COMPLETELY POSITIVE DEFINITE FUNCTIONS AND BOCHNER'S THEOREM FOR LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. We prove two versions of Bochner's theorem for locally compact quantum groups. First, every completely positive definite "function" on a locally compact quantum group $\mathbb{G}$ arises as a transform of a positive functional on the universal C*-algebra $C_0^u(\hat{\mathbb{G}})$ of the dual quantum group. Second, when $\mathbb{G}$ is coamenable, complete positive definiteness may be replaced with the weaker notion of positive definiteness, which models the classical notion. A counterexample is given to show that the latter result is not true in general. To prove these results, we show two auxiliary results of independent interest: products are linearly dense in $L^1_\sharp(\mathbb{G})$, and when $\mathbb{G}$ is coamenable, the Banach $*$-algebra $L^1_\sharp(\mathbb{G})$ has a contractive bounded approximate identity.

Keywords: Quantum group, positive definite function, Bochner's Therorem.

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1. INTRODUCTION

Bochner's theorem (as generalised by Weil) tells us that any positive definite function on a locally compact abelian group G is the Fourier–Stieltjes transform of a positive measure on the dual group $\hat{G}$. In non-abelian harmonic analysis, we can replace the algebra $C_0(\hat{G})$ by the group C*-algebra $C^*(G)$, and hence replace positive measures on $\hat{G}$ by positive functionals on $C^*(G)$. Viewing $C^*(G)^*$ as $B(G)$, the Fourier–Stieltjes algebra, Bochner's theorem essentially says that positive definitive functions are precisely the positive elements of $B(G)$ (this viewpoint is taken in [14, Définition 2.2]).

For a locally compact quantum group $\mathbb{G}$, we replace functions on groups by elements of von Neumann (or C*-) algebras, which come equipped with extra structure reminiscent of an algebra which really arises from a group. Given $\mathbb{G}$, we can form the universal dual algebra $C_0^u(\hat{\mathbb{G}})$, which generalises the passage from G to the full group C*-algebra $C^*(G)$ (see the next section for further details on $C_0^u(\hat{\mathbb{G}})$ and so forth). Letting $\mathcal{W} \in M(C_0(\mathbb{G}) \otimes C_0^u(\hat{\mathbb{G}}))$ be the maximal unitary corepresentation of $\mathbb{G}$, we have an algebra homomorphism $C_0^u(\hat{\mathbb{G}})^* \to M(C_0(\mathbb{G})) \subseteq L^\infty(\mathbb{G}); \hat{\mu} \mapsto (\mathrm{id} \otimes \hat{\mu})(\mathcal{W}^*)$. In the commutative case, the image is precisely the Fourier–Stieltjes algebra (we remark that it is slightly a matter of convention if one uses $\mathcal{W}$ or $\mathcal{W}^*$ here). Motivated by this, there are perhaps two obvious notions for what a "positive definite" element of $L^\infty(\mathbb{G})$ should be; here we introduce some of our own terminology:

(1) A *positive definite function* is $x \in L^\infty(\mathbb{G})$ with $\langle x^*, \omega \star \omega^\sharp \rangle \geq 0$ for all $\omega \in L^1_\sharp(\mathbb{G})$.

(2) A *Fourier–Stieltjes transform of a positive measure* is $x \in L^\infty(\mathbb{G})$ such that there exists $\hat{\mu} \in C_0^u(\hat{\mathbb{G}})^*_+$ with $x = (\mathrm{id} \otimes \hat{\mu})(\mathcal{W}^*)$.

It seems that definition (2) is a better fit with the current literature, although the term "positive definite function" is not commonly used in this context (for examples where positive functionals in $\hat{\mu} \in C_0^u(\hat{\mathbb{G}})^*_+$, or their transforms, are used in place of positive definite functions (from the classical case), see [17] which studies Markov operators, [4, Section 4] and [3] which study various approximation properties for von Neumann algebras over quantum groups, or [24] and [25] which study property (T) for quantum groups; the latter reference actually uses the term "positive definite function" in an offhand way). Definition (1) is the most natural as it directly generalises the notion of a positive definite function on $L^1(G)$, see for example [11, Theorem 13.4.5]. This definition is rather briefly studied for Kac algebras in [13, Section 1.3]; however, it is mainly (2), in various guises, which is used in [13]. Indeed, we show in Example 17 below that even in the cocommutative case, definition (1) is problematic without some sort of amenability assumption – to be precise, that $\mathbb{G}$ is coamenable. Even when $\mathbb{G}$ is coamenable, we are required to deal with the unbounded antipode S, and our techniques are necessarily different from those used for Kac algebras. We remark that it is easy to see that always (2) $\implies$ (1), see Lemma 1 below.

Definition (2) applied in the cocommutative case suggests that a "positive definite" element of $VN(G)$ should come from a positive measure in $M(G) = C_0(G)^* = ML^1(G)$, the multiplier algebra of $L^1(G)$. On the dual side, De Cannière and Haagerup showed in [10] that *completely positive* multipliers of $A(G)$ coincide with positive definite functions on G. This suggests the following notions:

(3) A *completely positive multiplier* is $x \in L^\infty(\mathbb{G})$ such that there exists a completely positive left multiplier $L_x \colon L^1(\hat{\mathbb{G}}) \to L^1(\hat{\mathbb{G}})$ with $x\hat{\lambda}(\hat{\omega}) = \hat{\lambda}(L_x(\hat{\omega}))$ for every $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. Here $\hat{\lambda}$ denotes the map $\hat{\omega} \mapsto (\mathrm{id} \otimes \hat{\omega})(W^*) = (\hat{\omega} \otimes \mathrm{id})(\hat{W})$ where $W \in M(C_0(\mathbb{G}) \otimes C_0(\hat{\mathbb{G}}))$ is the left multiplicative unitary of $\mathbb{G}$.

(4) A *completely positive definite function* is $x \in L^\infty(\mathbb{G})$ such that there exists a normal completely positive map $\Phi \colon \mathcal{B}(L^2(\mathbb{G})) \to \mathcal{B}(L^2(\mathbb{G}))$ with

$$\langle x^*, \omega_{\xi,\alpha} \star \omega^\sharp_{\eta,\beta} \rangle = (\Phi(\theta_{\xi,\eta})\beta \mid \alpha)$$

for every $\xi, \eta \in D(P^{1/2})$ and $\alpha, \beta \in D(P^{-1/2})$. Here P is a densely defined, positive, injective operator on $L^2(\mathbb{G})$ implementing the scaling group of $\mathbb{G}$.

The first named author showed in [7] that (3) and (2) are equivalent notions, and that they imply (4). We note that while (2) is the notion mostly adopted in the literature (see above), it is actually the map L_x (or its adjoint) which is of interest (the point being that the implication (2) $\implies$ (3) is very easy to establish). Let us motivate (4) a little more. The unbounded involution $\sharp$ on $L^1(\mathbb{G})$ is given by $\omega^\sharp = \omega^* \circ S$, where this is bounded. Normal completely positive maps on $\mathcal{B}(L^2(\mathbb{G}))$ biject with the positive part of the extended (or weak*) Haagerup tensor product $\mathcal{B}(L^2(\mathbb{G})) \overset{eh}{\otimes} \mathcal{B}(L^2(\mathbb{G}))$ (see [2, 12]), where $a \otimes b$ is associated to Φ with

$$\Phi(\theta) = a\theta b \quad \Leftrightarrow \quad (\Phi(\theta_{\xi,\eta})\beta \mid \alpha) = \langle a, \omega_{\xi,\alpha} \rangle \langle b, \omega_{\beta,\eta} \rangle.$$

Hence (4) is equivalent to the existence of a positive $u \in \mathcal{B}(L^2(\mathbb{G})) \overset{eh}{\otimes} \mathcal{B}(L^2(\mathbb{G}))$ with

$$\langle \Delta(x^*), \omega_1 \otimes \omega_2^* \circ S \rangle = \langle u, \omega_1 \otimes \omega_2^* \rangle.$$

Hence, informally, this is equivalent to $(\mathrm{id} \otimes S)\Delta(x^*)$ being a positive member of $\mathcal{B}(L^2(\mathbb{G})) \overset{eh}{\otimes} \mathcal{B}(L^2(\mathbb{G}))$. In the commutative case, $x^* = F \in L^\infty(G)$ say, and this says that the function $(s,t) \mapsto F(st^{-1})$ is, in some sense, a "positive kernel", i.e. that F is positive definite. So (4) says that x^* defines a non-commutative, positive definite kernel.

We remark that as the "inverse" operator on $\mathbb{G}$ (the antipode S) and the adjoint $*$ on $L^\infty(\mathbb{G})$ do not commute, we have to be a little careful about using x or x^* in the above definitions.

The principle results of this paper are:

- For any $\mathbb{G}$, we have that (4) is equivalent to (3) and hence equivalent to (2).
- When $\mathbb{G}$ is coamenable (as is true in the commutative case!) all four conditions are equivalent.

Both these results may be interpreted as versions of Bochner's theorem for locally compact quantum groups. When given a condition like (1) the obvious thing to try is a GNS construction, in this case applied to the $*$-algebra $L^1_\sharp(\mathbb{G})$. In general, this algebra does not have an approximate identity, so we first show in Section 3 that products are always linearly dense in $L^1_\sharp(\mathbb{G})$, which enables a suitable GNS construction. In Section 4 we apply this, together with techniques similar to those used in [7] to show that (4) $\implies$ (3). One can take as a definition that $\mathbb{G}$ is coamenable if and only if $L^1(\mathbb{G})$ has a bounded approximate identity. In Section 5 we show that in this case, also $L^1_\sharp(\mathbb{G})$ has a (contractive) approximate identity. Indeed, we prove a slightly more general statement, adapting ideas of J. Kustermans, A. Van Daele and J. Verding from [19] (we wish to approximate the counit ϵ, which is *invariant* for the scaling group, and it is this invariance which is key; the argument in [19] is for multiplier algebras of C*-algebras and modular automorphism groups, and to our mind, works because the unit of $M(A)$ is invariant for the modular automorphism group). We suspect that the ideas of Sections 3 and 5 will prove to be useful in other contexts. In Section 6 we apply this to condition (1). In the final section we consider n-positive multipliers.

