), a\rangle,\end{aligned}$$

so that $S' = T_1$. Thus, for $\Phi, \Psi \in \mathcal{A}''$, we have

$$\langle S''(\Phi), \Psi\rangle = \langle \Phi, T_1(\Psi)\rangle = \langle T'(\kappa_{\mathcal{A}''}(\Psi)), \Phi\rangle = \langle T(\Phi), \Psi\rangle,$$

so that $S'' = T$. We know that the maps $L_T, R_T : \mathcal{A}'' \to \mathcal{B}(\mathcal{A}'', \mathcal{A}''')$, defined by $L_T(\Phi) = T \cdot \Phi$ and $R_T(\Phi) = \Phi \cdot T$ for $\Phi \in \mathcal{A}''$, are weakly-compact. Define $L_S, R_S : \mathcal{A} \to \mathcal{B}(\mathcal{A}, \mathcal{A}')$ is an analogous manner, using $S \in \mathcal{W}(\mathcal{A}, \mathcal{A}')$. For $a \in \mathcal{A}$, $S \cdot a \in \mathcal{W}(\mathcal{A}, \mathcal{A}')$, so for $\Psi \in \mathcal{A}''$ and $b \in \mathcal{A}$,

$$\langle (S \cdot a)'(\Psi), b\rangle = \langle \Psi, (S \cdot a)(b)\rangle = \langle \Psi, S(b) \cdot a\rangle = \langle a \cdot \Psi, S(b)\rangle = \langle S'(a \cdot \Psi), b\rangle.$$

Thus, for $a \in \mathcal{A}$ and $\Phi, \Psi \in \mathcal{A}''$, we have that

$$\langle \iota_{\mathcal{A}}(L_S(a))(\Phi), \Psi\rangle = \langle (S \cdot a)''(\Phi), \Psi\rangle = \langle \Phi, S'(a \cdot \Psi)\rangle = \langle S''(\Phi) \cdot a, \Psi\rangle,$$

so that $\iota_{\mathcal{A}}(L_S(a))(\Phi) = S''(\Phi) \cdot a$, and hence that $\iota_{\mathcal{A}}(L_S(a)) = S'' \cdot a = T \cdot a = T \cdot \kappa_{\mathcal{A}}(a) = L_T(\kappa_{\mathcal{A}}(a))$. Thus we have that $L_S = (\kappa_{\mathcal{A}} \otimes \kappa_{\mathcal{A}})' \circ R_T \circ \kappa_{\mathcal{A}}$, so that L_S is weakly-compact. A similar calculation shows that R_S is also weakly-compact, so that $S \in \mathrm{WAP}(\mathcal{W}(\mathcal{A}, \mathcal{A}'))$. This shows that (1) implies (2).

Conversely, if (2) holds, then L_S and R_S are weakly-compact. As S is weakly-compact, $T(\mathcal{A}'') = S''(\mathcal{A}'') \subseteq \kappa_{\mathcal{A}'}(\mathcal{A}')$ and $T'(\kappa_{\mathcal{A}''}(\mathcal{A}'')) = S'''(\kappa_{\mathcal{A}''}(\mathcal{A}'')) = \kappa_{\mathcal{A}'}(S'(\mathcal{A}'')) \subseteq \kappa_{\mathcal{A}'}(\mathcal{A}')$, and T is weakly-compact. Thus, to show (1), we are required to show that L_T and R_T are weakly-compact.

For $a, b \in \mathcal{A}$ and $\Phi \in \mathcal{A}'$, we have

$$\langle (a \cdot S)'(\Phi), b\rangle = \langle \Phi, S(ba)\rangle = \langle a \cdot S'(\Phi), b\rangle.$$

Then, for $\Phi, \Psi \in \mathcal{A}''$ and $a \in \mathcal{A}$, we thus have

$$\begin{aligned}\langle R_S'(\rho_{\mathcal{A}}(\Phi \otimes \Psi)), a\rangle &= \langle (a \cdot S)'', \Phi \otimes \Psi\rangle = \langle (a \cdot S)''(\Psi), \Phi\rangle = \langle \Psi, a \cdot S'(\Phi)\rangle \\ &= \langle \Psi \cdot a, S'(\Phi)\rangle = \langle \Psi \Box \kappa_{\mathcal{A}}(a), S'(\Phi)\rangle = \langle S'(\Phi) \cdot \Psi, a\rangle.\end{aligned}$$

Hence we see that $R_S'(\rho_{\mathcal{A}}(\Phi \otimes \Psi)) = S'(\Phi) \cdot \Psi$. Let $U = R_S' \circ \rho_{\mathcal{A}} : \mathcal{A}'' \widehat{\otimes} \mathcal{A}'' \to \mathcal{A}'$, so that as R_S is weakly-compact, so is U. Then, for $\Phi, \Psi, \Gamma \in \mathcal{A}''$, we have that

$$\langle U'(\Gamma), \Phi \otimes \Psi\rangle = \langle \Gamma, S'(\Phi) \cdot \Psi\rangle = \langle \Psi \Diamond \Gamma, S'(\Phi)\rangle = \langle S''(\Psi \Box \Gamma), \Phi\rangle = \langle (\Gamma \cdot S'')(\Psi), \Phi\rangle,$$

so that $U'(\Gamma) = \Gamma \cdot T$, that is, $U' = R_T$, so that R_T is weakly-compact. Similarly, we can show that L_T is weakly-compact, completing the proof. $\square$

Theorem 3.9. *Let $\mathcal{A}$ be an Arens regular Banach algebra. Then $\mathcal{A}''$ is Connes-amenable if and only if $\mathcal{A}''$ is unital and there exists $M \in (\mathcal{A} \widehat{\otimes} \mathcal{A})''$ such that:*

1. $\Delta_{\mathcal{A}}''(M) = e_{\mathcal{A}''}$, *the unit of* $\mathcal{A}''$*;*
2. $\langle M, a \cdot T - T \cdot a\rangle = 0$ *for each* $a \in \mathcal{A}$ *and each* $T \in \mathrm{WAP}(\mathcal{W}(\mathcal{A}, \mathcal{A}'))$.

Proof. By Theorem 3.6, we wish to show that the existence of such an M is equivalent to the existence of $N \in (\mathcal{A}''\widehat{\otimes}\mathcal{A}'')''$ such that:

(N1) $\kappa'_{\mathcal{A}'}\Delta''_{\mathcal{A}''}(N) = e_{\mathcal{A}''}$;

(N2) $\langle N, \Phi \cdot S - S \cdot \Phi\rangle = 0$ for each $\Phi \in \mathcal{A}''$ and each $S \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}'''))$.

We can verify that $\iota_{\mathcal{A}} \circ \Delta'_{\mathcal{A}} = \Delta'_{\mathcal{A}''} \circ \kappa_{\mathcal{A}'}$, so that (N1) is equivalent to $\Delta''_{\mathcal{A}}\iota'_{\mathcal{A}}(N) = e_{\mathcal{A}''}$. For $S \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}'''))$, we know that $S = T''$ for some $T \in \mathrm{WAP}(\mathcal{W}(\mathcal{A}, \mathcal{A}'))$, by Theorem 3.8. That is, the maps ϕ_r and ϕ_l, formed using T as in Proposition 3.2, are weakly-compact. Then, for $\Phi \in \mathcal{A}''$, $\phi'_r(\Phi), \phi'_l(\Phi) \in \mathcal{B}(\mathcal{A}, \mathcal{A}')$, and we can check that

$$\phi'_r(\Phi)(a) = \kappa'_{\mathcal{A}}T''(a \cdot \Phi), \quad \phi'_l(\Phi)(a) = T(a) \cdot \Phi \qquad (a \in \mathcal{A}).$$

Then $\phi'_r(\Phi)', \phi'_l(\Phi)' \in \mathcal{B}(\mathcal{A}'', \mathcal{A}')$ are the maps

$$\phi'_r(\Phi)'(\Psi) = \Phi \cdot T'(\Psi), \quad \phi'_l(\Phi)'(\Psi) = T'(\Phi\Box\Psi) \qquad (\Psi \in \mathcal{A}''),$$

where we remember that $T''(\mathcal{A}'') \subseteq \kappa_{\mathcal{A}'}(\mathcal{A}')$. Consequently $\phi'_r(\Phi)'', \phi'_l(\Phi)'' \in \mathcal{B}(\mathcal{A}'', \mathcal{A}''')$ are given by

$$\phi'_r(\Phi)''(\Psi) = T''(\Psi\Box\Phi), \quad \phi'_l(\Phi)''(\Psi) = T''(\Psi) \cdot \Phi \qquad (\Psi \in \mathcal{A}''),$$

where $\mathcal{A}'''$ is an $\mathcal{A}''$-bimodule, as $\mathcal{A}''$ is Arens regular. That is, $\phi'_r(\Phi)'' = \Phi \cdot S$ and $\phi'_l(\Phi)'' = S \cdot \Phi$. Hence (N2) is equivalent to

$$0 = \langle N, \phi'_r(\Phi)'' - \phi'_l(\Phi)''\rangle = \langle N, \iota_{\mathcal{A}}(\phi'_r(\Phi) - \phi'_l(\Phi))\rangle = \langle \iota'_{\mathcal{A}}(N), \phi'_r(\Phi) - \phi'_l(\Phi)\rangle,$$

for each $\Phi \in \mathcal{A}''$ and $S \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}'''))$. That is, (N2) is equivalent to

$$\phi''_r\iota'_{\mathcal{A}}(N) - \phi''_l\iota'_{\mathcal{A}}(N) = 0 \qquad (S \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}'''))).$$

As ϕ_r and ϕ_l are weakly-compact, ϕ''_r and ϕ''_l take values in $\kappa_{\mathcal{A}'}(\mathcal{A}')$, and so (N2) is equivalent to

$$0 = \langle \phi''_r\iota'_{\mathcal{A}}(N) - \phi''_l\iota'_{\mathcal{A}}(N), \kappa_{\mathcal{A}}(a)\rangle = \langle \iota'_{\mathcal{A}}(N), \phi'_r(\kappa_{\mathcal{A}}(a)) - \phi'_l(\kappa_{\mathcal{A}}(a))\rangle,$$

for each $a \in \mathcal{A}$ and each $S \in \sigma WC(\mathcal{B}(\mathcal{A}'', \mathcal{A}'''))$. However, $\phi'_r(\kappa_{\mathcal{A}}(a)) - \phi'_l(\kappa_{\mathcal{A}}(a)) = a \cdot T - T \cdot a$, so that (N2) is equivalent to

$$0 = \langle \iota'_{\mathcal{A}}(N), a \cdot T - T \cdot a\rangle \qquad (a \in \mathcal{A}),$$

for each $T \in \mathcal{W}(\mathcal{A}, \mathcal{A}')$ such that ϕ_r and ϕ_l are weakly-compact.

