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A GENERALIZED FREQUENCY RESPONSE FOR NONLINEAR SYSTEMS

by

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Abstract

The principal aim of this paper is to present the idea of a 'generalized frequency response' of a nonlinear input-output map S_{x_0}. It is defined as $\mathcal{J} S_{x_0} \mathcal{J}^{-1}$ where $\mathcal{J}$ is the usual isomorphism from $L^2(o,\infty)$ to ℓ^2. Realization results are presented pertaining to linear, bilinear and nonlinear systems.

1. Introduction

The theory of linear systems has been developed from two points of view.- the time-domain state-space theory and the frequency-domain transfer-function approach. Each method has its advantages and disadvantages and, as is well-known, the classical theory of control is developed mainly in the (complex) frequency domain, while the theory of optimal control was initially developed in state space. The two methods can, however, be regarded as equivalent since the Four ier transform provides an isomorphism between the two representations of a system. Thus it is not surprising that state-space methods have frequency-domain counterparts and vice-versa ; consider, for example, the recent H^∞ methods for frequency-domain optimisation.

When we come to nonlinear systems, however, we are faced with what is essentially a state-space theory since it does not appear to be sensible even to consider the frequency response of such a system. Nevertheless, there have been attempts in the literature to define some kind of 'frequency domain' theory for nonlinear systems (see [1] , [6]). The method consists of finding a Volterra series expansion of the input-output map of the system and associating a sequence of transfer functions $H_k(s_1,\dots,s_k)$, $k\geq 1$, each one being the multi-dimensionsl Laplace transform of the corresponding Volterra series kernel. Setting $s_i=j\omega_i$ gives the 'frequency response' of the k^{th} kernel, namely $H_k(j\omega_1,\dots,j\omega_k)$. However this approach has two immediate difficulties; firstly, we obtain frequency spaces of increasing dimension and secondly, it is not at all clear how the functions $H_k(j\omega_1,\dots,j\omega_k)$ relate to the responses of the system to standard inputs (e.g. complex exponentials).

In this paper we shall take a more pragmatic approach and define a 'generalized frequency response' for a nonlinear system in terms of the way it responds to any input in terms of the components of the input with respect to some 'standard functions'. Thus we shll suppose that the input and output functions belong to $L^2[o,\infty]$ and take as standard functions some basis of this space (as a Hilbert space). Using the induced isomorphism of $L^2[o,\infty]$ with ℓ^2 we obtain a 'generalized frequency response' which is just an analytic map $s:\ell^2\to\ell^2$.

Throughout the paper we shall discuss the method relative to linear, bilinear and general nonlinear analytic systems. In particular, in the linear case we shall show that the method reduces essentially to the classical frequency response of a linear system in the sense that the map $s:\ell^2\to\ell^2$ is linear and has a block diagonal matrix representation, each block being a 2 x 2 matrix which can be decomposed into a rotation and di lation corresponding to the phase shift and amplitude response of the system at the frequency of the corresponding basis function.

In section 2 we shall specify some notation and in section 3 we shall present a new input-output representation for a nonlinear analytic system. In section 4 we shall define the generalized frequency response of a nonlinear system and in section 5 the realization of frequency response maps $s:\ell^2\to\ell^2$ will be discussed.

2. Notation and terminology:

In this paper we shall use the following notation. An n-multi-index is an n-tuple $i=(i_1,\ldots,i_n)$ of non-negative integers; its length (or order) is given by $|i|=i_1+\ldots+i_n$. The sum of two multi-indices i and ℓ is defined as $i+\ell=(i_1+\ell_1,\ldots,i_n+\ell_n)$. We say that $i\leq\ell$ if $i_k\leq\ell_k$ for $k=1,\ldots,n$. When $i\leq\ell$ we define $\ell-i$ as $(\ell_1-i_1,\ldots,\ell_n-i_n)$. We also define $i! = i_1!\ldots i_n!$ and $x^i=x_1^{i_1}\ldots x_n^{i_n}$ for $x = (x_1,\ldots,x_n)^T$ R^n. $1(r)$ will denote the n-multi-index with 1 in the r^{th} place and zero elsewhere, and $\delta_i^\ell = \delta_{i_1}^{\ell_1}\ldots\delta_{i_n}^{\ell_n}$ where $\delta_{i_k}^{\ell_k} = 1$ if $i_k = \ell_k$ and $\delta_{i_k}^{\ell_k} = o$ if $i_k \neq \ell_k$.

For an analytic function $h: R^n\to R$, Taylor's formula becomes

$$h(x) = \sum_{|i|\geq o} \frac{x^i}{i!} h^{(|i|)}(o)$$

where

$$h^{(|i|)}(x)= \partial^{|i|} h(x)/\partial x_1^{i_1}\ldots\partial x_n^{i_n}$$

Let ℓ^2 denote the standard Banach space of square summable sequences and let ℓ_e^2 [2] denote the Banach space of sequences $(\alpha_n)_{n\geq o}$ such that the sequence $(\alpha_n/n!)_{n\geq o}$ belongs to ℓ^2. Define a norm on ℓ_e^2 by

$$||(\alpha_n)_{n\geq o}||= \Big(\sum_{n\geq o} \frac{\alpha_n^2}{(n!)^2}\Big)^{1/2}, \quad (\alpha_n)_{n\geq o} \in \ell_e^2$$

Now consider the algebraic tensor product of n copies of ℓ_e^2, $\mathcal{L}_n = \otimes_n \ell_e^2$, and let $||.||$ be any cross norm on $\mathcal{L}_n$. For a simple tensor ϕ we have $\phi= (\phi_{i_1\ldots i_n}) = (\alpha_{i_1}^1\ldots\alpha_{i_n}^n) = \alpha^1\otimes\ldots\otimes\alpha^n$ where $\alpha^n = (\alpha_{i_k}^k)_{i_k\geq o}\in \ell_e^2$, $k=1,\ldots,n$.

Then, $||\phi|| =\prod_{k=1}^n ||\alpha^k||_e$.

The standard $L^2[o,\infty]$ and ℓ^2 spaces will also be used.