2. Preliminaries

Throughout the paper $\mathbb{G}$ denotes a locally compact quantum group [21, 22, 23]. Its comultiplication Δ is implemented by the left multiplicative unitary $W \in \mathcal{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$:

$$\Delta(x) = W^*(1 \otimes x)W \qquad (x \in L^\infty(\mathbb{G})).$$

The reduced C*-algebra $C_0(\mathbb{G})$ is the norm closure of

$$\{\, (\mathrm{id} \otimes \omega)W : \omega \in \mathcal{B}(L^2(\mathbb{G}))_* \,\}.$$

On the other hand, the norm closure of

$$\{\, (\omega \otimes \mathrm{id})W : \omega \in \mathcal{B}(L^2(\mathbb{G}))_* \,\}$$

gives the reduced C*-algebra $C_0(\hat{\mathbb{G}})$ of the dual quantum group $\hat{\mathbb{G}}$. The left multiplicative unitary of the dual quantum group is just $\hat{W} = \sigma W^* \sigma$ where σ is the flip map on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$. The associated von Neumann algebras $L^\infty(\mathbb{G})$ and $L^\infty(\hat{\mathbb{G}})$ are the weak*-closures of the respective C*-algebras $C_0(\mathbb{G})$ and $C_0(\hat{\mathbb{G}})$. The predual $L^1(\mathbb{G})$ of $L^\infty(\mathbb{G})$ is a Banach algebra under the convolution product

$\omega \star \tau = (\omega \otimes \tau)\Delta$. Given $\xi, \eta \in L^2(\mathbb{G})$, let $\omega_{\xi,\eta} \in L^1(\mathbb{G})$ be the normal functional $x \mapsto (x\xi \mid \eta)$. As $L^\infty(\mathbb{G})$ is in standard position on $L^2(\mathbb{G})$, every member of $L^1(\mathbb{G})$ arises in this way.

The scaling group (τ_t) of $\mathbb{G}$ is implemented by a positive, injective, densely defined operator P on $L^2(\mathbb{G})$: we have $\tau_t(x) = P^{it} x P^{-it}$. Then the antipode S of $\mathbb{G}$ has a polar decomposition $S = R\tau_{-i/2}$, where R is the unitary antipode. In particular $\Delta R = (R \otimes R)\sigma\Delta$, where σ denotes the flip map, now on $L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$.

We follow [20, Section 3] to define the $*$-algebra $L^1_\sharp(\mathbb{G})$. Recall that $\omega \in L^1(\mathbb{G})$ is a member of $L^1_\sharp(\mathbb{G})$ if and only if there exists $\omega^\sharp \in L^1(\mathbb{G})$ such that

$$\langle x, \omega^\sharp \rangle = \overline{\langle S(x)^*, \omega \rangle} \qquad (x \in D(S))$$

where $D(S)$ denotes the domain of S. Then $L^1_\sharp(\mathbb{G})$ is a dense subalgebra of $L^1(\mathbb{G})$. The natural norm of $L^1_\sharp(\mathbb{G})$ is $\|\omega\|_\sharp = \max(\|\omega\|, \|\omega^\sharp\|)$, and with $\sharp$ as the involution, $L^1_\sharp(\mathbb{G})$ is a Banach $*$-algebra. For $\omega \in L^1(\mathbb{G})$, let $\omega^* \in L^1(\mathbb{G})$ be the functional $\langle x, \omega^* \rangle = \overline{\langle x^*, \omega \rangle}$. Thus $\omega \in L^1_\sharp(G)$ if and only if $w^* \circ S$ is bounded. Notice that as Δ is a $*$-homomorphism, the map $\omega \mapsto \omega^*$ is an anti-linear homomorphism on $L^1(\mathbb{G})$, while $\omega \mapsto \omega^\sharp$ is an anti-linear anti-homomorphism on $L^1_\sharp(\mathbb{G})$.

The universal C*-algebra $C_0^u(\hat{\mathbb{G}})$ associated to $\hat{\mathbb{G}}$ is the universal C*-completion of $L^1_\sharp(\mathbb{G})$ (see [20] for details). The natural map (i.e. the universal representation) $\lambda_u \colon L^1_\sharp(\mathbb{G}) \to C_0^u(\hat{\mathbb{G}})$ is implemented as $\lambda_u(\omega) = (\omega \otimes \mathrm{id})(\mathcal{W})$ where $\mathcal{W} \in M(C_0(\mathbb{G}) \otimes C_0^u(\hat{\mathbb{G}}))$ is the maximal unitary corepresentation of $\mathbb{G}$ (which is denoted by $\hat{\mathcal{V}}$ in [20, Proposition 4.2]).

Lemma 1. *Let $x = (\mathrm{id} \otimes \hat{\mu})(\mathcal{W}^*)$ for some $\hat{\mu} \in C_0^u(\hat{\mathbb{G}})^*_+$. Then $\langle x^*, \omega \star \omega^\sharp \rangle \geq 0$ for every $\omega \in L^1_\sharp(\mathbb{G})$.*

Proof. Simply note that

$$\begin{aligned} \langle x^*, \omega \star \omega^\sharp \rangle &= \langle (\Delta \otimes \mathrm{id})\mathcal{W}, \omega \otimes \omega^\sharp \otimes \hat{\mu} \rangle = \langle \mathcal{W}_{13}\mathcal{W}_{23}, \omega \otimes \omega^\sharp \otimes \hat{\mu} \rangle \\ &= \langle \hat{\mu}, \lambda_u(\omega)\lambda_u(\omega^\sharp) \rangle = \langle \hat{\mu}, \lambda_u(\omega)\lambda_u(\omega)^* \rangle \geq 0, \end{aligned}$$

as required. □

Similarly to [18], we say that $x \in M(C_0(\mathbb{G}))$ is a *left multiplier of* $L^1(\hat{\mathbb{G}})$ if

$$x\hat{\lambda}(\hat{\omega}) \in \hat{\lambda}\big(L^1(\hat{\mathbb{G}})\big) \quad \text{whenever} \quad \hat{\omega} \in L^1(\hat{\mathbb{G}}),$$

where

$$\hat{\lambda} \colon C_0(\hat{\mathbb{G}})^* \to M(C_0(\mathbb{G})), \qquad \hat{\lambda}(\hat{\mu}) = (\hat{\mu} \otimes \mathrm{id})(\hat{W}) = (\mathrm{id} \otimes \hat{\mu})(W^*).$$

In this case we can define $L_x \colon L^1(\hat{\mathbb{G}}) \to L^1(\hat{\mathbb{G}})$ by

$$\hat{\lambda}(L_x(\hat{\omega})) = x\hat{\lambda}(\hat{\omega})$$

because $\hat{\lambda}$ is injective. We see immediately that L_x is a left multiplier (often termed a "left centraliser" in the literature) in the usual sense, that is, $L_x(\hat{\omega} \star \hat{\tau}) = L_x(\hat{\omega}) \star \hat{\tau}$ for every $\hat{\omega}, \hat{\tau} \in L^1(\hat{\mathbb{G}})$. The following lemma is shown for Kac algebras in [18], but since we need the (short) argument once more, we include a proof.

Lemma 2. *Let $x \in M(C_0(\mathbb{G}))$ be a left multiplier of $L^1(\hat{\mathbb{G}})$. Then $L_x \colon L^1(\hat{\mathbb{G}}) \to L^1(\hat{\mathbb{G}})$ is bounded.*

Proof. We apply the closed graph theorem. Suppose that $\hat{\omega}_n \to \hat{\omega}$ and $L_x(\hat{\omega}_n) \to \hat{\tau}$ in $L^1(\hat{\mathbb{G}})$. Then

$$\begin{aligned}\|\hat{\lambda}(L_x(\hat{\omega})) - \hat{\lambda}(\hat{\tau})\| &\leq \|x\hat{\lambda}(\hat{\omega}) - x\hat{\lambda}(\hat{\omega}_n)\| + \|\hat{\lambda}(L_x(\hat{\omega}_n)) - \hat{\lambda}(\hat{\tau})\| \\ &\leq \|x\|\|\hat{\omega} - \hat{\omega}_n\| + \|L_x(\hat{\omega}_n) - \hat{\tau}\| \to 0\end{aligned}$$

as $n \to \infty$. Since $\hat{\lambda}$ is injective, we have $L_x(\hat{\omega}) = \hat{\tau}$ and by the closed graph theorem, L_x is bounded. □

We say that $x \in M(C_0(\mathbb{G}))$ is an *n-positive multiplier* if it is a left multiplier of $L^1(\mathbb{G})$ and the map $L_x^*\colon L^\infty(\hat{\mathbb{G}}) \to L^\infty(\hat{\mathbb{G}})$ is n-positive. We shall consider n-positive multipliers more carefully in Section 7, but the main interest of the paper shall be the completely positive multipliers: that is, $x \in M(C_0(\mathbb{G}))$ that are n-positive multipliers for every $n \in \mathbb{N}$. When x is a completely positive multiplier, L_x^* extends to a normal, completely positive map $\Phi\colon \mathcal{B}(L^2(\mathbb{G})) \to \mathcal{B}(L^2(\mathbb{G}))$ (see [16, Proposition 4.3] or [7, Proposition 3.3]). Moreover, by [7, Proposition 6.1],

$$\langle x^*, \omega_{\xi,\alpha} \star \omega_{\eta,\beta}^\sharp \rangle = (\Phi(\theta_{\xi,\eta})\beta \mid \alpha)$$

for every $\xi, \eta \in D(P^{1/2})$ and $\alpha, \beta \in D(P^{-1/2})$, so x is completely positive definite. We shall prove the converse in Section 4, but first we need a bit of groundwork.

The following is similar to known results about cores for analytic generators (compare [29, Theorem X.49] for example) but we give the short proof for completeness. Let us just remark that as $S = R\tau_{-i/2}$ and $\tau_t(x) = P^{it}xP^{-it}$ for all t, the functional $\omega_{\xi,\alpha}$ is in $L^1_\sharp(\mathbb{G})$ whenever $\xi \in D(P^{1/2})$ and $\alpha \in D(P^{-1/2})$, and in this case $\omega_{\xi,\alpha}^\sharp = R_*(\omega_{P^{-1/2}\alpha, P^{1/2}\xi}) = \omega_{\hat{J}P^{1/2}\xi, \hat{J}P^{-1/2}\alpha}$; see [7, Section 6].

Lemma 3. *The set*

$$D = \{\, \omega_{\xi,\alpha} : \xi \in D(P^{1/2}), \alpha \in D(P^{-1/2}) \,\}$$

is dense in $L^1_\sharp(\mathbb{G})$ with respect to its natural norm (i.e. D is a core for $\omega \mapsto \omega^\sharp$).

Proof. For $\omega \in L^1(\mathbb{G})$, $r > 0$, define

$$\omega(r) = \frac{r}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2t^2} \omega \circ \tau_t \; dt.$$

See also Section 5 below. Since the modular group (τ_t) is implemented by P, it follows that D is invariant under (τ_t). On the other hand, since R commutes with (τ_t) and $S = R\tau_{-i/2}$, we have $(\omega \circ \tau_t)^\sharp = \omega^\sharp \circ \tau_t$ for every $\omega \in L^1_\sharp(\mathbb{G})$, and so $t \mapsto \omega \circ \tau_t$ is continuous with respect to the norm of $L^1_\sharp(\mathbb{G})$. Consequently, if $\omega \in D$, then $\omega(r)$ is in the $L^1_\sharp(\mathbb{G})$-closure of D for every $r > 0$.

