Published article:

Daws, M (2010) *A Note on Operator biprojectivity of compact quantum groups.* Proceedings of the American Mathematical Society, 138 (4). 1349 - 1359. ISSN 0002-9939

http://dx.doi.org/10.1090/S0002-9939-09-10220-4

A Note on Operator Biprojectivity of Compact Quantum Groups

Matthew Daws

Abstract

Given a (reduced) locally compact quantum group A, we can consider the convolution algebra $L^1(A)$ (which can be identified as the predual of the von Neumann algebra form of A). It is conjectured that $L^1(A)$ is operator biprojective if and only if A is compact. The "only if" part always holds, and the "if" part holds for Kac algebras. We show that if the splitting morphism associated with $L^1(A)$ being biprojective can be chosen to be completely positive, or just contractive, then we already have a Kac algebra. We give another proof of the converse, indicating how modular properties of the Haar state seem to be important.

Keywords: Compact quantum group, Biprojective, Kac algebra, Modular automorphism group.

2000 *Mathematical Subject Classification:* 46L89, 46M10 (primary), 22D25, 46L07, 46L65, 47L25, 47L50, 81R15.

1 Introduction

A Banach algebra A is *biprojective* if the multiplication map $\Delta_* : A\widehat{\otimes}A \to A$ has a right inverse in the category of A-bimodule maps. This can be thought of as a "finiteness condition". In particular, the group algebra $L^1(G)$ is biprojective if and only if G is compact, see [8, Chapter IV, Theorem 5.13].

When dealing with more non-commutative (or "quantum") algebras (here we focus on $L^1(G)^* = L^\infty(G)$ when we suggest that the classical situation is commutative) there is a large amount of evidence that *operator spaces* form the correct category to work in. For example, if we consider the Fourier algebra $A(G)$, then $A(G)$ is *operator* biprojective if and only if G is discrete, [22]. When G is abelian, as $A(G) \cong L^1(\hat{G})$, and $\hat{G}$ is compact if and only if G is discrete, this result is in full agreement with what we might expect. By contrast, if we ask when $A(G)$ is biprojective, then, if G is discrete and almost abelian (contains a finite-index abelian subgroup) then $A(G)$ is biprojective. Conversely, if $A(G)$ is biprojective, then G is discrete, and either almost abelian, or is non-amenable yet does not contain $\mathbb{F}_2$, see [15].

In this note, we shall continue the study of when the convolution algebra of a (reduced) compact quantum group is operator biprojective. It was shown in [1, Theorem 4.12] that if the convolution algebra of a locally compact quantum group $\mathbb{G}$ is operator biprojective, then $\mathbb{G}$ is already compact. Conversely, if $\mathbb{G}$ is a compact Kac algebra, then $\mathbb{G}$ is operator biprojective. We shall show that if the right inverse to Δ_* can be chosen to be completely contractive, then $\mathbb{G}$ must already be a Kac algebra. We make some remarks on the general case. We indicate that the modular theory of the Haar state seems to be important outside of the Kac case, and it seems likely that a better understanding of how to deal with how the coproduct iteracts with the modular automorphism group will be necessary to completely characterise when the convolution algebra of $\mathbb{G}$ is operator biprojective.

We shall follow the notation of [5], and in particular, write $\widehat{\otimes}$ for the operator space projective tensor product, and write $\mathcal{CB}(E,F)$ to denote the space of complete bounded linear maps between operator spaces E and F.

2 Locally compact quantum groups

Locally compact quantum groups [9, 11] are an axiomatic framework which encompass the $L^1(G)$ algebras, the Fourier algebra $A(G)$, and various "quantum" examples, for example, Woronowicz's compact quantum groups. Kac algebras [6] are an earlier axiomatic framework which fails to encompass many of the "quantum" examples, for example [25].

However, we shall concentrate on the compact case, which is technically easier. We shall follow the presentation of [20], which in turn closely follows Woronowicz's original papers [23] and [24]. See also readable, non-technical accounts in [11], and the survey [12], although be aware that these sources use different notation.

A compact quantum semigroup is a unital C*-algebra A equipped with a unital $*$-homomorphism $\Delta : A \to A \otimes_{\min} A$ such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. A compact quantum group is a compact quantum semigroup (A, Δ) which satisfies the *cancellation laws*, namely that

$$\Delta(A)(A \otimes 1) := \mathrm{lin}\{\Delta(a)(b \otimes 1) : a, b \in A\}, \quad \Delta(A)(1 \otimes A),$$

are both dense in $A \otimes_{\min} A$. If G is a compact semigroup, then we may set $A = C(G)$ and $\Delta(f)(s, t) = f(st)$ to get a compact quantum semigroup (A, Δ). Then the cancellation laws correspond to G having the cancellation laws: namely that if $st = sr$ for $s, t, r \in G$, then $t = r$, and similarly with the orders reversed. As sketched in [12], these are equivalent to G being a group.

From now on, fix a compact quantum group (A, Δ). These axioms imply that A carries a unique *Haar state*, that is, a state $\varphi \in A^*$ such that

$$(\varphi \otimes \iota)\Delta(a) = \varphi(a)1 = (\iota \otimes \varphi)\Delta(a) \qquad (a \in A).$$

We can form the GNS construction (H, Λ) for φ. We shall always suppose that (A, Δ) is *reduced*, that is, that φ is faithful. As such, we shall identify A with a concrete C*-algebra acting on H. If φ is not faithful, then we may quotient by its kernal $N = \{a \in A : \varphi(a^*a) = 0\}$ to obtain a reduced compact quantum group. Note that N is an ideal because φ is a KMS weight (see below), see the details in [3, Theorem 2.1].