Thus we have established that (N1) holds for N if and only if (1) holds for $M = \iota'_{\mathcal{A}}(N)$, and that (N2) holds for N if and only if (2) holds for $M = \iota'_{\mathcal{A}}(N)$, completing the proof. $\Box$

We immediately see that $\mathcal{A}$ amenable implies that $\mathcal{A}''$ is Connes-amenable. Furthermore, if $\mathcal{A}$ is itself a dual Banach algebra, then Corollary 3.5 shows that if $\mathcal{A}''$ is Connes-amenable, then $\mathcal{A}$ is Connes-amenable: notice that if $e_{\mathcal{A}''}$ is the unit of $\mathcal{A}''$, then

$$\langle \kappa'_{\mathcal{A}_*}(e_{\mathcal{A}''})a, \mu\rangle = \langle e_{\mathcal{A}''} \cdot a, \kappa_{\mathcal{A}_*}(\mu)\rangle = \langle \kappa_{\mathcal{A}}(a), \kappa_{\mathcal{A}_*}(\mu)\rangle = \langle a, \mu\rangle \qquad (a \in \mathcal{A}, \mu \in \mathcal{A}_*),$$

so that $\kappa'_{\mathcal{A}_*}(e_{\mathcal{A}''})$ is the unit of $\mathcal{A}$.

4 Injectivity of the predual module

Let $\mathcal{A}$ be a Banach algebra, and let E and F be Banach left $\mathcal{A}$-modules. We write ${}_{\mathcal{A}}\mathcal{B}(E,F)$ for the closed subspace of $\mathcal{B}(E,F)$ consisting of left $\mathcal{A}$-module homomorphisms, and similarly write $\mathcal{B}_{\mathcal{A}}(E,F)$ and ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{A}}(E,F)$ for right $\mathcal{A}$-module and $\mathcal{A}$-bimodule homomorphisms, respectively. We say that $T \in {}_{\mathcal{A}}\mathcal{B}(E,F)$ is *admissible* if both the kernel and image of T are closed, complemented subspaces of, respectively, E and F. If T is injective, this is equivalent to the existence of $S \in \mathcal{B}(F,E)$ such that $ST = I_E$.

Definition 4.1. Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach left $\mathcal{A}$-module. Then E is *injective* if, whenever F and G are Banach left $\mathcal{A}$-modules, $\theta \in {}_{\mathcal{A}}\mathcal{B}(F,G)$ is injective and admissible, and $\sigma \in {}_{\mathcal{A}}\mathcal{B}(F,E)$, there exists $\rho \in {}_{\mathcal{A}}\mathcal{B}(G,E)$ with $\rho \circ \theta = \sigma$.

We say that E is *left-injective* when we wish to stress that we are treating E as a left module. Similar definitions hold for right modules and bimodules (written *right-injective* and *bi-injective* where necessary).

Let $\mathcal{A}$ be a Banach algebra, let E be a Banach left $\mathcal{A}$-module, and turn $\mathcal{B}(\mathcal{A},E)$ into a left $\mathcal{A}$-module by setting

$$(a \cdot T)(b) = T(ba) \qquad (a,b \in \mathcal{A}, T \in \mathcal{B}(\mathcal{A},E)).$$

Then there is a canonical left $\mathcal{A}$-module homomorphism $\iota : E \to \mathcal{B}(\mathcal{A},E)$ given by

$$\iota(x)(a) = a \cdot x \qquad (a \in \mathcal{A}, x \in E).$$

Notice that if E is a closed submodule of $\mathcal{A}'$, then $\mathcal{B}(\mathcal{A},E)$ is a closed submodule of $(A\widehat{\otimes}A)' = \mathcal{B}(\mathcal{A},\mathcal{A}')$, and ι is the restriction of $\Delta'_{\mathcal{A}} : \mathcal{A}' \to \mathcal{B}(\mathcal{A},\mathcal{A}')$ to E.

Similarly, we turn $\mathcal{B}(\mathcal{A}\widehat{\otimes}\mathcal{A},E)$ into a Banach $\mathcal{A}$-bimodule by

$$(a \cdot T)(b \otimes c) = T(ba \otimes c), \ (T \cdot a)(b \otimes c) = T(b \otimes ac) \quad (a,b,c \in \mathcal{A}, T \in \mathcal{B}(\mathcal{A}\widehat{\otimes}\mathcal{A},E)).$$

We then define (with an abuse of notation) $\iota : E \to \mathcal{B}(\mathcal{A}\widehat{\otimes}\mathcal{A},E)$ by

$$\iota(x)(a \otimes b) = a \cdot x \cdot b \qquad (x \in E, a \otimes b \in \mathcal{A}\widehat{\otimes}\mathcal{A}),$$

so that ι is an $\mathcal{A}$-bimodule homomorphism.

We can also turn $\mathcal{B}(\mathcal{A},E)$ into a right $\mathcal{A}$-module by reversing the above (in particular, we need to take the other possible choice in Section 3 leading to different module actions as compared to those in (1).)

Proposition 4.2. *Let $\mathcal{A}$ be a Banach algebra, and let E be a* faithful *Banach left $\mathcal{A}$-module (that is, for each non-zero $x \in E$ there exists $a \in \mathcal{A}$ with $a \cdot x \neq 0$). Then E is injective if and only if there exists $\phi \in {}_{\mathcal{A}}\mathcal{B}(\mathcal{B}(\mathcal{A},E),E)$ such that $\phi \circ \iota = I_E$.*

Similarly, if E is a left and right faithful Banach $\mathcal{A}$-bimodule (that is, for each non-zero $x \in E$ there exists $a,b \in \mathcal{A}$ with $a \cdot x \neq 0$ and $x \cdot b \neq 0$). Then E is injective if and only if there exists $\phi \in {}_{\mathcal{A}}\mathcal{B}_{\mathcal{A}}(\mathcal{B}(\mathcal{A}\widehat{\otimes}\mathcal{A},E),E)$ such that $\phi \circ \iota = I_E$.

Proof. The first claim is [4, Proposition 1.7], and the second claim is an obvious generalisation. ∎

Again, there exists a similar characterisation for right modules.

Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$. It is simple to show (see [10]) that if $\mathcal{A}_*$ is bi-injective, then $\mathcal{A}$ is Connes-amenable. Helemskii showed in [7] that for a von Neumann algebra $\mathcal{A}$, the converse is true. However, Runde (see [10]) and Tabaldyev (see [15]) have shown that $M(G)$, the measure algebra of a locally compact group G, while being a dual Banach algebra with predual $C_0(G)$, has that $C_0(G)$ is a left-injective $M(G)$-module only when G is finite. Of course, Runde (see [11]) has shown that $M(G)$ is Connes-amenable if and only if G is amenable.

Similarly, it is simple to show (using a virtual diagonal) that if $\mathcal{A}$ is a Banach algebra with a bounded approximate identity, then $\mathcal{A}$ is amenable if and only if $\mathcal{A}'$ is bi-injective.

Let E and F be Banach left $\mathcal{A}$-modules, and let $\phi : E \to F$ be a left $\mathcal{A}$-module homomorphism which is bounded below. Then $\phi(E)$ is a closed submodule of F, so that $F/\phi(E)$ is a Banach left $\mathcal{A}$-module. Hence we have a *short exact sequence*:

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, shift left, "\phi"] & F \arrow[l, dashed, shift left, "P"] \arrow[r, two heads] & F/\phi(E) \arrow[r] & 0 . \end{tikzcd}$$

If there exists a bounded linear map $P : F \to E$ such that $P \circ \phi = I_E$, then we say that the short exact sequence is *admissible*. If, further, we may choose P to be a left $\mathcal{A}$-module homomorphism, then the short exact sequence is said to *split*. Similar definitions hold for right modules and bimodules.

Proposition 4.3. *Let $\mathcal{A}$ be a Banach algebra, let E be a Banach left $\mathcal{A}$-module, and consider the following admissible short exact sequence:*

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, shift left, "\iota"] & \mathcal{B}(\mathcal{A}, E) \arrow[l, dashed, shift left, "P"] \arrow[r, two heads] & \mathcal{B}(\mathcal{A}, E)/\iota(E) \arrow[r] & 0 . \end{tikzcd}$$

Then E is injective if and only if this short exact sequence splits.

Proof. See, for example, [13, Section 5.3]. ☐

Proposition 4.4. *Let $\mathcal{A}$ be a unital dual Banach algebra with predual $\mathcal{A}_*$, and consider the following admissible short exact sequence of $\mathcal{A}$-bimodules:*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}_* \arrow[r, shift left, "\Delta_{\mathcal{A}}'"] & \sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})') \arrow[l, dashed, shift left, "P"] \arrow[r, two heads] & \sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})')/\Delta_{\mathcal{A}}'(\mathcal{A}_*) \arrow[r] & 0 . \end{tikzcd} \tag{2}$$

Then $\mathcal{A}$ is Connes-amenable if and only if this short exact sequence splits.

Proof. Notice that $\Delta_{\mathcal{A}}'$ certainly maps $\mathcal{A}_*$ into $\sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})') = \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$, and that Corollary 3.5 shows that we can define $P : \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}')) \to \mathcal{A}_*$ by $P(T) = T(e_{\mathcal{A}})$ for $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$.

Suppose that we can choose P to be an $\mathcal{A}$-bimodule homomorphism. Then let $M = P'(e_{\mathcal{A}})$, so that for $a \in \mathcal{A}$ and $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$,

$$\langle a \cdot M - M \cdot a, T\rangle = \langle e_{\mathcal{A}}, P(T \cdot a - a \cdot T)\rangle = \langle a - a, P(T)\rangle = 0,$$

so that $a \cdot M - M \cdot a$. Also $\Delta_{\mathcal{A}}''(M) = (P \circ \Delta_{\mathcal{A}}')'(e_{\mathcal{A}}) = e_{\mathcal{A}}$, so that M is a σWC-virtual diagonal, and hence $\mathcal{A}$ is Connes-amenable by Runde's theorem.