3. The Nonlinear Input-Output map:

In this section we shall consider an analytic system of the form

$$\begin{cases} \dot{x} = f(x,u) \quad ,x(o)=x_o & (3.1) \\ y = g(x) & (3.2) \end{cases}$$

where $x_o \in R^n$, $u \in R$; $f : R^n \times R \to R^n$ and $g: R^n \to R$ are analytic functions. (Thus, we are considering the single-input, single-output case for simplicity). We shall assume that the solution of (3.1) exist for all time and for each $x_o \in R^n$, although systems with finite escape times can be treated similarly (except that the solutions $x(t,x_o)$ of (3.1) are not analytic in t for all t.).
We shall require the input-output map $S: u \to y$ for the system (3.1), (3.2) which can be obtained by using the method of Carleman Linearization [3]. (See also [5]).

To obtain this map we introduce the functions

$$\phi_{i_1 \dots i_n}(x) = x_1^{i_1} \dots x_n^{i_n} \triangleq x^i$$

Since f(x,.) is analytic we have

$$f(x,u) = \sum_{j \geq o} \frac{u^j}{j!} f^{(o,j)}(x,o)$$

where $f^{(o,j)}$ denotes the j^{th} derivation of f(x,u) with respect to u. By the analyticity of f with respect to x, we have

$$f(x,u) = \sum_{j \geq o} \frac{u^j}{j!} \sum_{|i| \geq o} \frac{1}{i!} f^{(|i|,j)}(o,o) x^i$$

where $i = (i_1, \dots, i_n)$.

Hence we have

$$\dot{\phi}_{i_1 \dots i_n} = \sum_{r=1}^{n} i_r x^{i-1(r)} \dot{x}_r$$

and so

$$\begin{aligned}
\dot{\phi}_{i_1 \dots i_n} &= \sum_{r=1}^{n} i_r x^{i-1(r)} f_r(x,u) \\
&= \sum_{r=1}^{n} i_r x^{i-1(r)} \sum_{j \geq o} \frac{u^j}{j!} \sum_{|\ell| \geq o} \frac{1}{\ell!} f_r^{(|\ell|,j)}(o,o) x^{\ell} \\
&= \sum_{j \geq o} u^j \sum_{r=1}^{n} i_r \sum_{|\ell| \geq o} \frac{f_r^{(|\ell|,j)}}{\ell! j!} (o,o) x^{i-1(r)+\ell} \\
&= \sum_{j \geq o} u^j \sum_{r=1}^{n} i_r \sum_{|\ell| \geq o} \sum_{|p| \geq o} \frac{f_r^{(|\ell|,j)}}{\ell! j!} (o,o) \delta_p^{i-1(r)+\ell} x^p
\end{aligned}$$

We therefore have

$$\dot{\phi}_{i_1 \dots i_n} = \sum_{j \geq o} u^j \sum_{|p| \geq o} a_i^p(j) x^p$$

where

$$a_i^p(j) = \begin{cases} \sum_{r=1}^{n} \frac{i_r}{(p-i+1(r))!\,j!} f_r^{(|p-i+1(r)|,j)}(o,o) & \text{if } p-i+1(r) \geq o \\ o & \text{otherwise} \end{cases}$$

Finally, we have

$$\dot{\Phi} = A_o \Phi + \sum_{j\geq 1} u^j A_j \Phi \tag{3.3}$$

where Φ is the tensor with components $\phi_{i_1 \ldots i_n}$ and A_j is the tensor operator defined by

$$(A_j\Phi)_i = \sum_{|p|\geq o} a_i^p(j)\, x^p = \sum_{|p|\geq o} a_i^p(j)\, \phi_p$$

We can solve equation (3.3) by standard Picard iteration ;thus define

$$\Psi_o(t) = e^{A_o t} \Phi_o \tag{3.4}$$

$$\Psi_k(t) = \sum_{j\geq 1} \int_o^t e^{A_o(t-\tau)} u^j(\tau) A_j \Psi_{k-1}(\tau)\, d\tau, k\geq 1$$

Using methods developed in [3], it can be shown that the solution of (3.3) is given by

$$\Phi(t) = \sum_{k\geq o} \Psi_k(t)$$

where the series on the right hand side converges in ℓ_e^2 for bounded controls.

Explicitly we have

$$\Psi_k(t) = \sum_{i_1\geq 1} \cdots \sum_{i_k\geq 1} \int_o^t \int_o^{\sigma_k} \cdots \int_o^{\sigma_2} \bar{v}_k^{i_1 \ldots i_k}(t,\sigma_1 \ldots,\sigma_k)\, u^{i_k}(\sigma_k) \ldots u^{i_1}(\sigma_1)\, d\sigma_1 \ldots d\sigma_k \tag{3.5}$$

where

$$\bar{v}_k^{i_1 \ldots i_k}(t,\sigma_1,\ldots,\sigma_k) = e^{A_o(t-\sigma_k)} A_{i_k} e^{A(\sigma_k - \sigma_{k-1})} A_{i_{k-1}} \ldots A_1 e^{A_o \sigma_1} \Phi_o$$

From (3.2) we have

$$y = g(x) = \sum_{|p|\geq o} \frac{g^{|p|}(o)}{p!} x^p = G\,\Phi$$

for some tensor operator G. Hence the input-output map of the system (3.1), (3.2) is given by

$$y(t) = v_o(t) + \sum_{k\geq 1} \sum_{i_1 \geq 1} \cdots \sum_{i_k \geq 1} \int_o^t \int_o^{\sigma_k} \cdots \int_o^{\sigma_2} v_k^{i_1 \ldots i_k}(t,\sigma_1,\ldots,\sigma_k)\, u^{i_k}(\sigma_k) \ldots u^{i_1}(\sigma_1)\, d\sigma_1 \ldots d\sigma_k \tag{3.6}$$

when

$$v_o(t) = G\, e^{A_o t}\, \Phi_o$$

and

$$v_k^{i_1 \ldots i_k}(t,\sigma_1,\ldots,\sigma_k) = G\, \bar{v}_k^{i_1 \ldots i_k}(t,\sigma_1,\ldots,\sigma_k)$$

4. Spectral theory of Nonlinear Input-Output Maps:

Consider a System S given in terms of an input-output map

$$S : R^n \times L^2[o,\infty] \rightarrow L^2[o,\infty] \tag{4.1}$$

defined by $\quad y(t) = S(x_o, u(\cdot))(t)$ (4.2)

We assume that the input u and the output y belong to $L^2[o,\infty]$. $x_o \in R^n$ is the initial state in some given state-space realization. Again for simplicity, we have assumed scalar inputs and outputs, the multivariable case will be considered in a future paper.