Let $M = A''$ be the von Neumann algebra generated by A. Then Δ extends to a normal $*$-homomorphism $\Delta : M \to M \overline{\otimes} M$. Then, by [5, Theorem 7.2.4], $(M \overline{\otimes} M)_* = M_* \widehat{\otimes} M_*$ and normality of Δ induces a complete contraction $\Delta_* : M_* \widehat{\otimes} M_* \to M_*$. That Δ is coassociative implies that Δ_* is associative, so M_* becomes a completely contractive Banach algebra. If we started with a compact group G, then M_* is nothing but $L^1(G)$, and so we refer to M_* as the *convolution algebra* of (A, Δ). For more on (locally) compact quantum groups in the von Neumann algebra setting see [10].

A *finite-dimensional corepresentation* of (A, Δ) is a matrix $u = (u_{i,j}) \in \mathbb{M}_n(A)$ such that

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj} \qquad (1 \le i, j \le n).$$

There are suitable notions of *intertwiner* between corepresentations, and what an *irreducible* corepresentation is. Every finite-dimensional corepresentation can be written as the direct sum of irreducible corepresentations. Using the Haar state, it can be shown that every finite-dimensional corepresentation is equivalent to a *unitary* one, that is, where $u \in \mathbb{M}_n(A)$ is unitary. The general corepresentation theory of (A, Δ) parallels the representation theory of compact groups very closely.

Let $\{u^\alpha = (u^\alpha_{ij})_{i,j=1}^{n_\alpha} : \alpha \in \mathbb{A}\}$ be a maximal family of finite-dimensional irreducible unitary co-representations of (A, Δ). Let $\alpha_0 \in \mathbb{A}$ be such that $v^{\alpha_0} = 1$, the trivial corepresentation. Let $\mathcal{A}$ be the algebra generated by $\{u^\alpha_{ij} : \alpha \in \mathbb{A}, 1 \leq i, j \leq n_\alpha\}$ in A. Then $\mathcal{A}$ is a *Hopf $*$-algebra*, and $\{u^\alpha_{ij} : \alpha \in \mathbb{A}, 1 \leq i, j \leq n_\alpha\}$ forms a basis for $\mathcal{A}$. This means that $\mathcal{A}$ is a $*$-algebra, that Δ restricts to give a $*$-homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (the algebraic tensor product) and there exist maps $\epsilon : \mathcal{A} \to \mathbb{C}$ and $S : \mathcal{A} \to \mathcal{A}$, the *counit* and *antipode*, satisfying the usual properties. Indeed, for $\alpha \in \mathbb{A}$ and $1 \leq i, j \leq n_\alpha$, we have that

$$\Delta\big(u^\alpha_{i,j}\big) = \sum_{k=1}^{n_\alpha} u^\alpha_{i,k} \otimes u^\alpha_{k,j}, \quad S(u^\alpha_{i,j}) = \big(u^\alpha_{j,i}\big)^*, \quad \epsilon\big(u^\alpha_{i,j}\big) = \delta_{ij}, \quad \varphi\big(u^\alpha_{i,j}\big) = \delta_{\alpha,\alpha_0}.$$

Furthermore, for each $\alpha \in \mathbb{A}$, there exists a unique positive invertible matrix $F^\alpha \in \mathbb{M}_{n_\alpha}$ with $\operatorname{Tr} F^\alpha = \operatorname{Tr}(F^\alpha)^{-1}$, and such that

$$\varphi\big((u^\beta_{ij})^* u^\alpha_{kl}\big) = \delta_{\alpha\beta}\delta_{jl}\frac{((F^\alpha)^{-1})_{ki}}{\operatorname{Tr}(F^\alpha)}, \quad \varphi\big(u^\beta_{ij}(u^\alpha_{kl})^*\big) = \delta_{\alpha\beta}\delta_{ik}\frac{F^\alpha_{lj}}{\operatorname{Tr}(F^\alpha)}.$$

The Hopf $*$-algebra $\mathcal{A}$ is norm dense in A, and is the unique such dense Hopf $*$-algebra, see [3, Appendix A].

These "F-matricies" allow us to define characters on $\mathcal{A}$. For $z \in \mathbb{C}$, define

$$f_z : \mathcal{A} \to \mathbb{C}, \quad u^\alpha_{ij} \mapsto \big((F^\alpha)^z\big)_{ij}.$$

As F^α is positive, the matrix $(F^\alpha)^z$ makes sense. Then, for $w, z \in \mathbb{C}$, define

$$\rho_{z,w} : \mathcal{A} \to \mathcal{A}, \quad u^\alpha_{ij} \mapsto \sum_{k,l=1}^{n_\alpha} f_w(u^\alpha_{ik}) f_z(u^\alpha_{lj}) u^\alpha_{kl}.$$

Then $\rho_{z,w}$ is an automorphism of $\mathcal{A}$ with inverse $\rho_{-z,-w}$, and if z and w are purely imaginary, then $\rho_{z,w}$ is a $*$-automorphism of $\mathcal{A}$.

In particular, set

$$\sigma_z = \rho_{iz,iz}, \quad \tau_z = \rho_{-iz,iz} \qquad (z \in \mathbb{C}).$$

Then $(\sigma_t)_{t\in\mathbb{R}}$ is the (restriction) of the modular automorphism group for φ, and $(\tau_t)_{t\in\mathbb{R}}$ is the (restriction) of the scaling group. For example, we can calculate that $\varphi(a\sigma_{-i}(b)) = \varphi(ba)$ for $a, b \in \mathcal{A}$, a relation which we expect, as φ is KMS for σ. See [19] for more details on modular theory of weights.