Conversely, let M be a σWC-virtual diagonal and define $P : \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}')) \to \mathcal{A}'$ by

$$\langle P(T), a\rangle = \langle M, a \cdot T\rangle \qquad (a \in \mathcal{A}, T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}')).$$

Let (a_α) be a bounded net in $\mathcal{A}$ which tends to $a \in \mathcal{A}$ in the $\sigma(\mathcal{A}, \mathcal{A}_*)$-topology. By definition, $a_\alpha \cdot T \to a \cdot T$ weakly, for each $T \in \sigma WC(\mathcal{B}(\mathcal{A}, \mathcal{A}'))$, so that $\langle P(T), a_\alpha\rangle \to \langle P(T), a\rangle$. This implies that P maps into $\mathcal{A}_*$, as required. Then, for $\mu \in \mathcal{A}_*$,

$$\langle a, P\Delta_{\mathcal{A}}'(\mu)\rangle = \langle M, a \cdot \Delta_{\mathcal{A}}'(\mu)\rangle = \langle M, \Delta_{\mathcal{A}}'(a \cdot \mu)\rangle = \langle e_{\mathcal{A}}, a \cdot \mu\rangle = \langle a, \mu\rangle \qquad (a \in \mathcal{A}),$$

so that $P\Delta'_{\mathcal{A}} = I_{\mathcal{A}_*}$. Finally, we note that

$$\begin{aligned}\langle P(a\cdot T\cdot b), c\rangle &= \langle M, ca\cdot T\cdot b\rangle = \langle b\cdot M, ca\cdot T\rangle = \langle M\cdot b, ca\cdot T\rangle \\ &= \langle P(T), bca\rangle = \langle a\cdot P(T)\cdot b, c\rangle \qquad (a,b,c\in\mathcal{A}, T\in \sigma WC(\mathcal{B}(\mathcal{A},\mathcal{A}'))),\end{aligned}$$

so that P is an $\mathcal{A}$-bimodule homomorphism, as required. □

Let $\mathcal{A}$ be an Arens regular Banach algebra. By reversing the argument Theorem 3.8, we can show that $\Delta'_{\mathcal{A}} : \mathcal{A}' \to \mathcal{B}(\mathcal{A},\mathcal{A}')$ actually maps into $\mathrm{WAP}(\mathcal{W}(\mathcal{A},\mathcal{A}'))$. Furthermore, if $\mathcal{A}''$ is unital, then we may define $P : \mathrm{WAP}(\mathcal{W}(\mathcal{A},\mathcal{A}')) \to \mathcal{A}'$ by

$$\langle P(T), a\rangle = \langle e_{\mathcal{A}''}, P(a)\rangle \qquad (a\in\mathcal{A}, T\in \mathrm{WAP}(\mathcal{W}(\mathcal{A},\mathcal{A}'))).$$

Then we have that

$$\langle P\Delta'_{\mathcal{A}}(\mu), a\rangle = \langle e_{\mathcal{A}''}, a\cdot\mu\rangle = \langle \mu, a\rangle \qquad (a\in\mathcal{A}, \mu\in\mathcal{A}').$$

Proposition 4.5. *Let $\mathcal{A}$ be an Arens regular Banach algebra such that $\mathcal{A}''$ is unital, and consider the following admissible short exact sequence of $\mathcal{A}$-bimodules:*

$$0 \longrightarrow \mathcal{A}' \underset{P}{\overset{\Delta'_{\mathcal{A}}}{\rightleftarrows}} \mathrm{WAP}(\mathcal{W}(\mathcal{A},\mathcal{A}')) \twoheadrightarrow \mathrm{WAP}(\mathcal{W}(\mathcal{A},\mathcal{A}'))/\Delta'_{\mathcal{A}}(\mathcal{A}') \longrightarrow 0\,. \tag{3}$$

Then $\mathcal{A}''$ is Connes-amenable if and only if this short exact sequence splits.

Proof. This follows in the same manner as the above proof, using Theorem 3.9. □

5 Beurling algebras

Let S be a discrete semigroup (we can extend the following definitions to locally compact semigroups, but for the questions we are interested in, the results for non-discrete groups are trivial). A *weight* on S is a function $\omega : S \to \mathbb{R}^{>0}$ such that

$$\omega(st) \le \omega(s)\omega(t) \qquad (s,t\in S).$$

Furthermore, if S is unital with unit u_S, then we also insist that $\omega(u_S) = 1$. This last condition is simply a normalisation condition, as we can always set $\hat\omega(s) = \sup\{\omega(st)\omega(t)^{-1} : t\in S\}$ for each $s\in S$. For $s,t\in S$, we have that $\omega(st)\le\hat\omega(s)\omega(t)$, so that

$$\hat\omega(st) = \sup\{\omega(str)\omega(r)^{-1} : r\in S\} \le \sup\{\hat\omega(s)\omega(tr)\omega(r)^{-1} : r\in S\} = \hat\omega(s)\hat\omega(t).$$

Clearly $\hat\omega(u_S) = 1$ and $\hat\omega(s)\le\omega(s)$ for each $s\in S$, while $\hat\omega(s) \ge \omega(s)\omega(u_S)^{-1}$, so that $\hat\omega$ is equivalent to ω.

We form the Banach space

$$l^1(S,\omega) = \Big\{(a_g)_{g\in S}\subseteq\mathbb{C} : \|(a_g)\| := \sum_{g\in S}|a_g|\omega(g) < \infty\Big\}.$$

Then $l^1(S,\omega)$, with the convolution product, is a Banach algebra, called a *Beurling algebra*. See [1] and [3] for further information on Beurling algebras and, in particular, their second duals.

It will be more convenient for us to think of $l^1(S,\omega)$ as the Banach space $l^1(S)$ together with a weighted algebra product. Indeed, for $g\in S$, let $\delta_g\in l^1(S)$ be the standard unit vector basis element which is thought of as a point-mass at g. Then each $x\in l^1(S)$

can be written uniquely as $x = \sum_{g\in S} x_g\delta_g$ for some family $(x_g) \subseteq \mathbb{C}$ such that $\|x\| = \sum_{g\in S} |x_g| < \infty$. We then define

$$\delta_g \star_\omega \delta_h = \delta_g \star \delta_h = \delta_{gh}\Omega(g,h) \qquad (g,h \in S),$$

where $\Omega(g,h) = \omega(gh)\omega(g)^{-1}\omega(h)^{-1}$, and extend $\star$ to $l^1(S)$ by linearity and continuity.

For example, if ω and $\hat{\omega}$ are equivalent weights on S, the define $\psi : l^1(S,\omega) \to l^1(S,\hat{\omega})$ by $\psi(\delta_s) = \hat{\omega}(s)\omega(s)^{-1}\delta_s$. As ω and $\hat{\omega}$ are equivalent, ψ is an isomorphism of Banach spaces. Then $\psi(\delta_s \star \delta_t) = \omega(st)\omega(s)^{-1}\omega(t)^{-1}\hat{\omega}(st)\omega(st)^{-1}\delta_{st} = \psi(\delta_s) \star \psi(\delta_t)$, so that ψ is a homomorphism.

For a set I, we define the space $c_0(I)$ as

$$c_0(I) = \Big\{ (a_i)_{i\in I} : \forall\, \epsilon > 0, |\{i \in I : |a_i| \geq \epsilon\}| < \infty \Big\},$$

where $|\cdot|$ is the cardinality of a set. We equip $c_0(I)$ with the supremum norm; then $c_0(I)' = l^1(I)$. For $i \in I$, we let $e_i \in c_0(I)$ be the point mass at i, that is, $\langle \delta_j, e_i\rangle = \delta_{i,j}$, the Kronecker delta, for $\delta_j \in l^1(I)$. Then $c_0(I)$ is the closed linear span of $\{e_i : i \in I\}$. We let $l^\infty(I)$ be the Banach space of all bounded families $(a_i)_{i\in I}$, with the supremum norm. Then $l^1(I)' = l^\infty(I)$, we can treat $c_0(I)$ as a subspace of $l^\infty(I)$, and the map $\kappa_{c_0(I)} : c_0(I) \to l^\infty(I)$ is just the inclusion map.

For a semigroup S and $s \in S$, we define maps $L_s, R_s : S \to S$ by

$$L_s(t) = st, \quad R_s(t) = ts \qquad (t \in S).$$

If, for each $s \in S$, L_s and R_s are finite-to-one maps, then we say that S is *weakly cancellative*. When L_s and R_s are injective for each $s \in S$, we say that S is *cancellative*. When S is abelian and cancellative, a construction going back to Grothendieck shows that S is a sub-semigroup of some abelian group. However, this can fail to hold for non-abelian semigroups.

Proposition 5.1. *Let S be a weakly cancellative semigroup, let ω be a weight on S, and let $\mathcal{A} = l^1(S,\omega)$. Then $c_0(S) \subseteq l^\infty(S) = \mathcal{A}'$ is a sub-$\mathcal{A}$-module of $\mathcal{A}'$, so that $l^1(S,\omega)$ is a dual Banach algebra with predual $c_0(S)$.*

Proof. For $g,h \in S$ and $a = (a_s)_{s\in S} \in l^1(S,\omega)$, we have

$$\langle e_g \cdot \delta_h, a\rangle = \langle e_g, \delta_h \star a\rangle = \langle e_g, \sum_{s\in S} a_s\delta_{hs}\Omega(h,s)\rangle = \sum_{\{s\in S: hs=g\}} a_s\Omega(h,s).$$

As S is weakly cancellative, there exists at most finitely many $s \in S$ such that $hs = g$, so that $e_g \cdot \delta_h$ is a member of $c_0(S)$. Thus we see that $c_0(S)$ is a right sub-$\mathcal{A}$-module of $\mathcal{A}'$. The argument on the left follows in an analogous manner. $\square$

Notice that the above result will hold for some semigroups S which are not weakly cancellative, provided that the weight behaves in a certain way. However, it would appear that the later results do not easily generalise to the non-weakly cancellative case.

Following [3, Definition 2.2], we have the following definition.

Definition 5.2. Let I and J be non-empty infinite sets, and let $f : I \times J \to \mathbb{C}$ be a function. Then f *clusters on* $I \times J$ if

$$\lim_{n\to\infty}\lim_{m\to\infty} f(x_m, y_n) = \lim_{m\to\infty}\lim_{n\to\infty} f(x_m, y_n),$$

whenever $(x_m) \subseteq I$ and $(y_n) \subseteq J$ are sequences of distinct elements, and both iterated limits exist.

Furthermore, f 0-*clusters on* $I \times J$ if f clusters on $I \times J$, and the iterated limits are always 0, when they exist. $\square$

From now on we shall exclude the trivial case when our (semi-)group is finite.

Theorem 5.3. *Let S be a discrete, weakly cancellative semigroup, and let ω be a weight on S. Then the following are equivalent:*

1. *$l^1(S,\omega)$ is Arens regular;*
2. *for sequences of distinct elements (g_j) and (h_k) in S, we have*
$$\lim_{j\to\infty}\lim_{k\to\infty}\Omega(g_j,h_k)=0,$$
whenever the iterated limit exists;
3. *Ω 0-clusters on $S\times S$.*

Proof. That (1) and (2) are equivalent for cancellative semigroups is [1, Theorem 1]. Close examination of the proof shows that this holds for weakly cancellative semigroups as well. That (1) and (3) are equivalent follows by generalising the proof of [3, Theorem 7.11], which is essentially an application of Grothendieck's criterion for an operator to be weakly-compact. Alternatively, it follows easily that (2) and (3) are equivalent by considering the *opposite semigroup* to S where we reverse the product. □

In [1] it is also shown that if G is a discrete, uncountable group, then $l^1(G,\omega)$ is not Arens regular for any weight ω. Furthermore, by [1, Theorem 2], if G is a non-discrete locally compact group, then $L^1(G,\omega)$ is never Arens regular.