We have seen an example of such an input-output map in section 3, generated by a nonlinear analytic differential equation. Two simple examples are given by the linear-convolution systems and by the Volterra series of a bilinear system.

Example 1: Linear Systems

Consider the Linear system

$$\begin{cases} \dot{x} = Ax + Bu\ , \quad x(o) = x_o \\ y = Cx \end{cases} \tag{4.3}$$

then the input-output map is given by

$$y(t) = Ce^{At}x_o + \int_o^t C\, e^{A(t-\tau)}\, Bu(\tau)\, d\tau \tag{4.4}$$

In this case

$$S(x_o, u(\cdot))(t) = g_o(t) + (g*u)(t) \tag{4.5}$$

where

$$g_o(t) = Ce^{At}x_o, \quad g(t) = C\, e^{At}\, B$$

and

* denotes the convolution operator.

Example 2: Bilinear Systems

Consider the bilinear system

$$\begin{cases} \dot{x} = Ax + uNx + Bu, \; x(o) = x_o \\ y = Cx \end{cases} \tag{4.6}$$

where A, N, B, C are constant matrices of suitable dimensions. Then the input-output map is given by [4]:

$$\begin{aligned} y(t) = C\, e^{At} x_o + \sum_{j\geq 1} \int_o^t \int_o^{\sigma_j} \cdots \int_o^{\sigma_2} v_j(t,\sigma_1,\ldots,\sigma_j)\,. \\ .u(\sigma_1) \ldots u(\sigma_j)\, d\sigma_1 \ldots d\sigma_j + \\ + \sum_{j\geq 1} \int_o^t \int_o^{\sigma_j} \cdots \int_o^{\sigma_2} w_j(t,\sigma_1,\ldots,\sigma_j)\,. \\ .u(\sigma_1) \ldots u(\sigma_j)\, d\sigma_1 \ldots d\sigma_j \end{aligned} \tag{4.7}$$

where

$$v_1(t,\sigma_1) = C\, e^{A(t-\sigma_1)} B$$

$$v_j(t,\sigma_1,\ldots,\sigma_j) = Ce^{A(t-\sigma_j)} Ne^{A(\sigma_j-\sigma_{j-1})} N \ldots$$

$$\ldots N\, e^{A(\sigma_2-\sigma_1)} B \quad , \quad j>1$$

and

$$w_1(t,\sigma_1) = Ce^{A(t-\sigma_1)} x_o$$

$$w_j(t,\sigma_1,\ldots,\sigma_j) = C\, e^{A(t-\sigma_j)} N\, e^{A(\sigma_j-\sigma_{j-1})} N \ldots$$

$$\ldots N\, e^{A(\sigma_2-\sigma_1)} N\, e^{A\sigma_1} x_o, \; j>1$$

so $\quad y(t) = S(x_o, u(\cdot))\,(t).$

Returning to (4-1), for each fixed initial state x_o, we have a map

$$S_{x_o} \triangleq S(x_o,\cdot): L^2[o,\infty] \to L^2[o,\infty] \tag{4.8}$$

Let $\{e_j\}_{j\geq o}$ be a basis of $L^2[o,\infty]$ and let $\mathcal{J}$ denote the usual isomorphism

$$\mathcal{J} : L^2\,[o,\infty] \to \ell^2$$

given by

$$\mathcal{J}(f) = \{f_j\}_{j\geq o}$$

where

$$f \in L^2[o,\infty], \; f = \sum_{j\geq o} f_j\, e_j$$

The S_{x_o} induces a map

$$s_{x_o} : \ell^2 \to \ell^2$$

such that the diagram

$$\begin{array}{ccc} L^2[o,\infty] & \xrightarrow{S_{x_o}} & L^2[o,\infty] \\ \downarrow & & \downarrow \\ \ell^2 & \xrightarrow{s_{x_o}} & \ell^2 \end{array} \tag{4.9}$$

commutes.

We shall call s_{x_o} the generalized frequency response of S_{x_o} (with respect to the basis $\{e_k\}_{j\geq o}$). Explicitly s_{x_o} is given by

$$s_{x_o}(\{u_k\}_{k\geq o}) = \{< S_{x_o}(\sum_{j\geq o} u_j e_j), e_k>\}_{k\geq o} \tag{4.10}$$

where $<.\ ,\ .>$ denotes a scalar product in $L^2(o,\infty)$.

As an example consider the linear system in example 1 above. The map S_{x_o}, and hence the map s_{x_o}, is affine and is linear if $x_o = o$. s_{x_o} has a matrix representation given by

$$s_{x_o}(\{u_k\}_{k\geq o})_\ell = w_\ell + \sum_{j\geq o} G_{\ell j} \cdot u_j$$

where

$$w_\ell = <C e^{At} x_o , e_\ell>$$

In order to see the relationship to the familiar frequency response consider the input and output functions over $[o,T]$ for fixed $T>o$. Let $x_o=o$ and introduce the basis

$$B = \{\frac{1}{\sqrt{T}}\} \cup \{\sqrt{\frac{2}{T}} \cos 2\, n\pi t/_T\}_{n\geq 1} \cup \{\sqrt{\frac{2}{T}} \sin 2\, n\pi t/_T\}_{n\geq 1}$$

of $L^2[o,T]$. Then, as is well-known from linear systems theory we can write $g*u$ in the form

$$(g*u)(t) = \int_o^t g(\tau)\, u(t-\tau)\, d\tau$$

and so if $u(t) = e^{iw_n t}$, where $w_n = 2\pi n/T$ we have

$$\begin{aligned}(g*u)(t) &= e^{iw_n t} \cdot \int_o^\infty g(\tau)\, e^{-iw_n\tau} d\tau - e^{iw_n t}\int_t^\infty g(\tau)\, e^{-iw_n\tau} d\tau \\ &\triangleq e^{iw_n t}\, G(iw_n) + E_n(t)\end{aligned} \tag{4.11}$$