Given $\omega \in L^1_\sharp(\mathbb{G})$, there exists $(\omega_n) \subseteq D$ such that $\omega_n \to \omega$ in $L^1(\mathbb{G})$, because $L^\infty(\mathbb{G})$ is in standard form on $L^2(\mathbb{G})$ and the domains of $P^{1/2}$ and $P^{-1/2}$ are dense in $L^2(\mathbb{G})$. By the beginning of the proof, $\omega_n(r)$ is in the $L^1_\sharp(\mathbb{G})$-closure of D. A simple calculation shows that

$$\|\omega(r) - \omega_n(r)\|_\sharp \leq e^{r^2/4}\|\omega - \omega_n\|,$$

and so $\omega(r)$ is in the $L^1_\sharp(\mathbb{G})$-closure of D. As $r \to \infty$, we have $\omega(r) \to \omega$ and $\omega(r)^\sharp = \omega^\sharp(r) \to \omega^\sharp$ (since $\omega \in L^1_\sharp(\mathbb{G})$). Therefore ω is in the $L^1_\sharp(\mathbb{G})$-closure of D, as claimed. □

3. Density of products in $L^1_\sharp(\mathbb{G})$

The main result of this section is that, in analogy to $L^1(\mathbb{G})$, the convolution products are linearly dense in $L^1_\sharp(\mathbb{G})$ with respect to its natural norm. We shall prove this result based on two (closely related) lemmas, the first of which is from [5, Proposition A.1].

Lemma 4. *Let $x, y \in L^\infty(\mathbb{G})$ satisfy $\langle y, \omega^\sharp\rangle = \langle x^*, \omega^*\rangle$ for all $\omega \in L^1_\sharp(\mathbb{G})$. Then $y \in D(S)$ and $S(y) = x^*$.*

Lemma 5. *Let $x, y \in L^\infty(\mathbb{G})$ satisfy $\langle y, (\omega_1 \star \omega_2)^\sharp\rangle = \langle x^*, \omega_1^* \star \omega_2^*\rangle$ for all $\omega_1, \omega_2 \in L^1_\sharp(\mathbb{G})$. Then $y \in D(S)$ with $S(y) = x^*$.*

Proof. For $n \in \mathbb{N}$ define the smear

$$y(n) = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-n^2t^2} \tau_t(y)\ dt$$

where (τ_t) is the scaling group. Define $x(n)$ similarly using x^*.

As R commutes with (τ_t) and $S = R\tau_{-i/2}$, it follows that $\omega^\sharp \circ \tau_t = (\omega \circ \tau_t)^\sharp$ for $\omega \in L^1_\sharp(\mathbb{G})$. Using that $\Delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Delta$ we find that for $\omega_1, \omega_2 \in L^1_\sharp(\mathbb{G})$

$$\begin{aligned}
\langle \Delta(y(n)), \omega_2^\sharp \otimes \omega_1^\sharp\rangle &= \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-n^2t^2} \langle \Delta(\tau_t(y)), \omega_2^\sharp \otimes \omega_1^\sharp\rangle\ dt \\
&= \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-n^2t^2} \langle \Delta(y), (\omega_2 \circ \tau_t)^\sharp \otimes (\omega_1 \circ \tau_t)^\sharp\rangle\ dt \\
&= \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-n^2t^2} \langle \Delta(x^*), (\omega_1 \circ \tau_t)^* \otimes (\omega_2 \circ \tau_t)^*\rangle\ dt \\
&= \langle \Delta(x(n)), \omega_1^* \otimes \omega_2^*\rangle.
\end{aligned}$$

As $y(n) \in D(S)$ the von Neumann algebraic version of [23, Lemma 5.25] shows that

$$\langle \Delta(y(n)), \omega_2^\sharp \otimes \omega_1^\sharp\rangle = \langle \Delta(S(y(n))), \omega_1^* \otimes \omega_2^*\rangle.$$

Thus $\Delta(S(y(n))) = \Delta(x(n))$, and as Δ is injective, $S(y(n)) = x(n)$. Now $y(n) \to y$ in the σ-weak topology, and $x(n) \to x^*$. As S is a σ-weakly closed operator, it follows that $y \in D(S)$ with $S(y) = x^*$, as required. ☐

Theorem 6. *Let $\mathbb{G}$ be a locally compact quantum group. Then the set $\{\, \omega \star \tau : \omega, \tau \in L^1_\sharp(\mathbb{G}) \,\}$ is linearly dense in $L^1_\sharp(\mathbb{G})$ in its natural norm.*

Proof. For a Banach space E, let $\overline{E}$ be the conjugate space to E. For $x \in E$ let $\overline{x} \in \overline{E}$ be the image of x, so $\overline{x} + \overline{y} = \overline{x+y}$ and $t\overline{x} = \overline{\overline{t}x}$ for $x, y \in E, t \in \mathbb{C}$. We identify $(\overline{E})^*$ with $\overline{E^*}$ via $\langle \overline{\mu}, \overline{x}\rangle = \overline{\langle \mu, x\rangle}$.

Then the map

$$L^1_\sharp(\mathbb{G}) \to L^1(\mathbb{G}) \oplus_\infty \overline{L^1(\mathbb{G})}, \quad \omega \mapsto (\omega, \overline{\omega^\sharp})$$

is a linear isometry. Thus the adjoint $L^\infty(\mathbb{G}) \oplus_1 \overline{L^\infty(\mathbb{G})} \to L^1_\sharp(\mathbb{G})^*$ is a quotient map. So any member of $L^1_\sharp(\mathbb{G})^*$ is induced by a pair $(x, \overline{y})$ with $x, y \in L^\infty(\mathbb{G})$, and the dual pairing is

$$\langle (x, \overline{y}), \omega\rangle = \langle x, \omega\rangle + \langle \overline{y}, \overline{\omega^\sharp}\rangle = \langle x, \omega\rangle + \overline{\langle y, \omega^\sharp\rangle}.$$

Firstly, $(x, \overline{y}) = 0$ if and only if $\langle -x^*, \omega^*\rangle = \langle y, \omega^\sharp\rangle$ for all $\omega \in L^1_\sharp(\mathbb{G})$ if and only if, by Lemma 4, $y \in D(S)$ with $S(y) = -x^*$.

Now let $(x, \overline{y})$ annihilate all elements of the form $\omega \star \tau$, with $\omega, \tau \in L^1_\sharp(\mathbb{G})$. Then

$$0 = \langle x, \omega \star \tau\rangle + \overline{\langle y, \tau^\sharp \star \omega^\sharp\rangle} \implies \langle -x^*, \omega^* \star \tau^*\rangle = \langle y, \tau^\sharp \star \omega^\sharp\rangle.$$

By Lemma 5, $y \in D(S)$ with $S(y) = -x^*$. That is, $(x, \overline{y}) = 0$. So by the Hahn–Banach theorem, the result follows. $\square$

4. Completely positive definite functions

The definition of completely positive definite functions on a locally compact quantum group $\mathbb{G}$ was proposed by the first named author in [7], and in this section we show that, as conjectured in [7], such elements are precisely the completely positive multipliers. This result may be viewed as a version of Bochner's theorem because the completely positive multipliers are known by [7] to be of the form $(\mathrm{id} \otimes \hat{\mu})(\mathcal{W}^*)$, with $\hat{\mu} \in C_0^u(\hat{\mathbb{G}})^*_+$.

We begin with a preliminary result, also of independent interest.

Proposition 7. *Let $x \in L^\infty(\mathbb{G})$ be completely positive definite. Then $x^* \in D(S)$ and $S(x^*) = x$.*

Proof. For every $\xi, \eta \in D(P^{1/2})$ and $\alpha, \beta \in D(P^{-1/2})$

$$\begin{aligned}\langle x^*, \omega_{\xi,\alpha} \star \omega^\sharp_{\eta,\beta}\rangle &= \langle \Phi(\theta_{\xi,\eta}), \omega_{\beta,\alpha}\rangle = \overline{\langle \Phi(\theta^*_{\xi,\eta}), \omega^*_{\beta,\alpha}\rangle} = \overline{\langle \Phi(\theta_{\eta,\xi}), \omega_{\alpha,\beta}\rangle} \\ &= \overline{\langle x^*, \omega_{\eta,\beta} \star \omega^\sharp_{\xi,\alpha}\rangle} = \langle x, \omega^*_{\eta,\beta} \star \omega^{\sharp *}_{\xi,\alpha}\rangle = \langle x, (\omega_{\xi,\alpha} \star \omega^\sharp_{\eta,\beta})^{\sharp *}\rangle.\end{aligned}$$

It follows from Theorem 6 and Lemma 3 that

$$\langle x^*, \omega\rangle = \langle x, \omega^{\sharp *}\rangle$$

for every $\omega \in L^1_\sharp(\mathbb{G})$. Then it follows from Lemma 4 that $x^* \in D(S)$ and $S(x^*) = x$. $\square$

The following lemma shows that a GNS-type construction works for positive definite functions. Compare with [11, Proposition 2.4.4], but note that $L^1_\sharp(\mathbb{G})$ does not necessarily have a bounded approximate identity, the lack of which is remedied by the density of products in $L^1_\sharp(\mathbb{G})$ (Theorem 6).

Lemma 8. *Let $x \in L^\infty(\mathbb{G})$ be positive definite. Then*

$$(\omega \mid \tau) = \langle x^*, \tau^\sharp \star \omega\rangle$$

defines a pre-inner-product on $L^1_\sharp(\mathbb{G})$; let $\Lambda\colon L^1_\sharp(\mathbb{G}) \to H$ be the associated map to the Hilbert space H obtained by completion. Then,

$$\pi(\omega)\Lambda(\tau) = \Lambda(\omega \star \tau)$$

defines a non-degenerate $*$-representation π of $L^1_\sharp(\mathbb{G})$ on H.

Proof. Such GNS-type results are usually stated for algebras with an approximate identity (see for example [6, Section 3.1] or [11, Section 2.4]) so for completeness, we give the details in this slightly more general setting. When $x \in L^\infty(\mathbb{G})$ is positive definite,

$$(\omega \mid \tau) = \langle x^*, \tau^\sharp \star \omega\rangle.$$

defines a positive sesquilinear form on $L^1_\sharp(\mathbb{G})$. As in the statement, let H be the Hilbert space completion of $L^1_\sharp(\mathbb{G})/\mathcal{N}$, where $\mathcal{N}$ denotes the associated null space, and let $\Lambda\colon L^1_\sharp(\mathbb{G}) \to H$ be the map taking $\omega \in L^1_\sharp(\mathbb{G})$ to the image of $\omega + \mathcal{N}$ in H. We are left to show that

$$\pi(\omega)\Lambda(\tau) = \Lambda(\omega \star \tau)$$

defines a non-degenerate $*$-representation of $L^1_\sharp(\mathbb{G})$ on H. If we can show that $\pi(\omega)$ is well-defined and bounded, then it follows easily that π is a $*$-homomorphism.