Proposition 2.1. *There exists a maximal family of finite-dimensional irreducible unitary co-representations of (A, Δ), say $\{v^\alpha = (v^\alpha_{ij})_{i,j=1}^{n_\alpha} : \alpha \in \mathbb{A}\}$, with the property that the associated F-matricies are all diagonal, say F^α has diagonal entries $(\lambda^\alpha_i)_{i=1}^{n_\alpha}$, so that $\sum_i \lambda^\alpha_i = \sum_i (\lambda^\alpha_i)^{-1} = Tr_\alpha$, say.*

Proof. Start with some maximal family $\{u^\alpha = (u^\alpha_{ij})_{i,j=1}^{n_\alpha} : \alpha \in \mathbb{A}\}$ as before. As each F^α is positive it can be diagonalised by some unitary matrix $Q^\alpha \in \mathbb{M}_{n_\alpha}$. Let $(\lambda^\alpha_i)_{i=1}^{n_\alpha}$ be the eigenvalues of F^α, so that $\operatorname{Tr}(F^\alpha) = \sum_i \lambda^\alpha_i = \operatorname{Tr}((F^\alpha)^{-1}) = \sum_i (\lambda^\alpha_i)^{-1}$. Then $(Q^\alpha)^* F^\alpha Q^\alpha$ is the diagonal matrix with entries $(\lambda^\alpha_i)_{i=1}^{n_\alpha}$. Set

$$v^\alpha_{ij} = \big((Q^\alpha)^* u^\alpha Q^\alpha\big)_{ij} = \sum_{k,l=1}^{n_\alpha} \overline{Q^\alpha_{ki}} u^\alpha_{kl} Q^\alpha_{lj} \qquad (\alpha \in \mathbb{A}, 1 \leq i, j \leq n_\alpha).$$

It is now routine to check that v^α is a unitary corepresentation matrix, and that the properties above still hold for the family $\{v^\alpha_{ij}\}$. For example, we see that

$$\begin{aligned}
\varphi\big((v^\beta_{ij})^* v^\alpha_{kl}\big) &= \varphi\Big(\Big(\sum_{r,s} \overline{Q^\beta_{ri}} u^\beta_{rs} Q^\beta_{sj}\Big)^* \sum_{t,p} \overline{Q^\alpha_{tk}} u^\alpha_{tp} Q^\alpha_{pl}\Big) = \sum_{r,s,t,p} Q^\beta_{ri}\overline{Q^\beta_{sj}}\,\overline{Q^\alpha_{tk}} Q^\alpha_{pl}\varphi\big((u^\beta_{rs})^* u^\alpha_{tp}\big) \\
&= \delta_{\alpha\beta}\frac{1}{\mathrm{Tr}(F^\alpha)} \sum_{r,s,t} Q^\beta_{ri}\overline{Q^\beta_{sj}}\,\overline{Q^\alpha_{tk}} Q^\alpha_{sl}((F^\alpha)^{-1})_{tr} \\
&= \delta_{\alpha\beta}\frac{1}{\mathrm{Tr}_\alpha} \sum_s (Q^\alpha)^*_{js} Q^\alpha_{sl}\big((Q^\alpha)^*(F^\alpha)^{-1}Q^\alpha\big)_{ki} = \delta_{\alpha\beta}\delta_{jl}\delta_{ki}\frac{1}{\mathrm{Tr}_\alpha}\frac{1}{\lambda^\alpha_i}.
\end{aligned}$$

Similar calculations show that

$$\varphi\big(v^\beta_{ij}(v^\alpha_{kl})^*\big) = \delta_{\alpha\beta}\delta_{ik}\delta_{jl}\frac{\lambda^\alpha_j}{\mathrm{Tr}_\alpha}.$$

and also

$$f_z(v^\alpha_{ij}) = \delta_{ij}(\lambda^\alpha_i)^z, \qquad \rho_{z,w}(v^\alpha_{ij}) = (\lambda^\alpha_i)^w(\lambda^\alpha_j)^z v^\alpha_{ij}.$$

□

3 Biprojectivity

Let (A,Δ) be a reduced compact quantum group, with associated Haar state φ, GNS construction (H,Λ), von Neumann algebra M and convolution algebra M_*. We shall study when M_* is operator biprojective, that is, whether there is a completely bounded right inverse to $\Delta_* : M_*\widehat{\otimes}M_* \to M_*$ which is also an M_*-bimodule homomorphism. Henceforth, we shall term such a map θ_* a *splitting morphism*.

See [1, 2] for further details on the operator space case, and [8, Chapter IV] or [16, Section 4.3] for the classical Banach space setting.

Lemma 3.1. *M_* is biprojective if and only if there exists a normal completely bounded map $\theta : M\overline{\otimes}M \to M$ with*

$$\theta\Delta = \mathrm{id}, \quad \Delta\theta = (\theta\otimes\mathrm{id})(\mathrm{id}\otimes\Delta) = (\mathrm{id}\otimes\theta)(\Delta\otimes\mathrm{id}).$$

Proof. Suppose that such a θ exists, so as θ is normal, there exists $\theta_* : M_* \to M_*\widehat{\otimes}M_*$ with $\Delta_*\theta_* = \mathrm{id}$. Then, for $\omega,\tau\in M_*$ and $x\in M$,

$$\begin{aligned}
\langle x, \theta_*(\omega*\tau)\rangle &= \langle \theta(x), \Delta_*(\omega\otimes\tau)\rangle = \langle (\theta\otimes\mathrm{id})(\mathrm{id}\otimes\Delta)(x), \omega\otimes\tau\rangle \\
&= \langle (\mathrm{id}\otimes\Delta)(x), \theta_*(\omega)\otimes\tau\rangle = \langle x, \theta_*(\omega)*\tau\rangle.
\end{aligned}$$

Here we write $*$ for both the product in M_*, and the bimodule action of M_* on $M_*\widehat{\otimes}M_*$. Similarly, $\theta_*(\omega*\tau) = \omega*\theta_*(\tau)$, so we see that θ_* is a M_*-bimodule homomorphism.

The converse is simply a case of reversing the argument. □

In the following section, we shall carefully study the structure of normal completely bounded maps $M\overline{\otimes}M\to M$. From now on, fix such a map $\theta : M\overline{\otimes}M \to M$ and let $\{(v^\alpha_{ij})^{n_\alpha}_{i,j=1} : \alpha\in\mathbb{A}\}$ be as in Proposition 2.1.