for $t \in [o,T]$. Here, $E_n(t)$ is called the transient term. Of course, this is just the familiar expression for the frequency response $G(iw_n)$ (i.e., the Fourier transform of the impulse response). If we order the basis B in the following way:

$$1/\sqrt{T}\ ,\ \sqrt{\frac{2}{T}}\cos\pi(2\pi t/T)\ ,\ \sqrt{\frac{2}{T}}\sin(2\pi t/T),$$
$$\sqrt{\frac{2}{T}}\cos(4\pi t/T),\ \sqrt{\frac{2}{T}}\sin(4\pi t/T).\ \ldots$$

then the commutative diagram (4.9) induces by the map $\mathcal{J}$ a one-to-one correspondence of this basis with the basis

$$(1,0,0,\ldots),\ (0,1,0,\ldots),\ (0,0,1,0,\ldots),\ldots \text{ of } \ell^2,$$

From (4.10), we therefore see that the matrix representation of the linear operator $s_{x_o} : \ell^2 \to \ell^2$ for the linear system above, with respect to the basis given is

$$\begin{pmatrix} \gamma & & & 0 \\ & \Gamma_1 & & \\ & & \Gamma_2 & \\ 0 & & & \ddots \end{pmatrix} + \Delta \tag{4.12}$$

(assuming the system is stable). Here

$$\gamma = \int_o^T g(t)\,dt$$

is just the gain of the system, Γ_k $(1 \leq k < \infty)$ is a 2 x 2 matrix given by

$$\Gamma_k = \begin{pmatrix} \mathrm{Re}\,G(iw_k) & -\mathrm{Im}\,G(iw_k) \\ \mathrm{Im}\,G(iw_k) & \mathrm{Re}\,G(iw_k) \end{pmatrix}$$

and Δ is the matrix representation of the isomorphic image of the transient term in ℓ^2.

If we write $\Gamma_k = (\det\Gamma_k)^{1/2} \cdot \begin{pmatrix} \cos\Theta_k & -\sin\Theta_k \\ \sin\Theta_k & \cos\Theta_k \end{pmatrix}$

then Γ_k represents a rotation and a dilation corresponding to the phase shift and the amplitude response of the system.

To illustrate the expression (4.10) for the bilinear input-output map (4.7) let $\{e_i\}$ be a basis of $L^2[o,\infty]$ and define

$$w_o(t) = Ce^{At}x_o$$

and for $j \geq 1$

$$V_{jk_1,\dots,k_j}(t) = \int_0^t \int_0^{\sigma_j} \dots \int_0^{\sigma_2} v_j(t,\sigma_1,\dots,\sigma_j)\, e_{k_1}(\sigma_1)\dots e_{k_j}(\sigma_j) . d\sigma_1 \dots d\sigma_j \tag{4.13}$$

$$W_{jk_1,\dots,k_j}(t) = \int_0^t \int_0^{\sigma_j} \dots \int_0^{\sigma_2} w_j(t,\sigma_1,\dots,\sigma_j)\, e_{k_1}(\sigma_1)\dots e_{l_j}(\sigma_j) . d\sigma_1 \dots d\sigma_j$$

then, if $u = \sum_{k\geq o} u_k e_k$, $u \in L^2 [o,\infty]$

the input-output map (4.7) becomes

$$y(t) = w_o(t) + \sum_{j\geq 1} \sum_{k_1\geq o} \dots \sum_{k_j\geq o} u_{k_1} \dots u_{k_j} V_{jk_1\dots k_j}(t) + \sum_{j\geq 1} \sum_{k_1\geq o} \dots \sum_{k_j\geq o} u_{k_1} \dots u_{k_j} W_{jk_1\dots k_j}(t) \tag{4.14}$$

and so

$$<y(t) , e_\ell> = <w_o(t) , e_\ell> + \sum_{j\geq 1} \sum_{k_1\geq o} \dots \sum_{k_j\geq o} u_{k_1}\dots u_{k_j} . <V_{jk_1\dots k_j}(t) , e_\ell> + \sum_{j\geq 1} \sum_{k_1\geq o} \dots \sum_{k_j\geq o} u_{k_1}\dots u_{k_j} <W_{jk_1\dots k_j}(t), e_\ell>$$

Hence

$$y_\ell = w_{o\ell} + \sum_{j\geq 1} \sum_{k_1\geq o} \dots \sum_{k_j\geq o} u_{k_1}\dots u_{k_j} V_{jk_1\dots k_j\ell} + \sum_{j\geq 1} \sum_{k_1\geq o} \dots \sum_{k_j\geq o} u_{k_1}\dots u_{k_j} W_{jk_1\dots k_j\ell}$$

where $y(t) = \sum_{\ell\geq o} y_\ell e_\ell(t)$, $w_o(t) = \sum_{\ell\geq o} w_{o\ell} e_\ell(t)$

$$V_{jk_1\dots k_j}(t) = \sum_{\ell\geq o} V_{jk_1\dots k_j\ell} e_\ell(t), \quad W_{jk_1\dots k_j}(t) = \sum_{\ell\geq o} W_{jk_1\dots k_j\ell} e_\ell(t)$$

It follows that the diagram (4.9) induces the map $s_{x_o} : \ell^2 \to \ell^2$ given by

$$s_{x_o}((u_o,u_1,\dots))_\ell = w_{o\ell} + \sum_{j\geq 1} \sum_{k_1\geq o} \dots \sum_{k_j\geq o} u_{k_1}\dots u_{k_j} . V_{jk_1\dots k_j\ell} + \sum_{j\geq 1} \sum_{k_1\geq o} \dots \sum_{k_j\geq o} . u_{k_1}\dots u_{k_j} W_{jk_1\dots k_j\ell} \tag{4.15}$$