Let r denote the spectral radius on $L^1_\sharp(\mathbb{G})$. By [6, Corollary 3.1.6],

$$\|\pi(\omega)\Lambda(\tau)\|^2 = \langle x^*, \tau^\sharp \star \omega^\sharp \star \omega \star \tau\rangle \le r(\omega^\sharp \star \omega)\langle x^*, \tau^\sharp \star \tau\rangle = r(\omega^\sharp \star \omega)\|\Lambda(\tau)\|^2.$$

This shows that $\pi(\omega)$ maps $\mathcal{N}$ to $\mathcal{N}$ and hence is well-defined. Moreover, we see that $\pi(\omega)$ defines a bounded operator on H.

Finally, since $L^1_\sharp(\mathbb{G})$ is the closed linear span of $L^1_\sharp(\mathbb{G}) \star L^1_\sharp(\mathbb{G})$ by Theorem 6 and Λ is continuous, the $*$-representation π is non-degenerate. ☐

Theorem 9. *An element $x \in L^\infty(\mathbb{G})$ is completely positive definite if and only if it is a completely positive multiplier. In particular, every completely positive definite $x \in L^\infty(\mathbb{G})$ is in $M(C_0(\mathbb{G}))$.*

Proof. As already noted, the "if" part is proved in [7], so we let $x \in L^\infty(\mathbb{G})$ be completely positive definite. Let $\Phi\colon \mathcal{B}(L^2(\mathbb{G})) \to \mathcal{B}(L^2(\mathbb{G}))$ be the associated completely positive map such that

$$\langle x^*, \omega_{\xi,\alpha} \star \omega^\sharp_{\eta,\beta}\rangle = (\Phi(\theta_{\xi,\eta})\beta \mid \alpha)$$

whenever $\xi, \eta \in D(P^{1/2})$ and $\alpha, \beta \in D(P^{-1/2})$. Then Φ has a Stinespring dilation of the form

$$\Phi(\theta) = V^*(\theta \otimes 1)V \qquad (\theta \in \mathcal{B}_0(L^2(\mathbb{G}))),$$

where $V\colon L^2(\mathbb{G}) \to L^2(\mathbb{G}) \otimes K$ is a bounded map for some Hilbert space K (see [7, Section 5] for details). Letting (e_i) be an orthonormal basis of K, we can define a family (a_i) in $\mathcal{B}(L^2(\mathbb{G}))$ with $\sum_i a_i^* a_i < \infty$ such that

$$V\xi = \sum_i a_i \xi \otimes e_i$$

and hence

$$\Phi(\theta) = \sum_i a_i^* \theta a_i.$$

We may also take the Stinespring dilation to be *minimal* so that

$$\{\, (\theta \otimes \mathrm{id})V\xi : \xi \in L^2(\mathbb{G}), \theta \in \mathcal{B}_0(L^2(\mathbb{G})) \,\}$$

is linearly dense in $L^2(\mathbb{G}) \otimes K$. This is equivalent to vectors of the form

$$\sum_i \langle a_i, \omega\rangle \eta \otimes e_i \qquad (\omega \in \mathcal{B}(L^2(\mathbb{G}))_*, \eta \in L^2(\mathbb{G}))$$

being linearly dense; equivalently that vectors of the form $\sum_i \langle a_i, \omega\rangle e_i$ are dense in K as $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$ varies.

Let (Λ, π, H) be the GNS construction for x from Lemma 8. Then, for $\xi, \eta \in D(P^{1/2})$ and $\alpha, \beta \in D(P^{-1/2})$,

$$\begin{aligned}\big(\Lambda(\omega^\sharp_{\xi,\alpha}) \bigm| \Lambda(\omega^\sharp_{\eta,\beta})\big)_H &= \langle x^*, \omega_{\eta,\beta} \star \omega^\sharp_{\xi,\alpha}\rangle = (\Phi(\theta_{\eta,\xi})\alpha \mid \beta) \\ &= \sum_i (a_i(\alpha) \mid \xi)\overline{(a_i(\beta) \mid \eta)}.\end{aligned}$$

Let $q : \mathcal{B}(L^2(\mathbb{G}))_* \to L^1(\mathbb{G})$ be the quotient map. Since the functionals of the form $\omega^\sharp_{\xi,\alpha}$ are dense in $L^1_\sharp(\mathbb{G})$ by Lemma 3, it follows that there is an isometry

$$v\colon H \to K; \quad \Lambda(q(\omega)^\sharp) \mapsto \sum_i \langle a_i, \omega^* \rangle e_i \qquad (\omega \in \mathcal{B}(L^2(\mathbb{G}))_*, q(\omega) \in L^1_\sharp(\mathbb{G}))$$

(note that $\| \sum_i \langle a_i, \omega^* \rangle e_i \|^2 \leq \sum_i |\langle a_i, \omega^* \rangle|^2 \leq \|\omega\|^2 \| \sum_i a_i^* a_i \|$). As we have a minimal Stinespring dilation, v has dense range and is hence unitary. A corollary of v even being well-defined is that for each i and each $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$, the value of $\langle a_i, \omega \rangle$ depends only on the value $q(\omega)$. Hence $a_i \in L^\infty(\mathbb{G})$ for each i.

Consider now the non-degenerate $*$-representation of $L^1_\sharp(\mathbb{G})$ on K given by $v\pi(\cdot)v^*$. By Kustermans [20, Corollary 4.3], there is an associated unitary corepresentation U of $\mathbb{G}$; so $U \in L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(K)$ and $v\pi(\omega)v^* = (\omega \otimes \mathrm{id})(U)$. As U is a unitary corepresentation, we know that $(\omega^\sharp \otimes \mathrm{id})(U) = (\omega \otimes \mathrm{id})(U)^* = (\omega^* \otimes \mathrm{id})(U^*)$. If $\omega^* \in L^1_\sharp(\mathbb{G})$, it follows that $(\omega \otimes \mathrm{id})(U^*) = v\pi(\omega^{*\sharp})v^*$, and thus

$$\begin{aligned}
\Big(U^* \big(\xi \otimes \sum_i \langle a_i, \omega \rangle e_i \big) \Bigm| \alpha \otimes e_j \Big) &= \Big((\omega_{\xi,\alpha} \otimes \mathrm{id})(U^*) \sum_i \langle a_i, \omega \rangle e_i \Bigm| e_j \Big) \\
&= \Big(v\pi(\omega^\sharp_{\alpha,\xi}) v^* \sum_i \langle a_i, \omega \rangle e_i \Bigm| e_j \Big) = \Big(v\pi(\omega^\sharp_{\alpha,\xi}) \Lambda(\omega^{*\sharp}) \Bigm| e_j \Big) \\
&= \big(v\Lambda(\omega^\sharp_{\alpha,\xi} \star \omega^{*\sharp}) \bigm| e_j \big) = \langle a_j, (\omega^* \star \omega_{\alpha,\xi})^* \rangle = \langle a_j, \omega \star \omega_{\xi,\alpha} \rangle \\
&= \Big(\sum_i (\omega \otimes \mathrm{id})\Delta(a_i)\xi \otimes e_i \Bigm| \alpha \otimes e_j \Big)
\end{aligned}$$

whenever $\xi \in D(P^{1/2})$, $\alpha \in D(P^{-1/2})$. As an aside, we note that this is precisely the way in which U^* is defined in [7, Proposition 5.2]. Thus, perhaps as expected, if x comes from a completely positive multiplier, then the two approaches to forming representations agree.

Since U is unitary, we have, for every $\xi, \eta \in D(P^{1/2})$ and $\omega_1, \omega_2 \in L^1_\sharp(\mathbb{G})^*$,

$$\begin{aligned}
(\xi \mid \eta) \sum_i \langle a_i, \omega_1 \rangle \overline{\langle a_i, \omega_2 \rangle} &= \Big(U^* \big(\xi \otimes \sum_i \langle a_i, \omega_1 \rangle e_i \big) \Bigm| U^* \big(\eta \otimes \sum_i \langle a_i, \omega_2 \rangle e_i \big) \Big) \\
&= \Big(\sum_i (\omega_1 \otimes \mathrm{id})\Delta(a_i)\xi \otimes e_i \Bigm| \sum_i (\omega_2 \otimes \mathrm{id})\Delta(a_i)\eta \otimes e_i \Big) \\
&= \sum_i \big((\omega_2 \otimes \mathrm{id})\Delta(a_i)^* (\omega_1 \otimes \mathrm{id})\Delta(a_i)\xi \bigm| \eta \big).
\end{aligned}$$

It follows that

$$\sum_i \langle a_i^* \otimes a_i \otimes 1, \omega_2^* \otimes \omega_1 \otimes \omega_{\xi,\eta} \rangle = \sum_i \langle \Delta(a_i^*)_{13} \Delta(a_i)_{23}, \omega_2^* \otimes \omega_1 \otimes \omega_{\xi,\eta} \rangle.$$

As this holds for a dense collection of $\omega_1, \omega_2, \xi, \eta$ it follows that

$$\sum_i a_i^* \otimes a_i \otimes 1 = \sum_i \Delta(a_i^*)_{13} \Delta(a_i)_{23}$$

(recall that $\sum_i a_i^* a_i < \infty$ so the sums on both sides do converge σ-weakly.). Then, for $\theta_{\xi,\eta} \in \mathcal{B}_0(L^2(\mathbb{G}))$ and $\alpha, \beta \in L^2(\mathbb{G})$,

$$\begin{aligned}(\Phi(\theta_{\xi,\eta})\beta \mid \alpha)1 &= \sum_i (a_i^* \theta_{\xi,\eta} a_i \beta \mid \alpha)1 = \sum_i (a_i^* \xi \mid \alpha)(a_i \beta \mid \eta)1 \\ &= \sum_i (\omega_{\xi,\alpha} \otimes \omega_{\beta,\eta} \otimes \mathrm{id})(a_i^* \otimes a_i \otimes 1) \\ &= \sum_i (\omega_{\xi,\alpha} \otimes \omega_{\beta,\eta} \otimes \mathrm{id})\Delta(a_i^*)_{13}\Delta(a_i)_{23} \\ &= \sum_i (\omega_{\xi,\alpha} \otimes \mathrm{id})\Delta(a_i^*)(\omega_{\beta,\eta} \otimes \mathrm{id})\Delta(a_i) \\ &= \sum_i (\omega_{\beta,\alpha} \otimes \mathrm{id})\big(\Delta(a_i^*)(\theta_{\xi,\eta} \otimes 1)\Delta(a_i)\big).\end{aligned}$$

It follows that

$$\Phi(\theta) \otimes 1 = \sum_i \Delta(a_i^*)(\theta \otimes 1)\Delta(a_i) \qquad (\theta \in \mathcal{B}_0(L^2(\mathbb{G}))).$$