Proposition 3.2. *We have that $\theta\Delta = \mathrm{id}$ and $\Delta\theta = (\theta\otimes\mathrm{id})(\mathrm{id}\otimes\Delta) = (\mathrm{id}\otimes\theta)(\Delta\otimes\mathrm{id})$ if and only if there exists a family $\{X^\alpha \in \mathbb{M}_{n_\alpha} : \alpha \in \mathbb{A}\}$ such that, for $\alpha,\beta\in\mathbb{A}$, $1\leq i,j\leq n_\alpha$ and $1\leq k,l\leq n_\beta$,*

$$\theta\big(v^\alpha_{ij}\otimes v^\beta_{kl}\big) = \delta_{\alpha\beta}X^\alpha_{jk}v^\alpha_{il}, \qquad \sum_{r=1}^{n_\alpha} X^\alpha_{rr} = 1.$$

Proof. The "if" part follows as $\mathcal{A}$ generates M and θ is normal.

Conversely, let $x\in M$ and $\alpha\in\mathbb{A}$. For $1\leq i,j\leq n_\alpha$,

$$\Delta\theta\big(x\otimes v^\alpha_{ij}\big) = (\theta\otimes\mathrm{id})(\mathrm{id}\otimes\Delta)\big(x\otimes v^\alpha_{ij}\big) = \sum_{r=1}^{n_\alpha}\theta\big(x\otimes v^\alpha_{ir}\big)\otimes v^\alpha_{rj}.$$

Let $a_{ij} = \theta(x\otimes v^\alpha_{ij})$, so that $\Delta(a_{ij}) = \sum_r a_{ir}\otimes v^\alpha_{rj}$. As Δ is a $*$-homomorphism, for $1\leq k,l\leq n_\alpha$, we have that

$$\Delta\big(a_{ij}(v^\alpha_{kl})^*\big) = \sum_{r,s=1}^{n_\alpha} a_{ir}(v^\alpha_{ks})^*\otimes v^\alpha_{rj}(v^\alpha_{sl})^*.$$

Applying $(\iota\otimes\varphi)$, we see that, by the calculations in Proposition 2.1,

$$\varphi\big(a_{ij}(v^\alpha_{kl})^*\big)1 = \sum_{r,s=1}^{n_\alpha} a_{ir}(v^\alpha_{ks})^*\varphi\big(v^\alpha_{rj}(v^\alpha_{sl})^*\big) = \delta_{jl}\frac{\lambda^\alpha_j}{\mathrm{Tr}_\alpha}\sum_{r=1}^{n_\alpha} a_{ir}(v^\alpha_{kr})^*.$$

As v^α is a unitary matrix, we see that $1 = \sum_k (v^\alpha_{kr})^* v^\alpha_{ks} = \delta_{rs}1$ for $1\leq r,s\leq n_\alpha$. Thus

$$\sum_{k=1}^{n_\alpha}\varphi\big(a_{ij}(v^\alpha_{kl})^*\big)v^\alpha_{ks} = \delta_{jl}\frac{\lambda^\alpha_j}{\mathrm{Tr}_\alpha}\sum_{r,k=1}^{n_\alpha} a_{ir}(v^\alpha_{kr})^*v^\alpha_{ks} = \delta_{jl}\frac{\lambda^\alpha_j}{\mathrm{Tr}_\alpha}a_{is}.$$

It follows that

$$a_{is} = \frac{\mathrm{Tr}_\alpha}{\lambda^\alpha_j}\sum_{k=1}^{n_\alpha}\varphi\big(a_{ij}(v^\alpha_{kj})^*\big)v^\alpha_{ks} \qquad \big(\alpha\in\mathbb{A}, 1\leq i,j,s\leq n_\alpha\big).$$

Similarly, if we set $b_{ij} = \theta(v^\alpha_{ij}\otimes x)$, then $\Delta(b_{ij}) = \sum_r v^\alpha_{ir}\otimes b_{rj}$, and we can show that

$$b_{sj} = \lambda^\alpha_i\,\mathrm{Tr}_\alpha\sum_{k=1}^{n_\alpha}\varphi\big((v^\alpha_{ik})^*b_{ij}\big)v^\alpha_{sk} \qquad \big(\alpha\in\mathbb{A}, 1\leq i,j,s\leq n_\alpha\big).$$

In particular, we see that $\theta(v^\alpha_{ij}\otimes v^\beta_{kl})$ is in the linear span of $\{v^\alpha_{is} : 1\leq s\leq n_\alpha\}$, and the linear span of $\{v^\beta_{rl} : 1\leq r\leq n_\beta\}$. Hence $\theta(v^\alpha_{ij}\otimes v^\beta_{kl}) = 0$ if $\alpha\neq\beta$. If $\alpha=\beta$, then by linear independence, we see immediately that

$$\theta(v^\alpha_{ij}\otimes v^\alpha_{kl}) = X^\alpha_{jk}v^\alpha_{il},$$

for some scalar X^α_{jk}. Finally, as $\sum_k\theta(v^\alpha_{ik}\otimes v^\alpha_{kj}) = v^\alpha_{ij}$, it follows $\sum_k X^\alpha_{kk} = 1$, as required. ☐

Theorem 3.3. *Let (A,Δ) be a compact quantum group with associated von Neumann algebra M. Let $\theta_*: M_*\to M_*\widehat{\otimes}M_*$ be a splitting morphism, and suppose further that $\theta = \theta_*^*$ is an M-bimodule map, in the sense that $\theta(\Delta(a)x\Delta(b)) = a\theta(x)b$ for $x\in M\overline{\otimes}M$ and $a,b\in M$. Then the Haar state φ is tracial, so (M,Δ) is a Kac algebra.*