We shall require the generalization of the Taylor's formula of an analytic function $f: R^n \to R$ to an analytic function $f: \ell^2 \to R$ defined on the infinite-dimensional space ℓ^2. From the general theory of higher-order derivatives of functions defined on an infinite-dimensional space [7] and from the familiar finite-dimensional formula, we can write an analytic function $f: \ell^2 \to R$ in the form

$$f(u_o, u_1, u_2, \ldots) = \sum_{|i| \geq o} \frac{f^{|i|}(o)}{i!} u^i \tag{4.16}$$

where

$$u = (u_o, u_1, u_2, \ldots) \in \ell^2$$

$i = (i_o, i_1, i_2, \ldots)$ with only a finite number of non-zero terms.

$$u^i = u_o^{i_o} . u_1^{i_1} . u_2^{i_2} \ldots$$

$$i! = i_o!\, i_1!\, i_2! \ldots$$

$$|i| = \sum_{\ell \geq o} i_\ell$$

$$o = (o, o, o, \ldots)$$

and

$$f^{|i|}(o) = \frac{\partial^i f(o)}{\partial u_o^{i_o} \partial u_1^{i_1} \partial u_2^{i_2} \ldots}$$

Define the map $s_{x_o, j} : \ell^2 \to R$ in the usual way by

$$s_{x_o, j}((u_o, u_1, \ldots)) = (s_{x_o}((u_o, u_1, \ldots)))_j$$

then, comparing (4.14) with (4.16) we obtain

$$w_{o\ell} = s_{x_o \ell}((o, o, \ldots))$$

$$V_{jk_1 \ldots k_{j\ell}} + W_{jk_1 \ldots k_{j\ell}} = \frac{\partial^j}{\partial u_o^{p_1} \partial u_1^{p_1} \ldots} s_{x_o \ell}((o, o, \ldots)) \tag{4.17}$$

where p_r is the number of terms in the sequence $k_1, \ldots, k_j$ equal to r (note that $\sum_{r \geq o} p_r = j$).

It follows that in the 'frequency domain' representation a bilinear system is just an analytic map from ℓ^2 to ℓ^2 (or if we vary the initial value from $R \times \ell^2$ to ℓ^2).

Consider, finally, the general non-linear analytic system (3.1), (3.2) and its input-output map (3.6). As before let $u = \sum_{j \geq o} u_j e_j$.

Since (3.6) involves $u^i(\sigma)$, we consider $u^i(\sigma) = \sum_{j\geq 0} U_{ij}\, e_j(\sigma)$ where

$$U_{ij} = < u^i, e_j >$$

we have

$$U_{ij} = < u^{i-1} \sum_{k\geq 0} u_k e_k, e_j > = \sum_{k\geq 0} u_k <u^{i-1} e_k, e_j >$$

$$= \sum_{k\geq 0} u_k <u^{i-1}, \bar{e}_k e_j>$$

let $\beta^j_{kp} = <\bar{e}_k e_j, e_p>$ (4.18)

then $U_{ij} = \sum_{k\geq 0} u_k <u^{i-1}, \sum_{p\geq 0} \beta^j_{kp} e_p>$

$$= \sum_{k\geq 0} \sum_{p\geq 0} u_k \beta^j_{kp} <u^{i-1}, e_p>$$

that is

$$\begin{cases} U_{ij} = \sum_{k\geq 0} \sum_{p\geq 0} u_k \beta^j_{kp} U_{i-1,p} \\ U_o = <1, e_j> \end{cases} \tag{4.19}$$

therefore

$$U_{ij} = \sum_{k_1\geq 0} \sum_{p_1\geq 0} \dots \sum_{k_i\geq 0} \sum_{p_i\geq 0} u_{k_1} \beta^j_{k_1 p_1} u_{k_2} \beta^{p_1}_{k_2 p_2} \dots u_{k_i} \beta^{p_{i-1}}_{k_i p_i} U_{op_i}$$

$$= \sum_{k_1\geq 0} \dots \sum_{k_i\geq 0} V^{k_1\dots k_i}_{ij} u_{k_1} \dots u_{k_i}$$

where

$$V^{k_1\dots k_i}_{ij} = \sum_{p_1\geq 0} \dots \sum_{p_i\geq 0} \beta^j_{k_1 p_1} \beta^{p_1}_{k_2 p_2} \dots \beta^{p_{i-1}}_{k_i p_i} u_{op_i}$$

Thus (3.6) becomes

$$y(t) = v_0(t) + \sum_{k\geq 1} \sum_{i_1\geq 1} \dots \sum_{i_k\geq 1} \sum_{j_1\geq 0} \dots \sum_{j_k\geq 0} U_{i_1 j_1} \dots U_{i_k j_k} V^{*j_1\dots j_k}_{k_1 i_1 \dots i_k}(t)$$

where

$$V^{*j_1\dots j_k}_{k, i_1\dots i_k}(t) = \int_0^t \int_0^{\sigma_k} \dots \int_0^{\sigma_2} V_k^{i_1\dots i_k}(t, \sigma_1, \dots, \sigma_k) e_{jk}(\sigma_k) \dots e_{j1}(\sigma_1) d\sigma_1 \dots d\sigma_k \tag{4.20}$$

ie.,

$$y(t) = v_0(t) + \sum_{k\geq 1} \sum_{i_1\geq 1} \dots \sum_{i_k\geq 1} \sum_{j_1\geq 0} \dots \sum_{j_k\geq 0} \sum_{\ell_{1,i_1}\geq 0} \dots \sum_{\ell_{i_1,i_1}\geq 0} \dots$$

$$\dots \sum_{\ell_{1,i_k}\geq 0} \dots \sum_{\ell_{i_k,i_k}\geq 0} V^{\ell_{1,i_1}\dots\ell_{i_1,i_1}}_{i_1 j_1} \dots V^{\ell_{1,i_k}\dots\ell_{i_k,i_k}}_{i_k j_k}.$$

$$\cdot u_{\ell_{1,i_1}} \cdots u_{\ell_{i_1,i_1}} \cdots u_{\ell_{1,i_k}} \cdots u_{\ell_{i_k,i_k}} V^{*j_1 \cdots j_k}_{ki_1 \cdots i_k}(t)$$