By normality, and using that $\Delta(\cdot) = W^*(1 \otimes \cdot)W$, we see that for $y \in \mathcal{B}(L^2(\mathbb{G}))$,

$$\Phi(y) \otimes 1 = \sum_i W^*(1 \otimes a_i^*)W(y \otimes 1)W^*(1 \otimes a_i)W$$

and hence

$$\begin{aligned}1 \otimes \Phi(y) &= \sum_i \hat{W}(a_i^* \otimes 1)\hat{W}^*(1 \otimes y)\hat{W}(a_i \otimes 1)\hat{W}^* \\ &= \hat{W}\big((\Phi \otimes \mathrm{id})(\hat{W}^*(1 \otimes y)\hat{W})\big)\hat{W}^*.\end{aligned} \tag{1}$$

In particular, for $\hat{x} \in L^\infty(\hat{\mathbb{G}})$,

$$\hat{W}^*(1 \otimes \Phi(\hat{x}))\hat{W} = (\Phi \otimes \mathrm{id})\hat{\Delta}(\hat{x}).$$

Now, the left-hand-side is a member of $L^\infty(\hat{\mathbb{G}}) \overline{\otimes} \mathcal{B}(L^2(\mathbb{G}))$, and the right-hand-side is a member of $\mathcal{B}(L^2(\mathbb{G})) \overline{\otimes} L^\infty(\hat{\mathbb{G}})$, and so both sides are really in $L^\infty(\hat{\mathbb{G}}) \overline{\otimes} L^\infty(\hat{\mathbb{G}})$ (by taking bicommutants for example). Then

$$(\Phi \otimes \mathrm{id})\big(\hat{\Delta}(\hat{x})(1 \otimes \hat{y})\big) = \big((\Phi \otimes \mathrm{id})\hat{\Delta}(\hat{x})\big)(1 \otimes \hat{y}) \in L^\infty(\hat{\mathbb{G}}) \overline{\otimes} L^\infty(\hat{\mathbb{G}}),$$

and so, as $\{\, \hat{\Delta}(\hat{x})(1 \otimes \hat{y}) : \hat{x}, \hat{y} \in L^\infty(\hat{\mathbb{G}}) \,\}$ is a σ-weakly, linearly dense subset of $L^\infty(\hat{\mathbb{G}}) \overline{\otimes} L^\infty(\hat{\mathbb{G}})$, it follows that Φ maps $L^\infty(\hat{\mathbb{G}})$ to $L^\infty(\hat{\mathbb{G}})$. (We remark that the fact that $\{\, \hat{\Delta}(\hat{x})(1 \otimes \hat{y}) : \hat{x}, \hat{y} \in L^\infty(\hat{\mathbb{G}}) \,\}$ is linearly σ-weakly dense is shown directly in the remark after [32, Proposition 1.21]. One can prove this by noting that $D(S)$ is σ-weakly dense, and then using the von Neumann algebraic version of the characterisation of S given by [23, Corollary 5.34].)

Let L be the restriction of Φ to $L^\infty(\hat{\mathbb{G}})$. Then the calculation above shows that

$$\hat{\Delta}L = (L \otimes \mathrm{id})\hat{\Delta},$$

and so L is the adjoint of a completely positive left multiplier on $L^1(\hat{\mathbb{G}})$. Comparing (1) with [7, Proposition 3.3] we see that Φ coincides with the extension of L used in [7, 16]. In particular, by [7, Propositions 3.2] there is $x_0 \in M(C_0(\mathbb{G}))$ such that

$$(x_0 \otimes 1)W^* = (\mathrm{id} \otimes L)(W^*) = (\mathrm{id} \otimes \Phi)(W^*).$$

Equivalently,

$$x_0 \otimes 1 = \sum_i (1 \otimes a_i^*)\Delta(a_i).$$

Moreover, by [7, Proposition 6.1], x_0 satisfies

$$\langle x_0, \omega_{\alpha,\eta} \star \omega_{\xi,\beta}^{\sharp *}\rangle = (\Phi(\theta_{\xi,\eta})\alpha \mid \beta) = \langle x^*, \omega_{\xi,\beta} \star \omega_{\eta,\alpha}^{\sharp}\rangle = \langle S(x^*), \omega_{\alpha,\eta} \star \omega_{\xi,\beta}^{\sharp *}\rangle$$

for every $\xi, \eta \in D(P^{1/2})$ and $\alpha, \beta \in D(P^{-1/2})$ because $x^* \in D(S)$ and $S(x^*) = x$ by Proposition 7. It follows, by density of such ξ, η, α, β, that $x_0 = x$. In particular, x is a completely positive multiplier. □

Two immediate corollaries of the proof are the following.

Corollary 10. *Let x be completely positive definite, this being witnessed by the completely positive map Φ. Then Φ restricted to $L^\infty(\hat{\mathbb{G}})$ is the adjoint of a completely bounded left multiplier L of $L^1(\hat{\mathbb{G}})$, and Φ is the canonical extension of L^*.*

Corollary 11. *Let x be positive definite, with GNS construction (H, Λ, π), and let (f_i) be an orthonormal basis of H. Suppose there is a dense subset $D \subseteq L^1_\sharp(\mathbb{G})$ and a family (a_i) in $\mathcal{B}(L^2(\mathbb{G}))$ with $(\Lambda(\omega^\sharp) \mid f_i)_H = \langle a_i, \omega^*\rangle$ for all $\omega \in D$ and all i. If $\sum_i a_i^* a_i < \infty$, then x is a completely positive multiplier.*

Supposing that $L^1_\sharp(\mathbb{G})$ is separable, the Gram–Schmidt process allows us to find an orthonormal basis of the form $f_i = \Lambda(\tau_i)$ for some sequence $(\tau_i) \subseteq L^1_\sharp(\mathbb{G})$. Then

$$(f_i \mid \Lambda(\omega^\sharp))_H = \langle x^*, \omega \star \tau_i\rangle = \langle (\mathrm{id} \otimes \tau_i)\Delta(x^*), \omega\rangle,$$

and so we may set $a_i^* = \tau_i \star x^* := (\mathrm{id} \otimes \tau_i)\Delta(x^*)$. Thus the existence of (a_i) is not the key fact; rather whether $\sum_i a_i^* a_i$ converges is the key issue. Below we shall see that when $\mathbb{G}$ is coamenable, this is automatic, while Example 17 shows that the condition does not always hold.

5. Bounded approximate identity for $L^1_\sharp(\mathbb{G})$

To prove an analogue of Bochner's theorem for coamenable locally compact quantum groups, we shall need a bounded approximate identity for $L^1_\sharp(\mathbb{G})$. In this section we show that when $\mathbb{G}$ is coamenable, $L^1_\sharp(\mathbb{G})$ has a bounded approximate identity in its natural norm – in fact a contractive one. (The converse is obviously true.) The proof is inspired by Propositions 2.25 and 2.26 in Kustermans's paper [19], where the proof of the latter proposition is credited to A. Van Daele and J. Verding. We prove a more general fact which is inspired by the underlying idea in the proof (as we see it).

For $\omega \in L^1(\mathbb{G})$, $z \in \mathbb{C}$ and $r > 0$, let

$$\omega(r, z) = \frac{r}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2(t-z)^2} \omega \circ \tau_t \, dt.$$

The integral converges in norm, as the function $t \mapsto \omega \circ \tau_t$ is norm-continuous. A little calculation, and some complex analysis, shows that

$$\omega(r, z) \in L^1_\sharp(\mathbb{G}), \qquad \omega(r, z)^\sharp = \omega^*(r, z - i/2) \circ R.$$

See [21, Section 5] for example. Notice that

$$\|\omega(r, z)^\sharp\| \le \frac{r}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left| e^{-r^2(t-z+i/2)^2} \right| \|\omega\| \, dt = e^{r^2(\Im(z)-1/2)^2}\|\omega\|. \tag{2}$$

By analogy with the definition of $L^1_\sharp(\mathbb{G})$, define $M_\sharp(\mathbb{G})$ to be the collection of those $\mu \in M(\mathbb{G})$ such that there is $\mu^\sharp \in M(\mathbb{G})$ with

$$\overline{\langle \mu, S(a)^*\rangle} = \langle \mu^\sharp, a\rangle \qquad (a \in D(S) \subseteq C_0(\mathbb{G})).$$

As $S\circ * \circ S \circ * = \mathrm{id}$, it is easy to show that $\mu \mapsto \mu^\sharp$ defines an involution on $M_\sharp(\mathbb{G})$. Set $\|\mu\|_\sharp = \max(\|\mu\|, \|\mu^\sharp\|)$ for $\mu \in M_\sharp(\mathbb{G})$.

In the next theorem, we look at $\mu \in M(\mathbb{G})$ such that $\mu \circ \tau_t = \mu$ for all t. Then the constant function $F : \mathbb{C} \to M(\mathbb{G}); z \mapsto \mu$ is holomorphic and satisfies that $F(t) = \mu \circ \tau_t$ for all $t \in \mathbb{R}$. Hence μ is invariant for the analytic continuation of the group (τ_t). In particular, $\mu \in M_\sharp(\mathbb{G})$ and $\mu^\sharp = \mu^* \circ R$.

Theorem 12. *Let $\mu \in M(\mathbb{G})$ with $\mu\circ\tau_t = \mu$ for all t. There is a net (ω_α) in $L^1_\sharp(\mathbb{G})$ such that $\|\omega_\alpha\|_\sharp \le \|\mu\|$ for all α, and for every $x \in M(C_0(\mathbb{G}))$,*

$$\langle \omega_\alpha, x\rangle \to \langle \mu, x\rangle \qquad \textit{and} \qquad \langle \omega_\alpha^\sharp, x\rangle \to \langle \mu^\sharp, x\rangle = \overline{\langle \mu, R(x)^*\rangle}.$$

Proof. Assume without loss of generality that $\|\mu\| = 1$. Let $\mathcal{F}$ be the collection of finite subsets of $M(C_0(\mathbb{G}))$, and turn $\Lambda = \mathcal{F} \times \mathbb{N}$ into a directed set for the order $(F_1, n_1) \le (F_2, n_2)$ if and only if $F_1 \subseteq F_2$ and $n_1 \le n_2$. For each $\alpha = (F, n) \in \Lambda$ choose $r > 0$ so that $1 - e^{-r^2/4} < 1/n$. Choose N so that

$$\frac{r}{\sqrt{\pi}} \int_{-\infty}^{-N} e^{-r^2t^2}\, dt = \frac{r}{\sqrt{\pi}} \int_{N}^{\infty} e^{-r^2t^2}\, dt \le \frac{1}{n}. \tag{3}$$

By the Cohen Factorisation Theorem, there are $a \in C_0(\mathbb{G})$ and $\nu \in C_0(\mathbb{G})^*$ such that $\|a\| \le 1$, $\|\mu - \nu\| \le 1/n$ and $\mu = \nu a$, where $\nu a(b) = \nu(ab)$ for $b \in C_0(\mathbb{G})$ (see, for example, [27, Proposition A2]). By the Hahn–Banach and Goldstine Theorems, we can find a net $(\sigma_i)_{i\in I}$ in $L^1(\mathbb{G})$ converging weak* to ν in $M(\mathbb{G})$, and with $\|\sigma_i\| \le \|\nu\|$ for each i.