Proof. Let $\alpha \in \mathbb{A}$ and $1 \leq i, j, k \leq n_\alpha$. As $\theta(x\Delta(b)) = \theta(x)b$ for $x \in M\overline{\otimes}M$ and $b \in M$, using the notation of the last proposition, we see that

$$X^\alpha_{jk} v^\alpha_{ij}(v^\alpha_{ij})^* = \theta\big(v^\alpha_{ij} \otimes v^\alpha_{kj}\big)(v^\alpha_{ij})^* = \sum_{l=1}^{n_\alpha} \theta\big(v^\alpha_{ij}(v^\alpha_{il})^* \otimes v^\alpha_{kj}(v^\alpha_{lj})^*\big). \tag{1}$$

Now, as $\{v^\beta_{rs}\}$ forms a basis for the $*$-algebra $\mathcal{A}$, and as φ picks out the trivial corepresentation $v^{\alpha_0} = 1$, by the calculations of Proposition 2.1, we see that

$$v^\alpha_{ij}(v^\alpha_{il})^* \otimes v^\alpha_{kj}(v^\alpha_{lj})^* = \delta_{jl}\frac{\lambda^\alpha_j}{\mathrm{Tr}_\alpha} 1 \otimes \delta_{kl}\frac{\lambda^\alpha_j}{\mathrm{Tr}_\alpha} 1 + \text{other terms.}$$

By the structure of θ established in the last proposition, it follows that

$$\sum_{l=1}^{n_\alpha} \varphi\theta\big(v^\alpha_{ij}(v^\alpha_{il})^* \otimes v^\alpha_{kj}(v^\alpha_{lj})^*\big) = \delta_{jk}\Big(\frac{\lambda^\alpha_j}{\mathrm{Tr}_\alpha}\Big)^2 1 + \text{other terms.}$$

By applying φ to (1), we conclude that

$$X^\alpha_{jk}\frac{\lambda^\alpha_j}{\mathrm{Tr}_\alpha} = \delta_{jk}\Big(\frac{\lambda^\alpha_j}{\mathrm{Tr}_\alpha}\Big)^2 \quad \text{so that} \quad X^\alpha_{jk} = \delta_{jk}\frac{\lambda^\alpha_j}{\mathrm{Tr}_\alpha}.$$

We now repeat this argument on the right, so we find that

$$\begin{aligned} X^\alpha_{jk}(v^\alpha_{ij})^* v^\alpha_{ij} = (v^\alpha_{ij})^*\theta\big(v^\alpha_{ij} \otimes v^\alpha_{kj}\big) &= \sum_s \theta\big((v^\alpha_{is})^* v^\alpha_{ij} \otimes (v^\alpha_{sj})^* v^\alpha_{kj}\big) \\ &= \sum_s \delta_{sj}\frac{1}{\lambda^\alpha_i \,\mathrm{Tr}_\alpha}\delta_{sk}\frac{1}{\lambda^\alpha_k \,\mathrm{Tr}_\alpha} 1 + \text{other terms} \end{aligned}$$

Again, by applying φ we see that

$$X^\alpha_{jk}\frac{1}{\lambda^\alpha_i \,\mathrm{Tr}_\alpha} = \delta_{jk}\frac{1}{\lambda^\alpha_i \,\mathrm{Tr}_\alpha}\frac{1}{\lambda^\alpha_k \,\mathrm{Tr}_\alpha} \quad \text{so that} \quad X^\alpha_{jk} = \delta_{jk}\frac{1}{\lambda^\alpha_k \,\mathrm{Tr}_\alpha}.$$

We hence see that for all α and $1 \leq k \leq n_\alpha$, we have $\lambda^\alpha_k = 1/\lambda^\alpha_k$. As $\lambda^\alpha_k > 0$, we see that $\lambda^\alpha_k = 1$. In particular, the modular automorphism group σ is trivial, and so φ is tracial, as claimed.

Indeed, if φ is tracial, then from Proposition 2.1, we see that $\lambda^\alpha_j = (\lambda^\alpha_i)^{-1}$ for all i, j. Thus $\lambda^\alpha_i = 1$ for all i and α. It follows that the automorphism $\rho_{z,w}$ are trivial, and hence also the scaling group is trivial. So the antipode S is bounded. It is now easy to verify the axioms of a compact Kac algebra, see [6, Section 6.2]. ☐

We note that an argument of Soltan, [17, Remark A.2], shows that if a compact quantum group (A, Δ) has a faithful family of tracial states (that is, for non-zero $x \in A$ there is a tracial state ϕ with $\phi(x^*x) \neq 0$) then (M, Δ) is a Kac algebra.

Theorem 3.4. *Let (A, Δ) be a compact quantum group with associated von Neumann algebra M. Let $\theta_* : M_* \to M_*\widehat{\otimes}M_*$ be a splitting morphism. Suppose that $\theta = \theta_*^*$ is completely positive, or that $\Delta\theta$ is a contraction. Then (M, Δ) is a Kac algebra.*

Proof. As $\theta(1) = \theta\Delta(1) = 1$, if θ is positive, then θ is contractive, so $\Delta\theta$ is contractive.

We have that $\Delta\theta : M\overline{\otimes}M \to M\overline{\otimes}M$ is contractive, and is a projection of $M\overline{\otimes}M$ onto the subalgebra $\Delta(M)$. A result of Tomiyama, [21] or [18, Theorem 3.4, Chapter III], tells us that, in particular, $\Delta\theta(\Delta(a)x\Delta(b)) = \Delta(a)\theta(x)\Delta(b)$ for $a, b \in M$ and $x \in M\overline{\otimes}M$. As Δ is an injective homomorpshim, the above theorem applies. ☐

In the following section, we shall show the converse to this corollary: namely that for a compact Kac algebra (M, Δ), we can choose θ to be a complete contraction; alternatively, see [14] or [1].

It is shown in [4] that if we have a completely bounded map $\theta : M\overline{\otimes}M \to M$ with $\theta\Delta = \mathrm{id}$ then there exists a completely bounded map $\theta_1 : M\overline{\otimes}M \to M$ which is an M-bimodule map, in the above sense. However, there is no reason that θ_1 need be normal, and no reason that the other conditions on θ will carry over to θ_1, so that Proposition 3.2 need not apply to θ_1. We can even choose θ_1 to be completely positive, which were it also *faithful* would imply, by [19, Theorem 4.2, Chapter IX], the existence of a weight ω on $M\overline{\otimes}M$ with interesting modular properties. Again, there seems to be no reason to expect that we can choose θ_1 in such a way.