We shall consider both the Connes-amenability of $l^1(S,\omega)''$ and $l^1(S,\omega)$ (with respect to the canonical predual $c_0(S)$) as, with reference to Corollary 3.5 and Theorem 3.8, the calculations should be similar.

Proposition 5.4. *Let I be a non-empty set, and let $X\subseteq l^\infty(I)$ be a subset. Then the following are equivalent:*

1. *X is relatively weakly-compact;*
2. *X is relatively sequentially weakly-compact;*
3. *the absolutely convex hull of X is relatively weakly-compact;*
4. *if we define $f: I\times X\to\mathbb{C}$ by $f(i,x)=\langle x,\delta_i\rangle$ for $i\in I$ and $x\in X$, then f clusters on $I\times X$;*

Proof. That (1) and (2) are equivalent is the Eberlien-Smulian theorem; that (1) and (3) are equivalent is the Krein-Smulian theorem. That (1) and (4) are equivalent is a result of Grothendieck, detailed in, for example, [3, Theorem 2.3]. □

It is standard that for non-empty sets I and J, we have that $l^1(I)\widehat{\otimes}l^1(J)=l^1(I\times J)$, where, for $i\in I$ and $j\in J$, $\delta_i\otimes\delta_j\in l^1(I)\widehat{\otimes}l^1(J)$ is identified with $\delta_{(i,j)}\in l^1(I\times J)$. Thus we have $(l^1(I)\widehat{\otimes}l^1(J))'=\mathcal{B}(l^1(I),l^\infty(J))=l^1(I\times J)'=l^\infty(I\times J)$, where $T\in\mathcal{B}(l^1(I),l^\infty(J))$ is identified with $(T_{(i,j)})\in l^\infty(I\times J)$, where $T_{(i,j)}=\langle T(\delta_i),\delta_j\rangle$.

Is this paragraph used? Let S be a countable, discrete, unital semigroup, and let ω be a weight on S. Then $l^1(S\times S)$ is a Banach $l^1(S,\omega)$-bimodule, with module actions
$$\delta_k\cdot\delta_{(g,h)}=\delta_{(kg,h)}\Omega(k,g)\quad,\quad\delta_{(g,h)}\cdot\delta_k=\delta_{(g,hk)}\Omega(h,k)\qquad(g,h,k\in S).$$

For a non-empty set I, the unit ball of $l^1(I)$ is the closure of the absolutely-convex hull of the set $\{\delta_i : i\in I\}$, so that for a Banach space E, by the Krein-Smulian theorem, a map $T: l^1(I)\to E$ is weakly-compact if and only if the set $\{T(\delta_i): i\in I\}$ is relatively weakly-compact in E.

Proposition 5.5. *Let S be a weakly cancellative semigroup, let ω be a weight on S, and let $\mathcal{A} = l^1(S, \omega)$. Let $T \in \mathcal{B}(\mathcal{A}, \mathcal{A}')$ be such that $T(\mathcal{A}) \subseteq \kappa_{c_0(S)}(c_0(S))$ and $T'(\kappa_{\mathcal{A}}(\mathcal{A})) \subseteq \kappa_{c_0(S)}(c_0(S))$. Then $T \in \mathcal{W}(\mathcal{A}, \mathcal{A}')$, and $T \in \mathrm{WAP}(\mathcal{W}(\mathcal{A}, \mathcal{A}'))$ if and only if, for each sequence (k_n) of distinct elements of S, and each sequence (g_m, h_m) of distinct elements of $S \times S$ such that the repeated limits*

$$\lim_n \lim_m \langle T(\delta_{h_m}), \delta_{k_n g_m} \rangle, \;\; \lim_n \lim_m \Omega(k_n, g_m) \tag{4}$$

$$\lim_n \lim_m \langle T(\delta_{h_m k_n}), \delta_{g_m} \rangle, \;\; \lim_n \lim_m \Omega(h_m, k_n) \tag{5}$$

all exist, we have that at least one repeated limit in each row is zero.

Proof. That T is weakly-compact follows from Gantmacher's Theorem (compare with Corollary 3.5). To show that $T \in \mathrm{WAP}$, by Lemma 3.4, we are required to show that the maps ϕ_r and ϕ_l are weakly-compact. We shall show that ϕ_l is weakly-compact if and only if one of the repeated limits in the first line (4) is zero; the proof that ϕ_r is related to (5) follows in a similar way. We have that

$$\phi_l(\delta_{(g,h)}) = \phi_l(\delta_g \otimes \delta_h) = \delta_g \cdot T(\delta_h) \qquad (g, h \in S).$$

By Proposition 5.4, ϕ_l is weakly-compact if and only if the function

$$S \times (S \times S) \to \mathbb{C}; \; (k, (g, h)) \mapsto \langle \delta_g \cdot T(\delta_h), \delta_k \rangle = \langle T(\delta_h), \delta_{kg} \rangle \Omega(k, g) \qquad (g, h, k \in S)$$

clusters on $S \times (S \times S)$. As T is weakly-compact, the function

$$S \times S \to \mathbb{C}; \quad (g, h) \mapsto \langle T(\delta_g), \delta_h \rangle \qquad (g, h \in S)$$

does cluster on $S \times S$.

Let (k_n) be a sequence of distinct elements of S, and let (g_m, h_m) be a sequence of distinct elements of $S \times S$ such that the iterated limits

$$\lim_n \lim_m \langle T(\delta_{h_m}), \delta_{k_n g_m} \rangle \Omega(k_n, g_m), \quad \lim_m \lim_n \langle T(\delta_{h_m}), \delta_{k_n g_m} \rangle \Omega(k_n, g_m) \tag{6}$$

exist. We now investigate when these iterated limits are equal.

Suppose firstly that, by moving to a subsequence if necessary, we have that $g_m = g$ for all m. Further, by moving to a subsequence if necessary, we may suppose that $\lim_n \Omega(k_n, g) = \alpha$, say, and that $(k_n g)$ is a sequence of distinct elements (as S is weakly cancellative). Then

$$\begin{aligned}
\lim_n \lim_m \langle T(\delta_{h_m}), \delta_{k_n g_m} \rangle \Omega(k_n, g_m) &= \lim_n \Omega(k_n, g) \lim_m \langle T(\delta_{h_m}), \delta_{k_n g} \rangle \\
&= \alpha \lim_n \lim_m \langle T(\delta_{h_m}), \delta_{k_n g} \rangle = \alpha \lim_m \lim_n \langle T(\delta_{h_m}), \delta_{k_n g} \rangle \\
&= \lim_m \lim_n \langle T(\delta_{h_m}), \delta_{k_n g_m} \rangle \Omega(k_n, g_m),
\end{aligned}$$

where we can swap the order of taking limits, as T is weakly-compact.

Alternatively, if we cannot move to a subsequence such that (g_m) is constant, then we may move to subsequence such that (g_m) is a sequence of distinct elements, and such that the iterated limits

$$\lim_m \lim_n \Omega(k_n, g_m), \quad \lim_n \lim_m \Omega(k_n, g_m),$$
$$\lim_m \lim_n \langle T(\delta_{h_m}), \delta_{k_n g_m} \rangle, \quad \lim_n \lim_m \langle T(\delta_{h_m}), \delta_{k_n g_m} \rangle$$

all exists. As $T(\mathcal{A}) \subseteq \kappa_{c_0(S)}(c_0(S))$, we have that

$$\{g \in S : |\langle T(\delta_h), \delta_g \rangle| \geq \epsilon\} \text{ is finite} \qquad (\epsilon > 0, h \in S).$$

Consequently, and using the fact that S is weakly cancellative, we see that

$$\lim_n \langle T(\delta_{h_m}), \delta_{k_n g_m} \rangle = 0$$

for each m. Hence the iterated limits in (6) are equal if and only if we have that at least one repeated limit in (4) is zero. □

Proposition 5.6. *Let S be a discrete, unital, weakly cancellative semigroup, and let ω be a weight on S such that $\mathcal{A} = l^1(S, \omega)$ is Arens regular. Then* $\mathrm{WAP}(\mathcal{W}(\mathcal{A}, \mathcal{A}')) = \mathcal{W}(\mathcal{A}, \mathcal{A}')$.

Proof. Let $T \in \mathcal{W}(\mathcal{A}, \mathcal{A}')$. We can follow the above proof through until the point at which we use the fact that $T(\mathcal{A}) \subseteq \kappa_{c_0(S)}(c_0(S))$. However, as $l^1(S, \omega)$ is Arens regular, by Theorem 5.3, we have that

$$\lim_m \lim_n \Omega(k_n, g_m) = \lim_n \lim_m \Omega(k_n, g_m) = 0,$$

so that the iterated limits in (6) must be 0, implying that ϕ_l is weakly-compact. In a similar manner, ϕ_r is weakly-compact. □

Theorem 5.7. *Let S be a discrete weakly cancellative semigroup, and let ω be a weight on S such that $\mathcal{A} = l^1(S, \omega)$ is Arens regular and $\mathcal{A}''$ is unital with unit $e_{\mathcal{A}''}$. Then $\mathcal{A}''$ is Connes-amenable if and only if there exists $M \in (\mathcal{A} \widehat{\otimes} \mathcal{A})'' = l^\infty(S \times S)'$ such that:*

1. *$\langle M, (f_{gh} \Omega(g, h))_{(g,h) \in S \times S} \rangle = \langle e_{\mathcal{A}''}, f \rangle$ for each bounded family $(f_g)_{g \in S}$;*
2. *$\langle M, (f(hk, g)\Omega(h, k) - f(h, kg)\Omega(k, g))_{(g,h) \in S \times S} \rangle = 0$ for each $k \in S$, and each bounded function $f : S \times S \to \mathbb{C}$ which clusters on $S \times S$.*

Proof. We use Theorem 3.9 and Proposition 5.6. For $f = (f_g)_{g \in S} \in l^\infty(S)$, we have

$$\langle \Delta'_{\mathcal{A}}(f), \delta_g \otimes \delta_h \rangle = \langle f, \delta_{gh} \rangle \Omega(g, h) \qquad (g, h \in S),$$

so that $\Delta'_{\mathcal{A}}(f) = (\langle f, \delta_{gh} \rangle \Omega(g, h))_{(g,h) \in S \times S} \in l^\infty(S \times S)$. As $f \in l^\infty(S)$ was arbitrary, we have condition (1).