As the interval $[-N, N]$ is compact and the map $t \mapsto a\tau_t(x)$ is norm continuous for any $x \in F$ (since $t \mapsto \tau_t(x)$ is strictly continuous), it follows that we can find $i \in I$ with

$$\big|\langle \mu, \tau_t(x)\rangle - \langle \sigma_i a, \tau_t(x)\rangle\big| = \big|\langle \nu, a\tau_t(x)\rangle - \langle \sigma_i, a\tau_t(x)\rangle\big| \le \frac{\|x\|}{n} \tag{4}$$

whenever $t \in [-N, N]$ and $x \in F$. Set

$$\omega_\alpha = \frac{e^{-r^2/4}}{1+\frac{1}{n}} (\sigma_i a)(r, 0).$$

Since $\|\sigma_i a\| \le \|\nu\| \le (1 + 1/n)\|\mu\|$, we have $\|\omega_\alpha\| \le e^{-r^2/4}\|\mu\| \le 1$ and also $\|\omega_\alpha^\sharp\| \le 1$ by (2).

Now given $\epsilon > 0$ and $x \in M(C_0(\mathbb{G}))$, choose $\alpha_0 = (F, n_0)$ such that $\{x, R(x)^*\} \subseteq F$ and $n_0 \ge 1 + 7\|x\|/\epsilon$. Since $\mu\circ\tau_t = \mu$ for all t, we have for every $\alpha \ge \alpha_0$ that

$$\begin{aligned}
\big|\langle \mu, x\rangle - \langle \omega_\alpha, x\rangle\big| &= \Big|\frac{r}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-r^2t^2}\Big(\langle \mu, \tau_t(x)\rangle - \frac{e^{-r^2/4}}{1+\frac{1}{n}}\langle \sigma_i a, \tau_t(x)\rangle\Big)\, dt\Big| \\
&\le \frac{4\|x\|}{n} + \frac{r}{\sqrt{\pi}}\int_{-N}^{N} e^{-r^2t^2}\big|\langle \mu, \tau_t(x)\rangle - \langle \sigma_i a, \tau_t(x)\rangle\big|\, dt \qquad (5)\\
&\quad + \frac{r}{\sqrt{\pi}}\int_{-N}^{N} e^{-r^2t^2}\Big(1 - \frac{e^{-r^2/4}}{1+\frac{1}{n}}\Big)|\langle \sigma_i a, \tau_t(x)\rangle|\, dt
\end{aligned}$$

by (3). Now

$$\Big(1-\frac{e^{-r^2/4}}{1+\frac{1}{n}}\Big)|\langle\sigma_i a,\tau_t(x)\rangle| \leq (1+\frac{1}{n}-e^{-r^2/4})\|x\| \leq \frac{2\|x\|}{n}$$

by the choice of r. Continuing from (5) and applying also (4), we have

$$\big|\langle\mu,x\rangle-\langle\omega_\alpha,x\rangle\big| \leq \frac{4\|x\|}{n}+\frac{\|x\|}{n}+\frac{2\|x\|}{n}<\epsilon.$$

Now consider $\mu^\sharp=\mu^*\circ R$. Using the extension of μ to a strictly continuous functional on $M(C_0(\mathbb{G}))$, we can form $\mu(r,z)$ by an integral which converges weakly when measured against elements in $M(C_0(\mathbb{G}))$. As $\mu\circ\tau_t=\mu$ for all t, it follows that $\mu(r,0)=\mu$, and so

$$\mu^\sharp=\mu(r,0)^\sharp=\mu^*(r,-i/2)\circ R=\frac{r}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-r^2(t+i/2)^2}\mu^*\circ\tau_t\circ R\;dt.$$

Thus, by a similar calculation to the above,

$$\begin{aligned}
&\big|\langle\mu^\sharp,x\rangle-\langle\omega_\alpha^\sharp,x\rangle\big|\\
&=\Big|\frac{r}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-r^2(t+i/2)^2}\Big(\overline{\langle\mu,\tau_t(R(x)^*)\rangle}-\frac{e^{-r^2/4}}{1+\frac{1}{n}}\overline{\langle\sigma_i a,\tau_t(R(x)^*)\rangle}\Big)\;dt\Big|\\
&\leq\frac{4\|x\|}{n}e^{r^2/4}+\frac{r}{\sqrt{\pi}}\int_{-N}^{N}e^{-r^2t^2}e^{r^2/4}\big|\langle\mu,\tau_t(R(x)^*)\rangle-\langle\sigma_i a,\tau_t(R(x)^*)\rangle\big|\;dt\\
&\qquad+\frac{r}{\sqrt{\pi}}\int_{-N}^{N}e^{-r^2t^2}e^{r^2/4}\Big(1-\frac{e^{-r^2/4}}{1+\frac{1}{n}}\Big)|\langle\sigma_i a,\tau_t(R(x)^*)\rangle|\;dt\\
&\leq\frac{7\|x\|}{n}e^{r^2/4}\leq\frac{7\|x\|}{n-1}<\epsilon.
\end{aligned}$$

Hence $\omega_\alpha^\sharp\to\mu^\sharp$ weak* in $M(C_0(\mathbb{G}))^*$. □

We remark that a standard technique would allow us, in the case when $C_0(\mathbb{G})$ is separable, to replace the net (ω_α) by a sequence in the above.

Theorem 13. *Let $\mathbb{G}$ be coamenable. Then $L^1_\sharp(\mathbb{G})$ has a contractive approximate identity, in its natural norm.*

Proof. Recall that $\mathbb{G}$ is coamenable if and only if there is a state $\epsilon\in M(\mathbb{G})$ with $(\mathrm{id}\otimes\epsilon)\Delta=(\epsilon\otimes\mathrm{id})\Delta=\mathrm{id}$; see [1, Section 3]. It is easy to see that ϵ must be unique. For each t, as τ_t is a $*$-homomorphism which intertwines the coproduct, it follows by uniqueness that $\epsilon\circ\tau_t=\epsilon$. As above, then $\epsilon\in M_\sharp(\mathbb{G})$ with $\epsilon^\sharp=\epsilon^*\circ R=\epsilon$ (again by uniqueness). Thus we can apply the previous theorem to find a net (ω_α) in $L^1_\sharp(\mathbb{G})$ such that $\|\omega_\alpha\|_\sharp\leq 1$ for every α, and for every $x\in M(C_0(\mathbb{G}))$,

$$\langle\omega_\alpha,x\rangle\to\langle\epsilon,x\rangle\qquad\text{and}\qquad\langle\omega_\alpha^\sharp,x\rangle\to\langle\epsilon,x\rangle.$$

Now for every $\omega\in L^1(\mathbb{G})$ and $y\in L^\infty(\mathbb{G})$

$$\langle\omega_\alpha\star\omega,y\rangle=\langle\omega_\alpha,(\mathrm{id}\otimes\omega)\Delta(y)\rangle\to\langle\epsilon,(\mathrm{id}\otimes\omega)\Delta(y)\rangle$$

because $(\mathrm{id}\otimes\omega)\Delta(y)\in M(C_0(\mathbb{G}))$ (see [31, Lemma 5.2] or [30, Theorem 2.3]). Indeed, as $W\in M\big(C_0(\mathbb{G})\otimes\mathcal{K}(L^2(\mathbb{G}))\big)$, where $\mathcal{K}(L^2(\mathbb{G}))$ is the collection of compact operators on $L^2(\mathbb{G})$, we see that

$$\Delta(y)=W^*(1\otimes y)W\in M\big(C_0(\mathbb{G})\otimes\mathcal{K}(L^2(\mathbb{G}))\big)\cap L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G}),$$

thinking of both algebras concretely acting on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$. In particular, if $\tau \in \mathcal{K}(L^2(\mathbb{G}))^*$ is any functional which, when restricted to $L^\infty(\mathbb{G})$, agrees with ω, then

$$(\mathrm{id} \otimes \omega)\Delta(y) = (\mathrm{id} \otimes \tau)(W^*(1 \otimes y)W) \in M(C_0(\mathbb{G})).$$

Moreover, arguing using the strict topology, for $\omega_1, \omega_2 \in L^1(\mathbb{G})$,

$$\begin{aligned}\langle \epsilon, (\mathrm{id} \otimes (\omega_1 \star \omega_2))\Delta(y)\rangle &= \langle \epsilon, (\mathrm{id} \otimes \omega_1)\Delta((\mathrm{id} \otimes \omega_2)\Delta(y))\rangle \\ &= \langle \epsilon \star \omega_1, (\mathrm{id} \otimes \omega_2)\Delta(y)\rangle = \langle \omega_1 \star \omega_2, y\rangle,\end{aligned}$$

and as convolutions are linearly dense in $L^1(\mathbb{G})$ (due to Δ being injective), we have

$$\langle \epsilon, (\mathrm{id} \otimes \omega)\Delta(y)\rangle = \langle \omega, y\rangle.$$

So $\omega_\alpha \star \omega \to \omega$ weakly, and similarly $\omega \star \omega_\alpha \to \omega$, $\omega_\alpha^\sharp \star \omega \to \omega$ and $\omega \star \omega_\alpha^\sharp \to \omega$, all in the weak topology. Now we apply a standard convexity argument to obtain a contractive approximate identity in $L^1_\sharp(\mathbb{G})$. We give a short sketch of the argument; for more details, see for example Day [9, pages 523–524]. First, we obtain a bounded left approximate identity (τ_β) consisting of convex combinations of elements in (ω_α). Since the convex combinations are taken from elements further and further along the net (ω_α), we have $\omega \star \tau_\beta \to \omega$ weakly. Thus we can iterate the construction and obtain a two-sided approximate identity (σ_γ), still consisting of convex combinations of elements in (ω_α). Repeating this process twice more, we obtain a net (e_δ) such that both (e_δ) and $(e_\delta^\sharp)$ are a two-sided approximate identities in $L^1(\mathbb{G})$. Then (e_δ) is a contractive approximate identity in $L^1_\sharp(\mathbb{G})$ since for example

$$\|e_\delta \star \omega - \omega\|_\sharp = \max(\|e_\delta \star \omega - \omega\|, \|\omega^\sharp \star e_\delta^\sharp - \omega^\sharp\|) \to 0$$

for every $\omega \in L^1_\sharp(\mathbb{G})$. □

Proposition 14. *Let $\mathbb{G}$ be coamenable. For any $\mu \in M_\sharp(\mathbb{G})$, there is a net (ω_α) in $L^1_\sharp(\mathbb{G})$ which converges weak* to μ in $M(\mathbb{G})$, and with $\|\omega_\alpha\|_\sharp \le \|\mu\|_\sharp$ for all α.*

Proof. Let (τ_α) be a contractive approximate identity in $L^1_\sharp(\mathbb{G})$. We know from [23, pages 913–914] that $L^1(\mathbb{G})$ is a two-sided ideal in $M(\mathbb{G})$. Thus we can set $\omega_\alpha = \mu \star \tau_\alpha$, and we have that (ω_α) is a net in $L^1(\mathbb{G})$, bounded by $\|\mu\|$. As $\mu \in M_\sharp(\mathbb{G})$ and $\tau_\alpha \in L^1_\sharp(\mathbb{G})$, it is easy to see that $\omega_\alpha \in L^1_\sharp(\mathbb{G})$ with $\omega_\alpha^\sharp = \tau_\alpha^\sharp \star \mu^\sharp$, and so $\|\omega_\alpha^\sharp\| \le \|\mu^\sharp\|$.