Hence

$$y_j = w_{oj} + \sum_{k\geq 1} \sum_{i_1\geq o} \cdots \sum_{i_k\geq o} \sum_{\ell_{1,i_1}\geq o} \cdots \sum_{\ell_{i_1,i_1}\geq o} \cdots \sum_{\ell_{1,i_k}\geq o} \cdots \sum_{\ell_{i_k,i_k}\geq o} W^{\ell_{1,i_1}\cdots\ell_{i_1,i_1}\cdots\ell_{1,i_k}\cdots\ell_{i_k,i_k}}_{k,i_1\cdots i_k j} \, .$$

$$u_{\ell_{1,i_1}} \cdots u_{\ell_{i_1,i_1}} \cdots u_{\ell_{1,i_k}} \cdots u_{\ell_{i_k,i_k}} \triangleq S_{x_o}((u_o,u_1,\cdots))_j \qquad (4.21)$$

where $y_j = \langle y(t), e_j\rangle$

$w_{oj} = \langle v_o(t), e_j\rangle$

and

$$W^{\ell_{1,i_1}\cdots\ell_{i_1,i_1}\cdots\ell_{1,i_k}\cdots\ell_{i_k,i_k}}_{ki_1\cdots i_k j} = \sum_{j_1\geq o} \cdots \sum_{j_k\geq o} V^{\ell_{1,i_1}\cdots\ell_{i_1,i_1}}_{i_1 j_1} \cdots V^{\ell_{1,i_k}\cdots\ell_{i_k,i_k}}_{i_k j_k} \langle V^{*j_1\cdots j_k}_{ki_1\cdots i_k}, e_j\rangle$$

5. Realization theory.

In this section we shall consider the problem of the realizability and the state space realization of an analytic map $s:\ell^2\to\ell^2$, which defines a generalized frequency response. We shall again study the problem in the linear, bilinear and general nonlinear situations and the results will be expressed in the form of conditions on the multi-dimensional Laplace transforms of various kernel functions associated with S.

5A: Linear Systems.

Theorem 1:

A necessary and sufficient condition for a sequence $\{G_{\ell j}\}_{\ell,j\geq o}$ of numbers to be the 'generalized frequency response' of a linear system with zero initial condition (with respect to a given basis $\{e_k\}_{k\geq o}$ of $L^2[o,\infty])$ is that there exists a strictly proper rational function $G(s)$ such that

$$\sum_{\ell\geq o} G_{\ell j} E_\ell(s) = G(s) E_j(s) \qquad (5.1)$$

for all $j \geq o$. $G(s)$ is then the transfer function of the linear system.

Proof:

In section 4, we derived the 'generalized frequency response' s_{x_o} of linear systems. Taking $x_o = o$, this reduces to

$$y_\ell = \sum_{j \geq o} G_{\ell j} u_j$$

where

$$y(t) = \sum_{\ell \geq o} y_\ell e_\ell(t), \quad t \in [o, \infty)$$

and

$$\sum_{\ell \geq o} G_{\ell j} e_\ell(t) = \int_o^t C e^{A(t-\tau)} B e_j(\tau)\, d\tau$$

Taking Laplace transform, we obtain

$$\sum_{\ell \geq o} G_{\ell j} E_\ell(s) = C(sI - A)^{-1} B . E_j(s)$$

that is

$$\sum_{\ell \geq o} G_{\ell j} E_\ell(s) = G(s) E_j(s) .$$

5.B: Bilinear Systems:

Let $E_k(s)$ have only simple poles $\alpha_k^{j_k}$ $j_k = 1, \ldots, r_k$. The corresponding residues are denoted $E_k^{j_k}$, $k \geq o$.

Theorem 2:

A necessary and sufficient condition for a sequence

$$\{V_{jk_1 \ldots k_j \ell}\}_{j \geq 1, k_1 \ldots k_j, \ell \geq o}$$ of numbers to be the 'generalized frequency response' of a bilinear system with zero initial condition (with respect to a given basis $\{e_k\}_{k \geq o}$ of $L^2[o, \infty]$) is that there exist four matrices $G_o(s)$, $G_1(s)$, $G_2(s)$, $G_3(s)$ with dimensions respectively 1x1, 1xm, mxm, mx1 of strictly proper rational functions such that

$$\text{(i)} \quad \sum_{\ell \geq o} V_{1k_1\ell} E_\ell(s) = G_o(s) E_{k_1}(s) \tag{5.2}$$

(ii) $$\sum_{\ell\geq o} V_{jk_1\ldots k_j\ell}\, E_\ell(s) = \sum_{\ell_{k_j}=1}^{r_{k_j}} \ldots \sum_{\ell_{k_2}=1}^{r_{k_2}} E_{k_j}^{\ell_{k_j}} \ldots E_{k_2}^{\ell_{k_2}}\, G_1(s)\, G_2(s-\alpha_{k_j}^{\ell_{k_j}}) \ldots$$

$$\ldots G_2(s-\alpha_{k_j}^{\ell_{k_j}}-\ldots-\alpha_{k_3}^{\ell_{k_3}}) . G_3(s-\alpha_{k_j}^{\ell_{k_j}}-\ldots-\alpha_{k_2}^{\ell_{k_2}}) \tag{5.3}$$

$$. E_{k_1}(s-\alpha_{k_j}^{\ell_{k_j}}-\ldots-\alpha_{k_2}^{\ell_{k_2}})$$

for all $k_1,\ldots,k_j\geq o$, $j>1$.