For $T \in \mathcal{B}(\mathcal{A}, \mathcal{A}')$, we treat T as being a member of $l^\infty(S \times S)$. Then T is weakly-compact if and only if the family $(\langle T(\delta_g), \delta_h \rangle)_{(g,h) \in S \times S}$ clusters on $S \times S$. For $k \in S$, we have

$$\langle \delta_k \cdot T - T \cdot \delta_k, \delta_g \otimes \delta_h \rangle = \langle T(\delta_{hk}), \delta_g \rangle \Omega(h, k) - \langle T(\delta_h), \delta_{kg} \rangle \Omega(k, g).$$

Thus we have condition (2). □

Notice that if S is unital with unit u_S, then the unit of $\mathcal{A}$ (and hence $\mathcal{A}''$) is δ_{u_S}. In this case, condition (1) reduces to $\langle M, (f_{gh} \Omega(g, h))_{(g,h) \in S \times S} \rangle = f_{u_S}$.

Theorem 5.8. *Let S be a discrete unital semigroup, let ω be a weight on S, and let $\mathcal{A} = l^1(S, \omega)$. Then $\mathcal{A}$ is amenable if and only if there exists $M \in (\mathcal{A} \widehat{\otimes} \mathcal{A})'' = l^\infty(S \times S)'$ such that:*

1. *$\langle M, (f_{gh} \Omega(g, h))_{(g,h) \in S \times S} \rangle = f_{u_S}$, where $u_S \in S$ is the unit of S, for each bounded family $(f_g)_{g \in S}$;*
2. *$\langle M, (f(hk, g)\Omega(h, k) - f(h, kg)\Omega(k, g))_{(g,h) \in S \times S} \rangle = 0$ for each $k \in S$, and each bounded function $f : S \times S \to \mathbb{C}$.*

Proof. This follows from Theorem 2.1 in the same way that the above follows from Theorem 3.9. $\square$

Notice that condition (2) of Theorem 5.8 is strictly stronger than condition (2) of Theorem 5.7.

Theorem 5.9. *Let S be a discrete, weakly cancellative semigroup, let ω be a weight on S, and let $\mathcal{A} = l^1(S,\omega)$ be unital with unit $e_{\mathcal{A}}$. Then $\mathcal{A}$ is Connes-amenable, with respect to the predual $c_0(S)$, if and only if there exists $M \in (\mathcal{A}\widehat{\otimes}\mathcal{A})'' = l^\infty(S\times S)'$ such that:*

1. *$\langle M, (f_{gh}\Omega(g,h))_{(g,h)\in S\times S}\rangle = \langle e_{\mathcal{A}}, f\rangle$ for each family $(f_g)_{g\in S} \in c_0(S)$;*
2. *$\langle M, (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}\rangle = 0$ for each $k \in S$, and each bounded function $f : S\times S \to \mathbb{C}$ which satisfies the conclusions of Proposition 5.5.*

Proof. We now use Theorem 3.6. By f satisfying the conclusions of Proposition 5.5, we identify $f : S\times S\to\mathbb{C}$ with $T \in \mathcal{B}(\mathcal{A},\mathcal{A}')$ by $\langle T(\delta_g),\delta_h\rangle = f(g,h)$, for $g,h\in S$. $\square$

We shall now establish when $l^1(S,\omega)$ and $l^1(S,\omega)''$ are Connes-amenable. For a discrete group G, a weight ω on G and $h\in G$, define $J_h \in \mathcal{B}(l^\infty(G))$ by

$$J_h(f) = \big(f_{hg}\Omega(h,g)\omega(h)\Omega(g^{-1},h^{-1})\omega(h^{-1})\big)_{g\in G} \qquad (f = (f_g)_{g\in G}\in l^\infty(G)).$$

Notice then that, for $f\in l^\infty(G)$, we have

$$\|J_h(f)\| = \sup_g |f_{hg}|\omega(hg)\omega(g)^{-1}\omega(g^{-1}h^{-1})\omega(g^{-1})^{-1} \leq \|f\|\omega(h)\omega(h^{-1}),$$

so that J_h is bounded.

Definition 5.10. Let G be a discrete group, and let ω be a weight on G. We say that G is *ω-amenable* if there exists $N\in l^\infty(G)'$ such that:

1. $\langle N, (\Omega(g,g^{-1}))_{g\in G}\rangle = 1$, where Ω is defined by ω, and hence $(\Omega(g,g^{-1}))_{g\in G}$ is a bounded family forming an element of $l^\infty(G)$;
2. $J_h'(N) = N$ for each $h\in G$. $\square$

Notice that if ω is identically 1, then this condition reduces to the usual notion of a group being amenable (we usually require that N is a *mean*, in that N is a positive functional on $l^\infty(G)$, but by forming real and imaginary parts, and then positive and negative parts, we can easily generate a non-zero scalar multiple of a mean from a functional N satisfying the definition above).

Theorem 5.11. *Let G be a discrete group, let ω be a weight on G, and let $\mathcal{A} = l^1(G,\omega)$. Then the following are equivalent:*

1. *$\mathcal{A}$ is Connes-amenable, with respect to the predual $c_0(G)$;*
2. *$\mathcal{A}$ is amenable;*
3. *G is ω-amenable.*

Furthermore, if $\mathcal{A}$ is Arens regular, then these conditions are equivalent to $\mathcal{A}''$ being Connes-amenable.

Proof. It is clear that (2) implies (1). When $\mathcal{A}$ is Arens regular, (2) implies that $\mathcal{A}''$ is Connes-amenable, and $\mathcal{A}''$ Connes-amenable implies (1). We shall thus show that (1) implies (3), and that (3) implies (2).

If (1) holds, then let $M \in l^\infty(G \times G)'$ be given as in Theorem 5.9. Define $\phi : l^\infty(G) \to l^\infty(G \times G)$ by

$$\langle \phi(f), \delta_{(g,h)} \rangle = \begin{cases} f_g & : g = h^{-1}, \\ 0 & : g \neq h^{-1}, \end{cases} \qquad (f = (f_g)_{g \in G} \in l^\infty(G)).$$

Let $N = \phi'(M) \in l^\infty(G)'$. Then we have

$$\phi((\Omega(g, g^{-1}))_{g \in G}) = (\delta_{h,g^{-1}} \Omega(g,h))_{(g,h) \in G \times G} = (\delta_{gh,e_G} \Omega(g,h))_{(g,h) \in G \times G},$$

where δ is the Kronecker delta, so that

$$\langle N, (\Omega(g, g^{-1}))_{g \in G} \rangle = \delta_{e_G, e_G} = 1,$$

by condition (1) on M from Theorem 5.9; clearly $(\delta_{e_G,g})_{g \in G} \in c_0(G)$.

Fix $k \in G$ and $f \in l^\infty(G)$. Define $F : G \times G \to \mathbb{C}$ by

$$F(h,g) = \delta_{gh,k} f_g \omega(k) \omega(hk^{-1}) \omega(h)^{-1}. \qquad (g, h \in G).$$

Then we have $|F(h,g)| \leq |f_g||\omega(k)||\omega(hk^{-1})||\omega(h)|^{-1} \leq \|f\|_\infty |\omega(k)||\omega(k^{-1})|$, so that F is bounded. Let $T : \mathcal{A} \to \mathcal{A}'$ be the operator associated with F. For $g, h \in G$, we have that $F(h,g) \neq 0$ only when $gh = k$, so that $T(\mathcal{A}) \subseteq c_0(S)$ and $T'(\kappa_{\mathcal{A}}(\mathcal{A})) \subseteq c_0(S)$. Furthermore, if (k_n) is a sequence of distinct elements in G, and (g_m, h_m) is a sequence of distinct elements in $G \times G$, then $\lim_n \lim_m F(h_m, k_n g_m) = 0$. This follows, as for n_0 fixed, $k_{n_0} g_m h_m = k$ only if $g_m h_m = k_{n_0}^{-1} k$, so if this holds for all sufficiently large m, we have that $k_n g_m h_m \neq k$ for sufficiently large m and $n \neq n_0$. Similarly, $\lim_n \lim_m F(h_m k_n, g_m) = 0$, so that F satisfies the conditions of Proposition 5.5.

Notice that

$$\langle \phi(J_k(f)), \delta_{(g,h)} \rangle = \delta_{gh,e_G} \langle J_k(f), \delta_g \rangle = \delta_{gh,e_G} f_{kg} \omega(kg) \omega(g)^{-1} \omega(g^{-1}k^{-1}) \omega(g^{-1})^{-1}.$$

Thus we have

$$\begin{aligned} F(hk,g)&\Omega(h,k) - F(h,kg)\Omega(k,g) \\ &= \delta_{ghk,k} f_g \omega(k) \omega(hkk^{-1}) \omega(hk)^{-1} \Omega(h,k) - \delta_{kgh,k} f_{kg} \omega(k) \omega(hk^{-1}) \omega(h)^{-1} \Omega(k,g) \\ &= \delta_{gh,e_G} f_g - \delta_{gh,e_G} f_{kg} \omega(hk^{-1}) \omega(h)^{-1} \omega(kg) \omega(g)^{-1} \\ &= \langle \phi(f) - \phi(J_k(f)), \delta_{(g,h)} \rangle. \end{aligned}$$

So, by condition (2) from Theorem 5.9, we have that

$$\langle N, f - J_k(f) \rangle = 0,$$

which, as f was arbitrary, shows that $N = J_k'(N)$, as required.

Now suppose that G is ω-amenable. We shall show that $\mathcal{A}$ is amenable, which completes the proof. Define $\psi : l^\infty(G \times G) \to l^\infty(G)$ by

$$\langle \psi(F), \delta_g \rangle = F(g, g^{-1}) \qquad (F \in l^\infty(G \times G), g \in G).$$

Let $N \in l^\infty(G)'$ be as in Definition 5.10, and let $M = \psi'(N)$. Then let $(f_g)_{g \in G}$ be a bounded family in $\mathbb{C}$, so that

$$\langle M, (f_{gh} \Omega(g,h))_{(g,h) \in G \times G} \rangle = \langle N, (f_{e_G} \Omega(g, g^{-1}))_{g \in G} \rangle = f_{e_G},$$

verifying condition (1) of Theorem 5.8 for M.