For $a \in C_0(\mathbb{G})$ and $\lambda \in M(\mathbb{G})$, both $(\lambda \otimes \mathrm{id})\Delta(a)$ and $(\mathrm{id} \otimes \lambda)\Delta(a)$ are members of $C_0(\mathbb{G})$ (since $\lambda = \lambda' b$ for some $\lambda' \in M(\mathbb{G})$, $b \in C_0(\mathbb{G})$ and both $(b \otimes 1)\Delta(a)$ and $(1 \otimes b)\Delta(a)$ are in $C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$). Hence for $a \in C_0(\mathbb{G})$,

$$\lim_\alpha \langle \omega_\alpha, a\rangle = \lim_\alpha \langle \tau_\alpha, (\mu \otimes \mathrm{id})\Delta(a)\rangle = \langle \epsilon, (\mu \otimes \mathrm{id})\Delta(a)\rangle = \langle \mu, a\rangle,$$

as required. □

We do not see why the conclusions of this proposition require that $\mathbb{G}$ be coamenable; however, a proof in the general case has eluded us.

6. Bochner's theorem in the coamenable case

The following is an analogue of Bochner's theorem for coamenable locally compact quantum groups.

Theorem 15. *Suppose that $\mathbb{G}$ is a coamenable locally compact quantum group. Then $x \in L^\infty(\mathbb{G})$ is positive definite if and only if there is a positive functional $\hat{\mu} \in C_0^u(\hat{\mathbb{G}})^*$ such that $x = (\mathrm{id} \otimes \hat{\mu})(\mathcal{W}^*)$.*

Proof. As we have already noted that the converse holds, we only need to show that a positive definite $x \in L^\infty(\mathbb{G})$ is of the form $(\mathrm{id} \otimes \hat{\mu})(\mathcal{W}^*)$.

Let (Λ, H, π) be as in Lemma 8, applied to x. By Theorem 13, there is a contractive approximate identity (e_α) in $L^1_\sharp(\mathbb{G})$. As $\|\Lambda(e_\alpha)\| \leq \|x\|^{1/2}$ for all α, the net $\Lambda(e_\alpha)$ clusters weakly at some $\xi \in H$. Observe that then $\pi(\omega)\xi = \Lambda(\omega)$ for every $\omega \in L^1_\sharp(\mathbb{G})$.

Define $\hat{\mu} \in C_0^u(\mathbb{G})^*_+$ by $\hat{\mu} \circ \lambda_u = \omega_{\xi,\xi} \circ \pi$ where $\lambda_u \colon L^1_\sharp(\mathbb{G}) \to C_0^u(\hat{\mathbb{G}})$ is the universal representation defined by $\lambda_u(\omega) = (\omega \otimes \mathrm{id})(\mathcal{W})$. Then $\hat{\mu}$ is well-defined because λ_u is injective and bounded due to the universality of λ_u. Moreover, for every $\omega \in L^1_\sharp(\mathbb{G})$,

$$\langle (\mathrm{id} \otimes \hat{\mu})(\mathcal{W}^*)^*, \omega \rangle = \langle \hat{\mu}, \lambda_u(\omega) \rangle = (\pi(\omega)\xi \mid \xi) = (\Lambda(\omega) \mid \xi) = \langle x^*, \omega \rangle.$$

□

Remark 16. An alternative (and much less direct) way to prove this result is to use Corollary 11. Indeed, as $\pi(\omega)\xi = \Lambda(\omega)$, and by [20] there is a unitary corepresentation U such that $\pi(\omega) = (\omega \otimes \mathrm{id})(U)$, it follows that $(\Lambda(\omega^\sharp) \mid f_i) = (\pi(\omega)^*\xi \mid f_i) = (\xi \mid (\omega \otimes \mathrm{id})(U) f_i) = \overline{\langle a_i^*, \omega \rangle}$ with $a_i^* = (\mathrm{id} \otimes \omega_{f_i,\xi})(U)$. However, then $\sum_i a_i^* a_i = \sum_i (\mathrm{id} \otimes \omega_{f_i,\xi})(U)(\mathrm{id} \otimes \omega_{\xi,f_i})(U^*) = (\mathrm{id} \otimes \omega_{\xi,\xi})(UU^*) = \|\xi\|^2 < \infty$, as required to invoke Corollary 11.

The following example shows that without coamenability, positive definiteness is not enough for a Bochner type representation.

Example 17. Let $\mathbb{F}_2$ be the free group on two generators and let $\mathbb{G}$ be the dual of $\mathbb{F}_2$ (so that $L^\infty(\mathbb{G}) = VN(\mathbb{F}_2)$, the group von Neumann algebra, and $L^1(\mathbb{G}) = L^1_\sharp(\mathbb{G}) = A(\mathbb{F}_2)$, the Fourier algebra). There exists an infinite subset $E \subseteq \mathbb{F}_2$ such that the map given by restriction of functions induces a surjection $\theta \colon A(\mathbb{F}_2) \to \ell^2(E)$, see [26, equation (2.1)]. The adjoint $\theta^* \colon \ell^2(E) \to VN(\mathbb{F}_2)$ is hence an isomorphism onto its range. Let $x' \in \ell^2(E)$ and set $x = \theta^*(x')$. As the involution on $A(\mathbb{F}_2)$ is pointwise conjugation of functions, it follows that x is positive definite if and only if x' is pointwise non-negative.

For $a, b \in A(\mathbb{F}_2)$, we have that

$$(\Lambda(a) \mid \Lambda(b))_H = \langle x, b^\sharp \star a \rangle = \langle x', \theta(b^\sharp)\theta(a) \rangle = \sum_{s \in E} x'(s) a(s) \overline{b(s)}.$$

It follows that we have an isomorphism

$$H \to \ell^2(E); \quad \Lambda(a) \mapsto \big(a(s) x'(s)^{1/2}\big)_{s \in E}.$$

The action π of $A(\mathbb{F}_2)$ on $H \cong \ell^2(E)$ is just multiplication of functions. It follows that the set of coefficient functionals of π is precisely $\ell^1(E)$, and we can hence recover x if and only if $x' \in \ell^1(E) \subseteq \ell^2(E)$. This immediately tells us that the method of the proof of Theorem 15 fails for any $x' \in \ell^2(E) \setminus \ell^1(E)$.

Furthermore, we claim that there is no (positive) functional in $C_0^u(\hat{\mathbb{G}})^* = \ell^1(\mathbb{F}_2)$ which gives a Bochner type representation for such x (arising from $x' \in \ell^2(E) \setminus \ell^1(E)$). Indeed, suppose towards a contradiction that $\hat{\mu} \in \ell^1(\mathbb{F}_2)$ is such that $x = (\mathrm{id} \otimes \hat{\mu})(\mathcal{W}^*)$ (i.e. that x is the "Fourier–Stieltjes transform of a positive measure"

$\hat{\mu}$). In this setting $C_0^u(\hat{\mathbb{G}}) = C_0(\hat{\mathbb{G}}) = c_0(\mathbb{F}_2)$ and $\mathcal{W}^* = W^* \in \mathcal{B}(\ell^2(\mathbb{F}_2) \otimes \ell^2(\mathbb{F}_2))$ is the operator

$$W^*(e_s \otimes e_t) = e_{ts} \otimes e_t,$$

where $(e_t)_{t\in\mathbb{F}_2}$ is the canonical orthonormal basis of $\ell^2(\mathbb{F}_2)$. Let $a = \omega_{\xi,\eta} \in A(\mathbb{F}_2)$ where $\xi = (\xi_s), \eta = (\eta_t)$ are in $\ell^2(\mathbb{F}_2)$. Then

$$\begin{aligned}\langle x, a\rangle &= \big(x(\xi) \mid \eta\big) = \sum_t \hat{\mu}_t \big(W^*(\xi \otimes e_t) \mid \eta \otimes e_t\big) \\ &= \sum_{t,s,r} \hat{\mu}_t \xi_s \overline{\eta_r} \big(W^*(e_s \otimes e_t) \mid e_r \otimes e_t\big) = \sum_{t,s} \hat{\mu}_t \xi_s \overline{\eta_{ts}} = \sum_{t,r} \hat{\mu}_t \xi_{t^{-1}r} \overline{\eta_r}.\end{aligned}$$

On the other hand, if we view $A(\mathbb{F}_2)$ as a space of functions on $\mathbb{F}_2$ via the embedding $A(\mathbb{F}_2) \to c_0(\mathbb{F}_2); \omega_{\xi,\eta} \mapsto (\omega_{\xi,\eta} \otimes \mathrm{id})(W)$ then that $x = \theta^*(x')$ means that

$$\begin{aligned}\langle x, a\rangle &= \sum_{t\in E} x'(t) \langle (\omega_{\xi,\eta} \otimes \mathrm{id})(W), \omega_{e_t,e_t}\rangle \\ &= \sum_{t\in E} \sum_{s\in\mathbb{F}_2} x'(s) \xi_s \overline{\eta_{t^{-1}s}} = \sum_{t\in E} \sum_{r\in\mathbb{F}_2} x'(s) \xi_{tr} \overline{\eta_r}.\end{aligned}$$

If this is true for all ξ, η, it follows that $\hat{\mu}_t = x'(t^{-1})$ for all $t^{-1} \in E$ and $\hat{\mu}_t = 0$ otherwise, which is a contradiction, as $x' \in \ell^2(E) \setminus \ell^1(E)$. (We remark that [14] uses the embedding $A(\mathbb{F}_2) \to c_0(\mathbb{F}_2); \omega_{\xi,\eta} \mapsto (\omega_{\xi,\eta} \otimes \mathrm{id})(W^*)$ but this also leads to exactly the same contradiction.)

7. n-positive multipliers

The main result of this section is a characterisation of n-positive multipliers on coamenable locally compact quantum groups. The proof relies on Bochner's theorem from the previous section. The present section is inspired by the work of De Cannière and Haagerup [10], and the results are extensions of theirs concerning the commutative case when $\mathbb{G} = G$ is a locally compact group.