4 Completely bounded maps

There is a well-known structure theory for completely bounded maps, [5, Section 5.3]. If $\theta : N \to (M, H)$ is a completely positive normal map between von Neumann algebras, then the usual proof of the Stinespring theorem (for example, [18, Chapter IV, Theorem 3.6]) can be adapted to show that there exists a Hilbert space K, a *normal* $*$-homomorphism $\pi : N \to \mathcal{B}(K)$ and a bounded map $U : H \to K$ such that $\theta(x) = U^*\pi(x)U$ for $x \in N$.

Showing the same for completely bounded maps is not quite as simple, but the details are worked out in, for example, the proof of [7, Theorem 2.4]. In particular, given $\theta : N \to (M, H)$ a completely contractive normal map between von Neumann algebras, there exist unital completely positive *normal* maps $\phi_1, \phi_2 : N \to M$ such that

$$\sigma : \mathbb{M}_2(N) \to \mathbb{M}_2(M); \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \phi_1(a) & \theta(b^*)^* \\ \theta(c) & \phi_2(d) \end{pmatrix}$$

is unital completely positive and normal. One can now follow the presentation in [5, Theorem 5.33] or [13], essentially applying the Stinespring construction to σ. This yields a Hilbert space K, a normal $*$-homomorphism $\rho : \mathbb{M}_2(N) \to \mathcal{B}(K)$ and an isometry $U : H^2 \to K$ such that $\sigma(x) = U^*\rho(x)U$ for $x \in \mathbb{M}_2(N)$. Following the proof of [7, Theorem 2.3], there also exists a normal $*$-homomorphism $\rho' : \mathbb{M}_2(M)' \to \rho(\mathbb{M}_2(N))'$ such that $\rho'(y)U = Uy$ for $y \in \mathbb{M}_2(M)'$.

Define $\pi : M\overline{\otimes}M \to \mathcal{B}(K)$, $\pi' : M' \to \mathcal{B}(K)$ and $S, T : H \to K$ by

$$\pi(x) = \rho\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad \pi'(y) = \rho'\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \quad T(\xi) = \rho\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} U \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \quad S(\xi) = U\begin{pmatrix} 0 \\ \xi \end{pmatrix},$$

for $x \in M\overline{\otimes}M, y \in M'$ and $\xi \in H$. So π and π' are normal $*$-homomorphisms and S and T are contractions. Then, for $x \in M\overline{\otimes}M$ and $\xi, \eta \in H$,

$$\begin{aligned} \big(S^*\pi(x)T\xi \big| \eta\big) &= \left(\rho\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \rho\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} U \begin{pmatrix} \xi \\ 0 \end{pmatrix} \middle| U\begin{pmatrix} 0 \\ \eta \end{pmatrix}\right) \\ &= \left(\sigma\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ \eta \end{pmatrix}\right) = \big(\theta(x)\xi \big| \eta\big). \end{aligned}$$

So $\theta(x) = S^*\pi(x)T$. Then also, for $y \in M'$ and $\xi \in H$,

$$Ty\xi = \rho\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} U \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \rho\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \pi'(y) U \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \pi'(y)T\xi.$$

So $Ty = \pi'(y)T$ and similarly $Sy = \pi'(y)S$, for $y \in M'$.

Let M be a von Neumann algebra with a normal faithful state φ, leading to GNS construction (H,Λ) (here we identify M with a subalgebra of $\mathcal{B}(H)$). We can apply Tomita-Takesaki theory to find an anti-linear isometry $J : H \rightarrow H$ such that $M' = JMJ$ (see [19]). Let $(\sigma_t)_{t\in\mathbb{R}}$ be the modular automorphism group, and let $\mathcal{A} \subseteq M$ be a $*$-subalgebra of elements analytic for (σ_t) such that $\sigma_z(a) \in \mathcal{A}$ for $z \in \mathbb{C}$ and $a \in \mathcal{A}$. For $a \in \mathcal{A}$, write $a' = J\sigma_{i/2}(a)^* J$. Then

$$a'\Lambda(1) = J\sigma_{i/2}(a)^* J\Lambda(1) = \Lambda(a) \qquad (a \in \mathcal{A}).$$

Proposition 4.1. *Let M be a von Neumann algebra as above, and suppose that $\mathcal{A}'' = M$. Let N be a von Neumann algebra. If $\theta : N \rightarrow M$ is completely bounded normal map, then we can find a Hilbert space K, normal $*$-homomorphisms $\pi : N \rightarrow \mathcal{B}(K)$ and $\pi' : M' \rightarrow \pi(N)'$, and $\xi_0, \xi_1 \in K$ such that the maps*

$$\Lambda(a) \mapsto \pi'(a')\xi_0, \qquad \Lambda(a) \mapsto \pi'(a')\xi_1 \qquad (a \in \mathcal{A}),$$

are bounded, and

$$\varphi(\theta(x)a) = \big(\pi(x)\pi'(a')\xi_0 \big| \xi_1\big) \qquad (x \in N, a \in \mathcal{A}). \tag{2}$$

Conversely, given such K, π, π', ξ_0 and ξ_1, there exists a completely bounded normal map $\theta : N \rightarrow M$ satisfying (2).

Furthermore, θ is completely positive if and only if we can choose $\xi_0 = \xi_1$.