Let $f : G \times G \to \mathbb{C}$ be a bounded function, and let $k \in G$. Then

$$\begin{aligned}\psi\big((f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in G\times G}\big) \\ = \big(f(g^{-1}k,g)\Omega(g^{-1},k) - f(g^{-1},kg)\Omega(k,g)\big)_{g\in G}.\end{aligned}$$

Define $F : G \times G \to \mathbb{C}$ by

$$F(g,h) = f(hk,g)\Omega(h,k) \qquad (g,h \in G),$$

so that F is bounded. For $g \in G$, we have that

$$\begin{aligned}&\langle \psi(F) - J_k(\psi(F)), \delta_g\rangle \\ &= f(g^{-1}k,g)\Omega(g^{-1},k) - f((kg)^{-1}k,kg)\Omega((kg)^{-1},k)\omega(kg)\omega(g)^{-1}\omega(g^{-1}k^{-1})\omega(g^{-1})^{-1} \\ &= f(g^{-1}k,g)\Omega(g^{-1},k) - f(g^{-1},kg)\omega(k)^{-1}\omega(kg)\omega(g)^{-1} \\ &= f(g^{-1}k,g)\Omega(g^{-1},k) - f(g^{-1},kg)\Omega(k,g).\end{aligned}$$

Consequently, using condition (2) of Definition 5.10, we have established condition (2) of Theorem 5.8 for M. This shows that $l^1(G,\omega)$ is amenable. ☐

Example 5.12. If S is a semigroup which is not cancellative, then it is possible for $l^1(S)$ to be unital while S is not. For example, let S be $(\mathbb{N}, \max)$ (where $\mathbb{N} = \{1,2,3,\ldots\}$ say) with adjoined idempotents u and v such that $uv = vu = 1$ and $un = nu = vn = nv = n$ for $n \in \mathbb{N}$. Then S is a weakly cancellative, commutative semigroup without a unit, but $e = \delta_u + \delta_v - \delta_1$ is easily seen to be a unit for $l^1(S)$. Indeed, S is seen to be a finite semilattice of groups, so by the result of [6], $l^1(S)$ is amenable.

In [5, Theorem 2.3] it is shown that if $l^1(S,\omega)$ is amenable for a cancellative, unital semigroup S and some weight ω, then S is actually a group. We shall now show that this holds for Connes-amenability as well.

For a cancellative, unital semigroup S, with unit u_S, if $g \in S$ is invertible, then g has a unique inverse, denoted by g^{-1}. Furthermore, if g has a left inverse, say $hg = u_S$, then $ghg = g = u_S g$ so that $gh = u_S$; similarly, if $gh = u_S$ then $hg = u_S$.

Theorem 5.13. *Let S be a weakly cancellative semigroup, let ω be a weight on S, and let $\mathcal{A} = l^1(S,\omega)$. Suppose that $\mathcal{A}$ is Connes-amenable with respect to the predual $c_0(S)$. If S is cancellative or unital, then S is a group.*

Proof. As $\mathcal{A}$ is Connes-amenable, let $M \in (\mathcal{A}\widehat{\otimes}\mathcal{A})''$ be as in Theorem 5.9. Then $\mathcal{A}$ is unital, with unit $e_{\mathcal{A}} = (a_s)_{s\in S} \in l^1(S,\omega)$ say. For now, we shall not assume that $e_{\mathcal{A}}$ has norm one, as the standard renorming to ensure this will not (a priori) necessarily yield an $l^1(S,\hat{\omega})$ algebra for some weight $\hat{\omega}$. Suppose that S is cancellative. Fix $h \in S$, so that

$$\sum_{s\in S} a_s \delta_{sh}\Omega(s,h) = e_{\mathcal{A}} \star \delta_h = \delta_h = \delta_h \star e_{\mathcal{A}} = \sum_{s\in S} a_s \delta_{hs}\Omega(h,s).$$

In particular, for each $h \in S$ there is a unique $u_h \in S$ such that $hu_h = h$ (so that $hu_h h = h^2$ implying that $u_h h = h$), and we have that $a_{u_h}\omega(u_h)^{-1} = 1$. We also see that $a_s = 0$ for each $s \in S$ such that $sh \neq h$, that is, $s \neq u_h$. However, h was arbitrary, so that S is unital with unit u_S, and $e_{\mathcal{A}} = \omega(u_S)\delta_{u_S}$, where we can now assume that $\omega(u_S) = 1$ by a renorming.

Now suppose that S is a unital, weakly cancellative semigroup, so that the unit of $\mathcal{A}$ is δ_{u_S}. Suppose that $s \in S$ has no right inverse. Define $F : S \times S \to \mathbb{C}$ by

$$F(h,sg) = 0, \quad F(hs,g) = \begin{cases} \Omega(g,hs) & : gh = u_S, \\ 0 & : \text{otherwise}. \end{cases} \qquad (g,h \in S).$$

To show that this is well-defined, suppose that for $g, h, j, k \in S$, we have that $h = js$, $sg = k$ and $kj = u_S$. Then $s(gj) = kj = u_S$, so that s has a right inverse, a contradiction. Then F is bounded, so let $T : \mathcal{A} \to \mathcal{A}'$ be the operator associated with F. Then $F(a, b) \neq 0$ only when $ba = s$, so as S is weakly cancellative, we see that $T(\mathcal{A}) \subseteq c_0(S)$ and $T'(\kappa_{\mathcal{A}}(\mathcal{A})) \subseteq c_0(S)$.

Suppose that for sequences of distinct elements $(k_n) \subseteq S$ and $(g_m, h_m) \subseteq S \times S$, we have that

$$\lim_n \lim_m \langle T(\delta_{h_m}), \delta_{k_n g_m} \rangle = \lim_n \lim_m F(h_m, k_n g_m) \neq 0.$$

Then, for some $N > 0$ and $\epsilon > 0$, for each $n \geq N$, $\lim_m F(h_m, k_n g_m) \geq \epsilon$. Hence, for $n \geq N$, there exists $M_n > 0$ such that if $m \geq M_n$, then $k_n g_m h_m = s$ (as otherwise $F(h_m, k_n g_m) = 0$). This, however, contradicts S being weakly cancellative. Similarly, if $\lim_n \lim_m \langle T(\delta_{h_m k_n}), \delta_{g_m} \rangle \neq 0$, then we need $g_m h_m k_n = s$ for all n, m sufficiently large, which is a contradiction. Thus T satisfies all the conditions of Proposition 5.5.

Then, for $g, h \in S$, if $gh = u_S$, we have that $\Omega(h, s)\Omega(g, hs) = \omega(h)^{-1}\omega(g)^{-1} = \Omega(g, h)$, so that

$$F(hs, g)\Omega(h, s) - F(h, sg)\Omega(s, g) = \begin{cases} \Omega(g, h) & : gh = u_S, \\ 0 & : \text{otherwise.} \end{cases}$$

Hence condition (2) of Theorem 5.9 implies that $\langle M, (\delta_{gh,u_S}\Omega(g, h))_{(g,h)\in S\times S} \rangle = 0$, which contradicts condition (1) of this theorem. Hence every element of S has a right inverse.

By symmetry (or by repeating the argument on the left) we see that every element of S has a left inverse, and that hence S must be a group. $\square$

We hence have the following theorem, which shows that weighted semigroup algebras behave like C*-algebras with regards to Connes-amenability.

Theorem 5.14. *Let S be a discrete cancellative semigroup, and let ω be a weight on S. The following are equivalent:*

1. *$l^1(S, \omega)$ is amenable;*
2. *$l^1(S, \omega)$ is Connes-amenable, with respect to the predual $c_0(S)$;*

If $l^1(S, \omega)$ is Arens regular, then these conditions are equivalent to $l^1(S, \omega)''$ being Connes-amenable. These equivalent conditions imply that S is a group. $\square$

This result extends the result of [12], where it is shown that $M(G)$, the *measure algebra* of a locally compact group G, is Connes-amenable if and only if G is amenable. This follows as, for discrete groups G, $M(G) = l^1(G)$.

Example 5.15. Let ω be the weight on $\mathbb{Z}$ defined by $\omega(n) = 1 + |n|$ for $n \in \mathbb{Z}$. By Theorem 5.3, $\mathcal{A} = l^1(\mathbb{Z}, \omega)$ is Arens regular. For $m, n \in \mathbb{Z}$ and $f = (a_k)_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, we have that

$$\langle \delta_m \cdot f, \delta_n \rangle = \langle f, \delta_{n+m}\Omega(n, m) \rangle = f_{n+m} \frac{1 + |n + m|}{(1 + |n|)(1 + |m|)}.$$

Suppose that $M \Box \kappa_{\mathcal{A}}(\delta_m) = \kappa_{\mathcal{A}}(a)$ for some $m \in \mathbb{Z}$, $M \in l^\infty(\mathbb{Z})'$ and $a \in \mathcal{A}$. Then $\langle M, \delta_m \cdot f \rangle = \langle f, a \rangle$ for each $f \in l^\infty(\mathbb{Z})$, so by letting $f = \kappa_{c_0(\mathbb{Z})}(e_k) \in c_0(\mathbb{Z})$, we see that $a = \sum_{k\in\mathbb{Z}} a_k \delta_k$, where $a_k = \langle M, \delta_m \cdot \kappa_{c_0(\mathbb{Z})}(e_k) \rangle$. However, $\delta_m \cdot \kappa_{c_0(\mathbb{Z})}(e_k) \in \kappa_{c_0(\mathbb{Z})}(c_0(\mathbb{Z}))$ for each $k \in \mathbb{Z}$, so if $M \in c_0(\mathbb{Z})^\circ$, then $a = 0$.

Consequently, if $M \Box \kappa_{\mathcal{A}}(\delta_m) \in \kappa_{\mathcal{A}}(\mathcal{A})$ for each $m \in \mathbb{Z}$ and $M \in l^\infty(\mathbb{Z})'$, then $\delta_m \cdot f \in \kappa_{c_0(\mathbb{Z})}(c_0(\mathbb{Z}))$ for each $m \in \mathbb{Z}$ and $f \in l^\infty(\mathbb{Z})$. However, if $\mathbf{1} \in l^\infty(\mathbb{Z})$ is the constant 1 sequence, then

$$\lim_n \langle \delta_m \cdot \mathbf{1}, \delta_n \rangle = \lim_n \frac{1 + |n + m|}{(1 + |n|)(1 + |m|)} = \frac{1}{1 + |m|},$$

so that $\delta_m \cdot \mathbf{1} \notin \kappa_{c_0(\mathbb{Z})}(c_0(\mathbb{Z}))$.

We hence conclude that $\mathcal{A}$ is not an ideal in $\mathcal{A}''$, and so we cannot apply Theorem 1.4 in this case. □

Unfortunately, it is not possible for $l^1(S,\omega)$ to be both amenable and Arens regular.

Theorem 5.16. *Let G be discrete group, and let ω be a weight on G. Then $l^1(G,\omega)$ is amenable if and only if G is an amenable group, and* $\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$.