Proof:

For $x_o = o$, s_{x_o} given in (4.15) reduces to

$$y_\ell = \sum_{j\geq 1} \sum_{k_1\geq o} \ldots \sum_{k_j\geq o} u_{k_1} \ldots u_{k_j}\, V_{jk_1\ldots k_j\ell}$$

where

$$y(t) = \sum_{\ell\geq o} y_\ell\, e_\ell(t)$$

$$\text{and } V_{jk_1\ldots k_j}(t) = \sum_{\ell\geq o} V_{jk_1\ldots k_j\ell}\, e_\ell(t) \tag{5.4}$$

but (4.7) and (4.13) yield

$$V_{1k_1}(t) = \int_o^t Ce^{A(t-\sigma_1)} B\, e_{k_1}(\sigma_1) d\sigma_1 \tag{5.5}$$

and

$$V_{jk_1\ldots k_j}(t) = \int_o^t \int_o^{\sigma_j} \ldots \int_o^{\sigma_2} C\, e^{A(t-\sigma_j)} N\, e^{A(\sigma_j-\sigma_{j-1})} N \ldots$$

$$\ldots Ne^{A(\sigma_2-\sigma_1)} B\, e_{k_1}(\sigma_1) \ldots e_{k_j}(\sigma_j) \cdot \tag{5.6}$$

$$\cdot\, d\sigma_1 \ldots d\sigma_j \quad , \quad j>1$$

Taking the Laplace transform of (5.4) we obtain

$$\sum_{\ell\geq o} V_{jk_1\ldots k_j\ell}\, E_\ell(s) = V_{jk_1\ldots k_j}(s) \tag{5.7}$$

whereas the Laplace transform of (5.5) yield:

$$V_{1k_1}(s) = C\{sI-A\}^{-1}B.\ E_{k_1}(s)$$

that is

$$\sum_{\ell \geq o} V_{1k_1\ell} E_\ell(s) = G_o(s) E_{k_1}(s) \tag{5.8}$$

where $G_o(s) = C\{sI-A\}^{-1}B$ (5.9)

In order to find the Laplace transform of (5.6) we are going to write it in a different form, more suitable to the introduction of the multidimensional Laplace transform [5]. Consider the change of variables :

$$\sigma_1 = t - \tau_1, \ldots, \sigma_j = t - \tau_j$$

we obtain,

$$V_{jk_1\ldots k_j}(t) = \int_o^\infty \int_o^\infty \ldots \int_o^\infty Ce^{A\tau_j} N e^{A(\tau_{j-1}-\tau_j)} N \ldots N e^{A(\tau_1-\tau_2)} B .$$

$$e_{k_1}(t-\tau_1)\ldots e_{k_j}(t-\tau_j)\delta_{-1}(\tau_{j-1}-\tau_j)\ldots\delta_{-1}(\tau_1-\tau_2). \tag{5.10}$$

$$.d\tau_1\ldots d\tau_j$$

where δ_{-1} is the unit step, and the e_k's being zero for negative arguments. We shall introduce functions $\hat{V}_{jk_1\ldots k_j}(t_1,\ldots,t_j)$ defined by

$$\hat{V}_{jk_1\ldots k_j}(t_1,\ldots,t_j) = \int_o^\infty \ldots \int_o^\infty Ce^{A\tau_j} N e^{A(\tau_{j-1}-\tau_j)} N \ldots$$

$$\ldots N e^{A(\tau_1-\tau_2)} B . e_{k_1}(t_1-\tau_1)\ldots . e_{k_j}(t_j-\tau_j).$$

$$.\delta_{-1}(\tau_{j-1}-\tau_j)\ldots\delta_{-1}(\tau_1-\tau_2).\ d\tau_1\ldots d\tau_j \tag{5.11}$$

Then, taking the j-dimensional Laplace transform we obtain

$$\hat{V}_{jk_1\ldots k_j}(s_1,\ldots,s_j) = \bar{V}_j(s_1,\ldots,s_j) E_{k_1}(s_1)\ldots E_{k_j}(s_j) \tag{5.12}$$

where

$$\bar{V}_j(s_1,\ldots,s_j) = \int_o^\infty \ldots \int_o^\infty C e^{A\tau_j} N e^{A(\tau_{j-1}-\tau_j)} N.$$

$$\ldots N e^{A(\tau_1-\tau_2)} B \delta_{-1}(\tau_{j-1}-\tau_j)\ldots\delta_{-1}(\tau_1-\tau_2).$$

$$.e^{-(s_1\tau_1+\ldots+s_j\tau_j)} d\tau_1\ldots . d\tau_j \tag{5.13}$$

and

$$\hat{V}_{jk_1\ldots k_j}(s_1,\ldots,s_j) = \int_o^\infty \ldots . \int_o^\infty \hat{V}_{jk_1\ldots k_j}(t_1,\ldots t_j)\, e^{-(s_1t_1+\ldots+s_jt_j)}$$

$$.\ dt_1\ldots dt_j \tag{5.14}$$

We need the following lemma ([6]):

lemma 1:

The following relation holds between $V_{jk_1\ldots k_j}(s)$ and $\hat{V}_{jk_1\ldots k_j}(s_1,\ldots,s_j)$

$$V_{jk_1\ldots k_j}(s)=\frac{1}{(2\pi i)^{j-1}}\int_{p_1-i\infty}^{p_1+i\infty}\cdots\int_{p_{j-1}-i\infty}^{p_{j-1}+i\infty}\hat{V}_{jk_1\ldots k_j}(s-s_1-\ldots-s_{j-1},s_1,s_2,\ldots,s_{j-1})\,ds_{j-1}\ldots ds_1 \quad (5.15)$$

In the following, we shall be concerned with the evaluation of $V_{jk_1\ldots k_j}(s)$.

From (5.8) we obtain

$$\bar{V}_j(s_1,\ldots,s_j) = \int_0^\infty\ldots\int_0^\infty Ce^{A\sigma_j}N\,e^{A\sigma_{j-1}}N\ldots Ne^{A\sigma_1}B.$$

$$e^{-\{s_1[\sigma_1+\ldots+\sigma_j]+s_2[\sigma_2+\ldots+\sigma_j]+\ldots+s_j\sigma_j\}}.$$

$$.\,d\sigma_1\ldots d\sigma_j$$

Therefore

$$\bar{V}_j(s_1,\ldots,s_j) = C\{(s_1+\ldots+s_j)I-A\}^{-1}N\{(s_1+\ldots+s_{j-1})I-A\}^{-1}.$$

$$.N\ldots N\{s_1I-A\}^{-1}B \quad (5.16)$$

Now let $N = N_1 \,.\, N_2$ where N_1 is nxm and N_2 is mxn, and define $G_1(s)$, $G_2(s)$, $G_3(s)$ by

$$\begin{aligned} G_1(s) &= C\{sI-A\}^{-1}N_1 \\ G_2(s) &= N_2\{sI-A\}^{-1}N_1 \\ G_3(s) &= N_2\{sI-A\}^{-1}B \end{aligned} \quad (5.17)$$