Recall that

$$\hat{\lambda}\colon C_0(\hat{\mathbb{G}})^* \to M(C_0(\mathbb{G})), \qquad \hat{\lambda}(\hat{\mu}) = (\hat{\mu} \otimes \mathrm{id})(\hat{W}) = (\mathrm{id} \otimes \hat{\mu})(W^*).$$

The one-sided universal analogue of this is

$$\hat{\lambda}^u\colon C_0^u(\hat{\mathbb{G}})^* \to M(C_0(\mathbb{G})), \qquad \hat{\lambda}^u(\hat{\mu}) = (\mathrm{id} \otimes \hat{\mu})(\mathcal{W}^*).$$

Lemma 18. *Let $x \in M(C_0(\mathbb{G}))$ be such that $x\hat{\lambda}(\hat{\omega}) \in \hat{\lambda}^u\big(C_0^u(\hat{\mathbb{G}})^*\big)$ for every $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. Then x is a left multiplier of $L^1(\hat{\mathbb{G}})$.*

Proof. Let $\hat{\pi}\colon C_0^u(\hat{\mathbb{G}}) \to C_0(\hat{\mathbb{G}})$ denote the canonical quotient map. Note that $\hat{\lambda}(\hat{\omega}) = \hat{\lambda}^u(\hat{\omega} \circ \hat{\pi})$ for every $\hat{\omega} \in L^1(\hat{\mathbb{G}})$. Define $\widetilde{L}_x\colon L^1(\hat{\mathbb{G}}) \to C_0^u(\hat{\mathbb{G}})^*$ by

$$\hat{\lambda}^u(\widetilde{L}_x(\hat{\omega})) = x\hat{\lambda}(\hat{\omega}).$$

Since $\hat{\lambda}^u$ is injective (because left slices of $\mathcal{W}^*$ are dense in $C_0^u(\hat{\mathbb{G}})$) $\widetilde{L}_x$ is well-defined. The proof of Lemma 2 also works in this universal setting, and so $\widetilde{L}_x$ is bounded.

By hypothesis, for $\hat{\omega}, \hat{\tau} \in L^1(\hat{\mathbb{G}})$,

$$x\hat{\lambda}(\hat{\omega} \star \hat{\tau}) = \big(x\hat{\lambda}(\hat{\omega})\big)\hat{\lambda}(\hat{\tau}) \in \hat{\lambda}\big(L^1(\hat{\mathbb{G}})\big)$$

because $L^1(\hat{\mathbb{G}}) \circ \hat{\pi}$ is an ideal in $C_0^u(\hat{\mathbb{G}})^*$, see for example [8, Proposition 8.3]. Since convolution products are linearly dense in $L^1(\hat{\mathbb{G}})$, the above shows that for every $\hat{\omega} \in L^1(\hat{\mathbb{G}})$ there is a sequence $(\hat{\omega}_n) \subseteq L^1(\hat{\mathbb{G}})$ such that $\hat{\omega}_n \to \hat{\omega}$ in norm and $x\hat{\lambda}(\hat{\omega}_n) \in \hat{\lambda}\big(L^1(\hat{\mathbb{G}})\big)$. Since $\widetilde{L}_x$ is bounded, $\widetilde{L}_x(\hat{\omega}_n) \to \widetilde{L}_x(\hat{\omega})$ in $C_0^u(\hat{\mathbb{G}})^*$. As $\hat{\pi}$ is a metric surjection, $L^1(\hat{\mathbb{G}}) \circ \hat{\pi}$ is isometrically isomorphic to $L^1(\hat{\mathbb{G}})$, and so $\widetilde{L}_x(\hat{\omega}) \in L^1(\hat{\mathbb{G}}) \circ \hat{\pi}$. Hence $x\hat{\lambda}(\hat{\omega}) \in \hat{\lambda}\big(L^1(\hat{\mathbb{G}})\big)$ as claimed. □

The following result extends Proposition 4.3 of [10].

Proposition 19. *Suppose that $\mathbb{G}$ is coamenable. The following are equivalent for $x \in M(C_0(\mathbb{G}))$*
(1) x is an n-positive multiplier
(2) for every $(\alpha_i)_{i=1}^n \in L^2(\mathbb{G})^n$ and $(\omega_i)_{i=1}^n \in L^1_\sharp(\mathbb{G})^n$

$$\Big\langle x^*, \sum_{i,j=0}^n (\omega_j \star \omega_i^\sharp) \cdot \hat{\lambda}(\omega_{\alpha_j,\alpha_i})^* \Big\rangle \geq 0.$$

Here $\cdot$ denotes the action of $L^\infty(\mathbb{G})$ on $L^1(\mathbb{G})$.

Proof. Suppose that x is an n-positive multiplier. Let $(\omega_i)_{i=1}^n \in L^1_\sharp(\mathbb{G})^n$, and note that $[\lambda(\omega_i \star \omega_j^\sharp)] \geq 0$ in $M_n(C_0(\hat{\mathbb{G}})) \subseteq \mathcal{B}(L^2(\mathbb{G})^n)$. Since L_x^* is n-positive, we have for every $(\alpha_i)_{i=1}^n \in L^2(\mathbb{G})^n$

$$\begin{aligned}
0 &\leq \sum_{i,j=1}^n (L_x^*((\lambda(\omega_i \star \omega_j^\sharp)))\alpha_j \mid \alpha_i) = \sum_{i,j=1}^n \langle \lambda(\omega_i \star \omega_j^\sharp), L_x(\omega_{\alpha_j,\alpha_i})\rangle \\
&= \sum_{i,j=1}^n \big((\omega_i \star \omega_j^\sharp) \otimes L_x(\omega_{\alpha_j,\alpha_i})\big)(W) = \sum_{i,j=1}^n \big((\omega_i \star \omega_j^\sharp)^{\sharp *} \otimes L_x(\omega_{\alpha_j,\alpha_i})\big)(W^*) \\
&= \sum_{i,j=1}^n \langle \hat{\lambda}\big(L_x(\omega_{\alpha_j,\alpha_i})\big), (\omega_j \star \omega_i^\sharp)^* \rangle = \sum_{i,j=1}^n \langle x\hat{\lambda}\big(\omega_{\alpha_j,\alpha_i}\big), (\omega_j \star \omega_i^\sharp)^* \rangle \\
&= \overline{\Big\langle x^*, \sum_{i,j=0}^n (\omega_j \star \omega_i^\sharp) \cdot \hat{\lambda}(\omega_{\alpha_j,\alpha_i})^* \Big\rangle},
\end{aligned}$$

where we used the fact that $S\big((\mathrm{id} \otimes \tau)(W)\big) = (\mathrm{id} \otimes \tau)(W^*)$ for $\tau \in \mathcal{B}(L^2(\mathbb{G}))_*$. The calculation shows that (2) holds.

Conversely, suppose that (2) holds. For every $x \in M_n(L^\infty(\hat{\mathbb{G}}))_+$, there is a net $(a_\alpha) \in M_n(C_0(\hat{\mathbb{G}}))_+$ converging weak* to x, and by [28, Lemma 3.13], every a_α is a sum of n matrices of the form $[b_i b_j^*]_{i,j=1}^n$ with $(b_i)_{i=1}^n \in C_0(\hat{\mathbb{G}})^n$. Hence the density of $L^1_\sharp(\mathbb{G})$ in $L^1(\mathbb{G})$ implies that the linear span of matrices of the form $[\lambda(\omega_i \star \omega_j^\sharp)]$ with $(\omega_i)_{i=1}^n \in L^1_\sharp(\mathbb{G})^n$ is weak*-dense in $M_n(L^\infty(\hat{\mathbb{G}}))_+$. Therefore the calculation in the first part of the proof shows that L_x is n-positive, assuming that x is a left multiplier of $L^1(\hat{\mathbb{G}})$. We shall show that x is indeed a multiplier by applying Lemma 18. For $\alpha \in L^2(\mathbb{G})$, write $\omega_\alpha = \omega_{\alpha,\alpha}$. Since x is 1-positive, each $x\hat{\lambda}(\omega_\alpha)$ is positive definite. Hence, by Theorem 15, $x\hat{\lambda}(\omega_\alpha) \in \hat{\lambda}_u(C_0^u(\hat{\mathbb{G}})^*)$. But since $L^1(\hat{\mathbb{G}})$ is spanned by elements of the form ω_α, an application of Lemma 18 implies that x is a left multiplier of $L^1(\hat{\mathbb{G}})$. □

The following result extends Corollary 4.4 of [10].

Proposition 20. *Suppose that $\hat{\mathbb{G}}$ is coamenable. Every n-positive multiplier $x \in M(C_0(\mathbb{G}))$ is positive definite.*

Proof. Since x is, in particular, a 1-positive multiplier,

$$\left\langle x^*, (\omega \star \omega^\sharp) \cdot \hat{\lambda}(\omega_\alpha)^* \right\rangle \geq 0$$

for every $\alpha \in L^2(\mathbb{G})$ and $\omega \in L^1_\sharp(\mathbb{G})$ (by Proposition 19; this part does not rely on coamenability). That is, $x\hat{\lambda}(\omega_\alpha)$ is positive definite. By coamenability of $\hat{\mathbb{G}}$, there exists a net (α_i) of unit vectors in $L^2(\mathbb{G})$ such that

$$\|W^*(\xi \otimes \alpha_i) - (\xi \otimes \alpha_i)\| \to 0$$

for every $\xi \in L^2(\mathbb{G})$ ([1, Theorem 3.1], recall that $\hat{W} = \sigma W^* \sigma$). For every $\xi, \eta \in L^2(\mathbb{G})$, we have

$$|\langle \hat{\lambda}(\omega_{\alpha_i}), \omega_{\xi,\eta}\rangle - \langle 1, \omega_{\xi,\eta}\rangle| = |(W^*(\xi \otimes \alpha_i) - (\xi \otimes \alpha_i) \mid \eta \otimes \alpha_i)|,$$

and so $\hat{\lambda}(\omega_{\alpha_i}) \to 1$ weak* in $L^\infty(\mathbb{G})$. Since $x\hat{\lambda}(\omega_{\alpha_i})$ is positive definite and positive definiteness is preserved by weak* limits, it follows that x is positive definite. □

Corollary 21. *Suppose that both $\mathbb{G}$ and $\hat{\mathbb{G}}$ are coamenable. Then the following are equivalent for $x \in M(C_0(\mathbb{G}))$.*
(1) x is positive definite.
(2) x is an n-positive multiplier for some natural number n.
(3) x is completely positive definite.

Even without the assumptions on coamenability, a completely positive definite function is always positive definite and an n-positive multiplier. However, we have the following counterexamples:

- Example 17 shows that, in general, a positive define function need not be an n-positive multiplier for every n. In this example $\mathbb{G}$ is not coamenable but $\hat{\mathbb{G}}$ is.
- De Cannière and Haagerup showed in [10, Corollary 4.8], that if $\mathbb{G}$ is the free group $\mathbb{F}_N$ on $N \geq 2$ generators and $n \geq 1$, then there exist n-positive multipliers of $L^1(\hat{\mathbb{G}})$ which are not positive definite. In this example $\mathbb{G}$ is coamenable but $\hat{\mathbb{G}}$ is not.

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