Proof. This is [5, Theorem 3.2]. □

Proposition 5.17. *Let S be a discrete, unital semigroup, and let ω be a weight on S such that $\mathcal{A} = l^1(S,\omega)$ is Arens regular. Let $K > 0$ and $B \subseteq S$ be such that for each $g \in B$, g has a right inverse g^{-1} (which need not be unique), and $\omega(g)\omega(g^{-1}) \leq K$. Then B is finite.*

Proof. For $g \in B$ and $h \in S$, we have

$$\omega(g)\omega(h) = \omega(g)\omega(hgg^{-1}) \leq \omega(g)\omega(hg)\omega(g^{-1}) \leq K\omega(hg),$$

so that $\Omega(h,g) \geq K^{-1}$. Suppose now that B is infinite. Then we can easily construct sequences which violate condition (2) of Theorem 5.3, showing that $\mathcal{A}$ is not Arens regular. This contradiction shows that B must be finite. □

5.1 Injectivity of the predual module

Let S be a unital, weakly cancellative semigroup, let ω be a weight on S, and let $\mathcal{A} = l^1(S,\omega)$, $\mathcal{A}_* = c_0(S)$. Then $\mathcal{B}(\mathcal{A},\mathcal{A}_*) = \mathcal{B}(l^1,c_0) = l^\infty(c_0) \subseteq l^\infty(S \times S)$, where we identify $T : \mathcal{A} \to \mathcal{A}_*$ with the bounded family $(\langle \delta_s, T(\delta_t)\rangle)_{(s,t)\in S\times S}$. Let $\phi : \mathcal{B}(\mathcal{A},\mathcal{A}_*) \to \mathcal{A}_*$, so that ϕ is represented by a bounded family $(M_s)_{s\in S} \subseteq \mathcal{B}(\mathcal{A},\mathcal{A}_*)'$ using the relation

$$\langle \delta_s, \phi(T)\rangle = \langle M_s, T\rangle \qquad (s \in S, T \in \mathcal{B}(\mathcal{A},\mathcal{A}_*)).$$

Suppose further that ϕ is a left $\mathcal{A}$-module homomorphism. Then

$$\langle \delta_s, \phi(T)\rangle = \langle \delta_{u_s}, \phi(\delta_s \cdot T)\rangle = \langle M_{u_S}, \delta_s \cdot T\rangle = \langle M_s, T\rangle \qquad (s \in S, T \in \mathcal{B}(\mathcal{A},\mathcal{A}_*)), \quad (7)$$

so that $M_s = M_{u_S} \cdot \delta_s$ for each $s \in S$. We see also that ϕ maps into $c_0(S)$ (and not just $l^\infty(S)$) if and only if

$$\lim_{s\to\infty} \langle M_{u_S}, \delta_s \cdot T\rangle = 0 \qquad (T \in \mathcal{B}(\mathcal{A},\mathcal{A}_*)).$$

Conversely, if condition (7) holds, then for $s,t \in S$ and $T \in \mathcal{B}(\mathcal{A},\mathcal{A}_*)$, we have that

$$\begin{aligned}\langle \delta_s, \phi(\delta_t \cdot T)\rangle &= \langle M_s, \delta_t \cdot T\rangle = \langle M_{u_S}, \delta_s \cdot \delta_t \cdot T\rangle = \Omega(s,t)\langle M_{st}, T\rangle \\ &= \Omega(s,t)\langle \delta_{st}, \phi(T)\rangle = \langle \delta_s, \delta_t \cdot \phi(T)\rangle.\end{aligned}$$

Hence ϕ is a left $\mathcal{A}$-module homomorphism.

Notice that $c_0(S \times S) \subseteq \mathcal{B}(\mathcal{A},\mathcal{A}_*)$, so that $c_0(S \times S)^\circ \subseteq \mathcal{B}(\mathcal{A},\mathcal{A}_*)'$.

Definition 5.18. Let G be a group and ω be a weight on G such that for each $\epsilon > 0$, the set $\{g \in G : \omega(g)\omega(g^{-1}) < \epsilon^{-1}\}$ is finite. Then we say that the weight ω is *strongly non-amenable*.

Proposition 5.19. *Let G be a group, and let ω be a weight on G such that ω is not strongly non-amenable, and let $\phi : \mathcal{B}(\mathcal{A}, c_0(G)) \to c_0(G)$ be a left $\mathcal{A}$-module homomorphism. If ϕ is represented by $(M_g)_{g\in G}$ as above, then $M_{u_G} \in c_0(S \times S)^\circ$.*

Proof. We adapt the methods of [4] to the weighted, discrete case. As ω is not strongly non-amenable, there exists some $K > 0$ such that the set $X_K = \{g \in G : \omega(g)\omega(g^{-1}) \leq K\}$ is infinite. Let $M = M_{u_G}$, and suppose that $M \notin c_0(G \times G)^\circ$, so that for some $g, h \in G$, we have that $\delta := \langle M, e_{(g,h)} \rangle \neq 0$. We shall henceforth treat $e_{(g,h)}$ as a member of $\mathcal{B}(\mathcal{A}, c_0(G))$, noting that for $k \in G$,

$$\langle \delta_s, (\delta_k \cdot e_{(g,h)})(\delta_t) \rangle = \begin{cases} \Omega(t,k) & : s = g, t = hk^{-1}, \\ 0 & : \text{otherwise.} \end{cases}$$

We claim that we can find a sequence $(g_n)_{n\in\mathbb{N}}$ of distinct elements in G such that

$$|\langle M \cdot \delta_{g_m^{-1}g_n}, e_{(g,h)} \rangle| \leq K^{-1}2^{-2-|m-n|} \qquad (n \neq m),$$
$$\omega(g_n)\omega(g_n^{-1}) \leq K \qquad (n \in \mathbb{N}).$$

We can do this as ϕ must map into $c_0(G)$, so that for any $T : \mathcal{A} \to c_0(G)$, we have $\lim_{g\to\infty} \langle M \cdot \delta_g, T \rangle = 0$. Explicitly, let $g_1 \in X_K$ be arbitrary, and suppose that we have found $g_1, \dots, g_k$. Then notice that the sets

$$\{s \in G : |\langle M \cdot \delta_{s^{-1}g_n}, e_{(g,h)} \rangle| > K^{-1}2^{-2-|k+1-n|} : 1 \leq n \leq k\},$$
$$\{s \in G : |\langle M \cdot \delta_{g_m^{-1}s}, e_{(g,h)} \rangle| > K^{-1}2^{-2-|k+1-m|} : 1 \leq m \leq k\}$$

are finite, so as X_K is infinite, we can certainly find some x_{k+1}.

Then, for $x = (x_n) \in l^\infty(\mathbb{N})$, define $T_x : \mathcal{A} \to c_0(G)$ by setting $\langle \delta_g, T_x(\delta_{hg_n^{-1}}) \rangle = x_n\Omega(hg_n^{-1}, g_n)$ for $n \geq 1$, and $\langle \delta_s, T_x(\delta_t) \rangle = 0$ otherwise. Then clearly T_x does map into $c_0(G)$, and $\|T_x\| \leq \|x\|$. Notice that for $s, t \in G$, we have

$$\begin{aligned}\langle \delta_s, T_x(\delta_t) \rangle &= \begin{cases} x_n\Omega(t, g_n) & : s = g, t = hg_n^{-1}, \\ 0 & : \text{otherwise,} \end{cases} \\ &= \sum_n x_n \langle \delta_s, (\delta_{g_n} \cdot e_{(g,h)})(\delta_t) \rangle.\end{aligned}$$

Define $Q : l^\infty(\mathbb{N}) \to c_0(\mathbb{N})$ by

$$\langle \delta_n, Q(x) \rangle = \langle M, \delta_{g_n^{-1}} \cdot T_x \rangle \qquad (n \in \mathbb{N}),$$

so that Q is bounded and linear.

Let $n_0 \geq 1$ and let $x = e_{n_0} \in c_0(\mathbb{N}) \subseteq l^\infty(\mathbb{N})$. Then, $T_x = \delta_{g_{n_0}} \cdot e_{(g,h)}$, so that

$$\begin{aligned}\langle \delta_n, Q(x) \rangle = \langle M, \delta_{g_n^{-1}} \cdot T_x \rangle &= \langle M, \delta_{g_n^{-1}} \cdot (\delta_{g_{n_0}} \cdot e_{(g,h)}) \rangle \\ &= \begin{cases} \delta\, \Omega(g_{n_0}^{-1}, g_{n_0}) & : n = n_0, \\ \Omega(g_n^{-1}, g_{n_0}) \langle M \cdot e_{g_n^{-1}g_{n_0}}, e_{(g,h)} \rangle & : n \neq n_0. \end{cases}\end{aligned}$$

Define $Q_1 \in \mathcal{B}(c_0(\mathbb{N}))$ by

$$Q_1(x) = \big(\Omega(g_n^{-1}, g_n)x_n\big)_{n\in\mathbb{N}} \qquad (x = (x_n) \in c_0(\mathbb{N})).$$

Then, as each $g_n \in X_K$, Q_1 is an invertible operator. Let Q_2 be the restriction of Q to $c_0(\mathbb{N})$, so that $Q_2 \in \mathcal{B}(c_0(\mathbb{N}))$ and $Q_2 = \delta Q_1 + \delta Q_3 Q_1$ for some $Q_3 \in \mathcal{B}(c_0(\mathbb{N}))$. Thus

$Q_3 = \delta^{-1}Q_2Q_1^{-1} - I_{c_0(\mathbb{N})}$, so that for $x \in c_0(\mathbb{N})$, we have that

$$\begin{aligned}\|Q_3(x)\| &= \sup_n |\langle \delta_n, \delta^{-1}Q_2Q_1^{-1}(x) - x\rangle| = \sup_n \Big| \sum_m x_m \langle \delta_n, \delta^{-1}Q_2Q_1^{-1}(e_m) - e_m\rangle \Big| \\ &= \sup_n \Big| \sum_{m\neq n} x_m \Omega(g_m^{-1}, g_m)^{-1}\Omega(g_n^{-1}, g_m)\langle M \cdot \delta_{g_n^{-1}g_m}, e_{(g,h)}\rangle \Big| \\ &\leq K^{-1} \sup_n \sum_{m\neq n} |x_m| 2^{-2-|m-n|}\omega(g_m)\omega(g_m^{-1}) \leq \|x\|/2.\end{aligned}$$

Consequently $Q_3 - I_{c_0(\mathbb{N})}$ is invertible, so that $Q_2Q_1^{-1}$ is invertible, showing that Q_2 is invertible. However, this implies that $Q_2^{-1}Q : l^\infty(\mathbb{N}) \to c_0(\mathbb{N})$ is a projection, which is a well-known contradiction, completing the proof. □

Theorem 5.20. *Let G be a countable group, let ω be a weight which is not strongly non-amenable, and let $\mathcal{A} = l^1(G, \omega)$. Then $c_0(G)$ is not left-injective.*