Proof. As $\mathcal{A}'' = M$, it follows that $\mathcal{A}$ is strongly dense in M and hence that $\Lambda(\mathcal{A})$ is norm dense in H. If θ is of the form claimed, then the map $T : \Lambda(\mathcal{A}) \rightarrow K; \Lambda(a) \mapsto \pi'(a')\xi_0$ is bounded and so extends to a bounded linear map $T : H \rightarrow K$. Similarly, there exists $S \in \mathcal{B}(H,K)$ with $S\Lambda(a) = \pi'(a')\xi_1$. Then, for $a, b \in \mathcal{A}$ and $x \in N$,

$$\begin{aligned}\big(S^*\pi(x)T\Lambda(a) \big| \Lambda(b)\big) &= \big(\pi(x)\pi'(a')\xi_0 \big| \pi'(b')\xi_1\big) = \big(\pi(x)\pi'((b')^* a')\xi_0 \big| \xi_1\big) \\ &= \varphi\big(\theta(x)a\sigma_{-i}(b^*)\big),\end{aligned}$$

as $(b')^* = (J\sigma_{i/2}(b)^* J)^* = J\sigma_{i/2}(b)J = J\sigma_{i/2}(c)^* J = c'$ if $c = \sigma_{-i}(b^*)$, and $d \mapsto d'$ is an anti-homomorphism. By the KMS condition, we see that

$$\big(S^*\pi(x)T\Lambda(a) \big| \Lambda(b)\big) = \varphi\big(b^*\theta(x)a\big) = \big(\theta(x)\Lambda(a) \big| \Lambda(b)\big).$$

Hence θ is completely bounded, as $\theta(x) = S^*\pi(x)T$ for $x \in N$. If $\xi_0 = \xi_1$ then $S = T$ and θ is completely positive.

Conversely, given θ, from the discussion above, we can find normal $*$-homomorphisms $\pi : N \rightarrow \mathcal{B}(K)$ and $\pi' : M' \rightarrow \pi(N)'$, and bounded maps $S, T : H \rightarrow K$ with $\theta(x) = S^*\pi(x)T$ for $x \in N$ and $Sy = \pi'(y)S, Ty = \pi'(y)T$ for $y \in M'$. Thus, for $x \in N$ and $a \in \mathcal{A}$,

$$\varphi(\theta(x)a) = \big(S^*\pi(x)T\Lambda(a) \big| \Lambda(1)\big) = \big(S^*\pi(x)\pi'(a')T\Lambda(1) \big| \Lambda(1)\big),$$

so the proof is complete by setting $\xi_0 = T\Lambda(1)$ and $\xi_1 = S\Lambda(1)$. If θ is completely positive, then we can set $S = T$ and hence $\xi_0 = \xi_1$. ∎

Notice that by the KMS condition, the calculations above also show that if $x, y \in M$ are such that $\varphi(xa) = \varphi(ya)$ for all $a \in \mathcal{A}$, then $x = y$.

The following is proved using different methods in [14] and [1]. Our proof makes explicit how φ being tracial, for a Kac algebra, is central to the proof, and indicates that understanding the modular properties of φ for a general compact quantum group will be important in finding a completely bounded analogue of the following.

Theorem 4.2. *Let (M,Δ) be a compact Kac algebra. Then there exists a splitting morphism $\theta_*: M_* \to M_* \widehat{\otimes} M_*$ such that $\theta = \theta_*^*$ is completely positive.*

Proof. We have that φ is tracial. Let $\pi: M\overline{\otimes} M \to M\overline{\otimes} M \subseteq \mathcal{B}(H\otimes H)$ be the trivial representation, let $\xi_0 = \xi_1 = \Lambda(1)\otimes\Lambda(1)$, and define π' by

$$\pi'(y) = (J\otimes J)\Delta(JyJ)(J\otimes J) \qquad (y\in M').$$

This formula is derived from the natural coproduct on M', see [10, Section 4]. Let $\mathcal{A}$ be the Hopf $*$-algebra associated to (M,Δ), as before. Then we can apply the above proposition to see that there exists a completely positive normal map $\theta: M\overline{\otimes} M \to M$ such that

$$\varphi(\theta(x)a) = \big(x(J\otimes J)\Delta(a^*)(J\otimes J)\Lambda(1)\otimes\Lambda(1)\big|\Lambda(1)\otimes\Lambda(1)\big),$$

where we use that σ is trivial, as φ is tracial. Then $J\Lambda(a) = \Lambda(a)^*$ for $a\in\mathcal{A}$, and so, as $\Delta(a^*)\in\mathcal{A}\otimes\mathcal{A}$,

$$\begin{aligned}\varphi(\theta(x)a) &= \big(x(J\otimes J)\Delta(a^*)\Lambda(1)\otimes\Lambda(1)\big|\Lambda(1)\otimes\Lambda(1)\big)\\ &= \big(x(\Lambda\otimes\Lambda)\Delta(a)\big|\Lambda(1)\otimes\Lambda(1)\big) = (\varphi\otimes\varphi)\big(x\Delta(a)\big).\end{aligned}$$

In particular,

$$\varphi(\theta\Delta(x)a) = (\varphi\otimes\varphi)\big(\Delta(xa)\big) = \varphi(xa),$$

so by the observation above, $\theta\Delta = \mathrm{id}$. Indeed, one may calculate (thinking about Proposition 3.2) that

$$\theta(v^\alpha_{ij}\otimes v^\alpha_{kl}) = \frac{1}{n_\alpha}\delta_{jk}v^\alpha_{il},$$

using that $\lambda^\alpha_i = 1$ for all α and i. Thus also $\Delta\theta = (\theta\otimes\mathrm{id})(\mathrm{id}\otimes\Delta) = (\mathrm{id}\otimes\theta)(\Delta\otimes\mathrm{id})$, and so θ_*, the preadjoint to θ, is a splitting morphism, as required. $\square$