Proof. Suppose, towards a contradiction, that $c_0(G)$ is left-injective, so that there exists $M = M_{u_G} \in \mathcal{B}(\mathcal{A}, \mathcal{A}_*)'$ as above, with the additional condition that

$$\begin{aligned}\delta_{g,h} &= \langle \delta_g, \phi\Delta'_{\mathcal{A}}(e_h)\rangle = \langle M, \delta_g \cdot \Delta'_{\mathcal{A}}(e_h)\rangle = \Omega(hg^{-1}, g)\langle M, \Delta'_{\mathcal{A}}(e_{hg^{-1}})\rangle \\ &= \Omega(hg^{-1}, g)\langle M, \big(\delta_{st,hg^{-1}}\Omega(s,t)\big)_{(s,t)\in G\times G}\rangle \qquad (g, h \in G).\end{aligned}$$

This clearly reduces to

$$\delta_{g,u_G} = \langle M, \big(\delta_{st,g}\Omega(s,t)\big)_{(s,t)\in G\times G}\rangle \qquad (g \in G).$$

As G is countable, we can enumerate G as $G = \{g_n : n \in \mathbb{N}\}$. Then, for $g_n \in G$, let $X_{g_n} = \{g_1, \ldots, g_n\} \subseteq G$. Define $Q : l^\infty(G) \to \mathcal{B}(\mathcal{A}, c_0(G))$ by

$$\langle \delta_s, Q(x)(\delta_t)\rangle = \Omega(s,t) \sum_{g\in X_t} x_g \delta_{st,g} \qquad (s, t \in G, x \in l^\infty(G)).$$

Then, for each $t \in G$, as X_t is finite, we see that $Q(x)(\delta_t) \in c_0(G)$, so Q is well-defined. Clearly Q is linear, and we see that for $x \in l^\infty(G)$,

$$\|Q(x)\| = \sup_{s,t\in G} \Omega(s,t)\Big|\sum_{g\in X_t} x_g\delta_{st,g}\Big| \leq \sup_{s,t\in G} \sum_{\{g\in X_t : g=st\}} |x_g| = \|x\|,$$

so that Q is norm-decreasing. Then, for $h \in G$, we have that

$$\langle \delta_s, Q(e_h)(\delta_t)\rangle = \Omega(s,t)\sum_{g\in X_t} \delta_{g,h}\delta_{st,g} = \begin{cases} \langle \delta_s, \Delta'_{\mathcal{A}}(e_h)(\delta_t)\rangle & : h \in X_t, \\ 0 & : h \notin X_t. \end{cases}$$

Let $h = g_{n_0}$, so that $\{t \in G : h \notin X_t\} = \{g_n \in G : h \notin X_{g_n}\} = \{g_1, g_2, \ldots, g_{n_0-1}\}$. We hence see that $Q(e_{g_0}) - \Delta'_{\mathcal{A}}(e_{g_0}) \in c_0(G\times G)$. By the preceding proposition, we hence have that $I_{c_0(G)} = \phi \circ \Delta'_{\mathcal{A}} = \phi \circ (Q|_{c_0(G)})$. However, this implies that $\phi \circ Q : l^\infty(G) \to c_0(G)$ is a projection onto $c_0(G)$, giving us the required contradiction. □

Theorem 5.21. *Let S be a discrete, weakly cancellative semigroup, let ω be a weight on S, and let $\mathcal{A} = l^1(S, \omega)$. When S is unital, or S is cancellative, $c_0(G)$ is not a bi-injective $\mathcal{A}$-bimodule.*

Proof. Suppose, towards a contradiction, that $c_0(G)$ is bi-injective. Then $\mathcal{A}$ is Connes-amenable, so that Theorem 5.14 implies that $\mathcal{A}$ is amenable, and that $S = G$ is a group. By Theorem 5.16, we know that ω is not strongly non-amenable. Suppose that G is countable, so that the above theorem shows that $c_0(G)$ is not left-injective, and that hence $c_0(G)$ is certainly not bi-injective, a contradiction.

Suppose that G is not countable. Then let H be some countably infinite subgroup of G. Let $K = \sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$, and let $g, h \in G$. Then

$$\Omega(g,h) = \frac{\omega(gh)}{\omega(g)\omega(h)} = \frac{\omega(gh)}{\omega(g)\omega(g^{-1}gh)} \geq \frac{\omega(gh)}{\omega(g)\omega(g^{-1})\omega(gh)} = \frac{1}{\omega(g)\omega(g^{-1})} \geq K^{-1},$$

so that Ω is bounded below on $G \times G$, and hence on $H \times H$.

Then we can find $X \subseteq G$ such that $G = \bigcup_{x \in X} Hx$ and $Hx \cap Hy = \emptyset$ for distinct $x, y \in X$. Notice that if $g \in Hx$ then $g^{-1} \in x^{-1}H$, so that $G = \bigcup_{x\in X} x^{-1}H$ as well.

By the proof of Theorem 5.20, we see that $c_0(H)$ is not a left-injective $l^1(H,\omega)$-module. Suppose, towards a contradiction, that we do have some left $\mathcal{A}$-module homomorphism $\phi : \mathcal{B}(l^1(G,\omega), c_0(G)) \to c_0(G)$ with $\phi\Delta'_{\mathcal{A}} = I_{\mathcal{A}'}$. Notice that certainly $\mathcal{B}(l^1(G,\omega), c_0(G))$ and $c_0(G)$ are Banach left $l^1(H,\omega)$-modules, by restricting the action from $l^1(G,\omega)$.

Define a map $\psi : \mathcal{B}(l^1(H,\omega), c_0(H)) \to \mathcal{B}(l^1(G,\omega), c_0(G))$ by, for $g, k \in G$,

$$\langle \delta_g, \psi(T)(\delta_k)\rangle = \begin{cases} \frac{\omega(s)\omega(t)}{\omega(tx)\omega(k)}\langle \delta_t, T(\delta_s)\rangle & : g = tx, k = x^{-1}s \text{ for some } x \in X, s,t \in H, \\ 0 & : \text{otherwise.} \end{cases}$$

Certainly ψ is linear, while

$$\|\psi(T)\| \leq \|T\| \sup_{s,t\in H, x\in X} \frac{\omega(s)\omega(t)}{\omega(tx)\omega(x^{-1}s)} \leq \|T\| \sup_{s,t\in H,x\in X} \frac{\omega(s)\omega(t)}{\omega(txx^{-1}s)} = \|T\| \sup_{s,t\in H} \Omega(t,s)^{-1},$$

so that ψ is bounded. For $h, s, t \in H$, and $x \in X$, we have

$$\begin{aligned} &\langle \delta_{tx}, (\delta_h \cdot \psi(T))(\delta_{x^{-1}s})\rangle = \Omega(x^{-1}s, h)\langle \delta_{tx}, \psi(T)(\delta_{x^{-1}sh})\rangle \\ &= \frac{\Omega(x^{-1}s,h)\omega(sh)\omega(t)}{\omega(tx)\omega(x^{-1}sh)}\langle \delta_t, T(\delta_s)\rangle = \frac{\omega(sh)\omega(t)}{\omega(x^{-1}s)\omega(h)\omega(tx)}\langle \delta_t, T(\delta_s)\rangle \\ &= \omega(s)\omega(x^{-1}s)^{-1}\omega(t)\omega(tx)^{-1}\langle \delta_t, (\delta_h \cdot T)(\delta_s)\rangle = \langle \delta_{tx}, \psi(\delta_h \cdot T)(\delta_{x^{-1}s})\rangle. \end{aligned}$$

Thus ψ is a left $l^1(H,\omega)$-module homomorphism. For $h, s, t \in H$ and $x \in X$, we then have that

$$\begin{aligned} \langle \delta_{tx}, \psi(\Delta'_{l^1(H,\omega)}(e_h))(\delta_{x^{-1}s})\rangle &= \frac{\omega(t)\omega(s)}{\omega(x^{-1}s)\omega(tx)}\langle \delta_t, \delta_s \cdot e_h\rangle = \Omega(tx, x^{-1}s)\delta_{ts,h} \\ &= \langle \delta_{tx}, \delta_{x^{-1}s} \cdot e_h\rangle = \langle \delta_{tx}, \Delta'_{\mathcal{A}}(e_h)(\delta_{x^{-1}s})\rangle. \end{aligned}$$

If $g, k \in G$ are such that $gk \notin H$ then $g = tx$ and $k = y^{-1}s$ for some $s, t \in H$ and distinct $x, y \in X$. Then, for $h \in H$, we have that $gk \neq h$, so that

$$\langle \delta_g, \Delta'_{\mathcal{A}}(e_h)(\delta_k)\rangle = \Omega(g,k)\delta_{gk,h} = 0 = \langle \delta_g, \psi(\Delta'_{l^1(H,\omega)}(e_h))(\delta_k)\rangle.$$

Hence $\psi \circ \Delta'_{l^1(H,\omega)}$ is equal to $\Delta'_{\mathcal{A}}$ restricted to $l^1(H,\omega)$.

Let $P : c_0(G) \to c_0(H)$ be the natural projection, which is obviously an $l^1(H,\omega)$-module homomorphism. Then $Q = P \circ \phi \circ \psi : \mathcal{B}(l^1(H,\omega), c_0(H)) \to c_0(H)$ is a bounded left $l^1(H,\omega)$-module homomorphism, and $Q \circ \Delta'_{l^1(H,\omega)} = I_{c_o(H)}$. This contradiction completes the proof. $\square$

We note that just because Ω is bounded below does not imply that ω is bounded, so that $l^1(G,\omega)$ is not necessarily isomorphic to $l^1(G)$, and hence we cannot simply apply the results of [4].

We have not been able to establish if $c_0(S)$ can every be a left-injective $l^1(S,\omega)$-module for some semigroup S and weight ω.

6 Open questions

We state a few open questions of interest:

1. Let $\mathcal{A}$ be an Arens regular Banach algebra such that $\mathcal{A}''$ is Connes-amenable. Need $\mathcal{A}$ be amenable?
2. This is true for C*-algebras. Can we find a "simple" proof?
3. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$, and suppose that $\mathcal{A}_*$ is bi-injective. If $\mathcal{A}$ necessarily a von Neumann algebra or the bidual of an Arens regular Banach algebra $\mathcal{B}$ such that $\mathcal{B}$ is an ideal in $\mathcal{A}$?
4. Let S be a (weakly cancellative) semigroup, and let ω be a weight on S. Classify (up to isomorphism) the preduals of $l^1(S,\omega)$, and calculate which preduals yield a Connes-amenable Banach algebra.
5. This question was asked by Niels Grønbæk. In most of our examples, it is obvious that when $\mathcal{A}$ is a Connes-amenable dual Banach algebra, there is $\mathcal{B} \subseteq \mathcal{A}$ which is weak*-dense and amenable. Is this always true?

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