If φ is not tracial, then the above proof fails, as for $a\in\mathcal{A}$,

$$\Delta(\sigma_{i/2}(a)^*) = \big((\tau_{i/2}\otimes\sigma_{i/2})\Delta(a)\big)^*,$$

and hence, as $J\Lambda(b) = \Lambda(\sigma_{i/2}(b)^*)$ for $b\in\mathcal{A}$,

$$(J\otimes J)\Delta(\sigma_{i/2}(a)^*)(J\otimes J)(\Lambda(1)\otimes\Lambda(1)) = (\Lambda\otimes\Lambda)\big((\tau_{i/2}\sigma_{-i/2}\otimes\mathrm{id})\Delta(a)\big).$$

If we continus to form θ as above, then we find that

$$\theta\big(v^\alpha_{ij}\otimes v^\beta_{kl}\big) = \delta_{\alpha\beta}v^\alpha_{il}\frac{\delta_{jk}}{\mathrm{Tr}_\alpha} \qquad (\alpha,\beta\in\mathbb{A}, 1\le i,j\le n_\alpha, 1\le k,l\le n_\beta).$$

This is nearly of the correct form, but we find that

$$\theta\Delta\big(v^\alpha_{ij}\big) = \frac{n_\alpha}{\mathrm{Tr}_\alpha}v^\alpha_{ij} \qquad (\alpha\in\mathbb{A}, 1\le i,j\le n_\alpha).$$

Notice that $n_\alpha = \sum_i(\lambda^\alpha_i)^{1/2}(\lambda^\alpha_i)^{-1/2} \le \big(\sum_i\lambda^\alpha\big)^{1/2}\big(\sum_i(\lambda^\alpha)^{-1}\big)^{1/2} = \mathrm{Tr}_\alpha$, it follows that $\theta\Delta = \mathrm{id}$ if and only if $\lambda^\alpha_i = 1$ for all α, i, that is, again, φ is tracial.

References

[1] O. Yu. Aristov, *Amenability and compact type for Hopf-von Neumann algebras from the homological point of view.* In *Banach algebras and their applications*, Contemp. Math. **363**, 15–37. Amer. Math. Soc., Providence, RI, 2004.

[2] O. Yu. Aristov, *Biprojective algebras and operator spaces*, J. Math. Sci. (New York) **111** (2002), 3339–3386.

[3] E. Bédos, G. J. Murphy, and L. Tuset, *Co-amenability of compact quantum groups*, J. Geom. Phys. **402** (2001), 130–153.

[4] E. Christensen and A. M. Sinclair, *Module mappings into von Neumann algebras and injectivity*, Proc. London Math. Soc. **71** (1995), 618–640.

[5] E. G. Effros and Z.-J. Ruan, *Operator spaces*, London Math. Society Monographs, New Series **23**. Oxford University Press, New York, 2000.

[6] M. Enock and J.-M. Schwartz, *Kac algebras and duality of locally compact groups*, Springer-Verlag, Berlin, 1992.

[7] U. Haagerup and M. Musat, *Classification of hyperfinite factors up to completely bounded isomorphism of their preduals*, preprint, arXiv:0706.3463 [math.OA]

[8] A. Ya. Helemskii, *The homology of Banach and topological algebras*, Translated from the Russian by Alan West. Mathematics and its Applications (Soviet Series), **41**. Kluwer Academic Publishers Group, Dordrecht, 1989.

[9] J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Sci. École Norm. Sup. **336** (2000), 837–934.

[10] J. Kustermans and S. Vaes, *Locally compact quantum groups in the von Neumann algebraic setting*, Math. Scand. **92** (2003), 68–92.

[11] J. Kustermans, *Locally compact quantum groups*, In *Quantum independent increment processes. I* volume 1865 of *Lecture Notes in Math.* pages 99–180 (Springer, Berlin, 2005).

[12] A. Maes and A. Van Daele, *Notes on compact quantum groups*, Nieuw Arch. Wisk. **16** (1998), 73–112.

[13] V. I. Paulsen, *Completely bounded maps and dilations*, Pitman Research Notes in Mathematics Series, **146**. John Wiley & Sons, Inc., New York, 1986.

[14] Z.-J. Ruan and G. Xu, *Splitting properties of operator bimodules and operator amenability of Kac algebras.* In *Operator theory, operator algebras and related topics* (Timisoara, 1996), 193–216, Theta Found., Bucharest, 1997.

[15] V. Runde, *Biflatness and biprojectivity of the Fourier algebra*, preprint, arXiv:0808.1146 [math.FA]

[16] V. Runde, *Lectures on Amenability*, Springer-Verlag, Berlin, 2002.

[17] P. M. Sołtan, *Quantum Bohr compactification*, Illinois J. Math. **49** (2005), 1245–1270.

[18] M. Takesaki, *Theory of operator algebras. I.*, Reprint of the first (1979) edition. Encyclopaedia of Mathematical Sciences, **124**. Operator Algebras and Non-commutative Geometry, **5**. Springer-Verlag, Berlin, 2002.

[19] M. Takesaki, *Theory of operator algebras. II.*, Encyclopaedia of Mathematical Sciences, **125**. Operator Algebras and Non-commutative Geometry, **6**. Springer-Verlag, Berlin, 2003.

[20] T. Timmermann, *An invitation to quantum groups and duality. From Hopf algebras to multiplicative unitaries and beyond.* European Mathematical Society (EMS), Zürich, 2008.

[21] J. Tomiyama, *On the projection of norm one in W^*-algebras*, Proc. Japan Acad. **33** (1957), 608–612.

[22] P. J. Wood, *The operator biprojectivity of the Fourier algebra*, Canad. J. Math. **545** (2002), 1100–1120.

[23] S. L. Woronowicz, *Compact quantum groups*. In *Symétries quantiques (Les Houches, 1995)*, 845–884. North-Holland, Amsterdam, 1998

[24] S. L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. **1114** (1987), 613–665.

[25] S. L. Woronowicz, *Twisted* SU(2) *group. An example of a noncommutative differential calculus*, Publ. Res. Inst. Math. Sci. **231** (1987), 117–181.

Author's address: School of Mathematics,
University of Leeds,
Leeds LS2 9JT
United Kingdom

Email: matt.daws@cantab.net