(5.11) becomes

$$\begin{aligned}\bar{V}_j(s_1,\ldots,s_j) &= G_1(s_1+\ldots+s_j)\,G_2(s_1+\ldots+s_{j-1}).\\ &\ldots G_2(s_1+s_2)\,G_3(s_1)\end{aligned} \quad (5.18)$$

hence (5.7) and (5.10) yield

$$V_{jk_1\ldots k_j}(s) = \frac{1}{(2\pi i)^{j-1}}\int_{p_1-i\infty}^{p_1+i\infty}\cdots\int_{p_{j-1}-i\infty}^{p_{j-1}+i\infty}G_1(s)\,G_2(s-s_{j-1}).$$

$$\ldots G_2(s-s_2-\ldots-s_{j-1}) \cdot G_3(s-s_1-\ldots-s_{j-1}) \cdot$$

$$E_{k_1}(s-s_1-\ldots-s_{j-1})\, E_{k_2}(s_1)\ldots\; E_{k_j}(s_{j-1}) \cdot$$

$$ds_{j-1}\ldots ds_1 \tag{5.19}$$

therefore

$$V_{jk_1\ldots k_j}(s) = G_1(s)\int_0^\infty \ldots \int_0^\infty g_2(t_{j-1})\ldots g_2(t_2)g_3(t_1)e_{k_1}(t_o) \cdot$$

$$\cdot\left\{\frac{1}{(2\pi i)^{j-1}}\int_{p_1-i\infty}^{p_1+i\infty}\ldots\int_{p_{j-1}-i\infty}^{p_{j-1}+i\infty} E_{k_2}(s_1)\ldots E_{k_j}(s_{j-1})\exp-\left[(s-s_{j-1})t_{j-1} +\right.\right.$$

$$+\ldots+(s-s_2-\ldots-s_{j-1})t_2+(s-s_1-\ldots-s_{j-1})\,t_1 +$$

$$\left.\left.+(s-s_1-\ldots-s_{j-1})\,t_o\right]\,ds_{j-1}\ldots ds_1\right\}\,dt_{j-1}\ldots dt_o \tag{5.20}$$

Thus

$$V_{jk_1\ldots k_j}(s) = G_1(s)\int_0^\infty \ldots \int_0^\infty g_2(t_{j-1})\ldots g_2(t_2)g_3(t_1) \cdot$$

$$e_{k_1}(t_o)\{\frac{1}{(2\pi i)^{j-1}}\int_{p_1-i\infty}^{p_1+i\infty}\ldots\int_{p_{j-1}-i\infty}^{p_{j-1}+i\infty} E_{k_2}(s_1)\ldots E_{k_j}(s_{j-1})$$

$$.\exp\left[s_{j-1}(t_{j-1}+\ldots+t_o) + s_{j-2}\,(t_{j-2}+\ldots+t_o)+\ldots\right.$$

$$\left.+ s_2(t_2+ t_1 + t_o) + s_1(t_1+t_o)\right]ds_{j-1}\ldots ds_1\}.$$

$$.\exp\,\left[-s(t_{j-1}+\ldots+t_o)\right]\,dt_{j-1}\ldots dt_o \tag{5.21}$$

Applying residue theory this will yield,

$$V_{jk_1\ldots k_j}(s) = G_1(s)\int_0^\infty \ldots \int_0^\infty g_2(t_{j-1})\ldots g_2(t_2)g_3(t_1) \cdot$$

$$e_{k_1}(t_o)\left\{\sum_{\ell_{k_j}=1}^{r_{k_j}}\ldots\sum_{\ell_{k_2}=1}^{r_{k_2}} E_{k_j}^{\ell_{k_j}}\ldots E_{k_2}^{\ell_{k_2}}.\ \exp\left[\alpha_{k_j}^{\ell_{k_j}}\,(t_{j-1}+\ldots+t_o)+\right.\right.$$

$$\left.\left.+\ldots+\alpha_{k_2}^{\ell_{k_2}}\,(t_1+t_o) - s(t_{j-1}+\ldots+t_o)\right]\right\}\,dt_{j-1}\ldots dt_o \tag{5.22}$$

hence we obtain (5.3).

Remark 1:

To obtain the actual matrices involved in the bilinear representation we proceed as follows [4]. Consider the matrix

$$G(s) = \begin{pmatrix} G_o(s) & G_1(s) \\ G_3(s) & G_2(s) \end{pmatrix} \quad (5.23)$$

Since all the elements in G(s) are proper rational functions, G(s) may be considered as the transfer function of a constant linear system of finite order with m+1 outputs and m+1 inputs. Therefore there exist three matrices A, R, S respectively nxn, nx(m+1), (m +1) x n such that

$$S\ \{sI - A\}^{-1}\ R = G(s) \quad (5.24)$$

By partitioning S and R in the form

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix},\ R = (R_1\ R_2) \quad (5.25)$$

where S_1 is a 1xn matrix and R_1 an nx1 matrix we obtain

$$\begin{aligned} G_o(s) &= S_1\ \{sI - A\}^{-1}\ R_1 \\ G_1(s) &= S_1\ \{sI - A\}^{-1}\ R_2 \\ G_2(s) &= S_2\ \{sI - A\}^{-1}\ R_2 \\ G_3(s) &= S_2\ \{sI - A\}^{-1}\ R_1 \end{aligned} \quad (5.26)$$

Now substitute $G_o(s),\ldots, G_3(s)$ in (5.2) and (5.3), and let $C=S_1$, $B = R_1$, $N=R_2S_2$. {A, N, B, C} will constitute the bilinear realization.

Remark 2:

If we assume further that the basis functions have only single simple poles and that

$$\sum_{\ell\geq o} V_{jk_1\ldots k_j\ell} E_\ell(s)/\ E^1_{k_j} \ldots \ldots E^1_{k_2}\ E_1(s-\alpha^1_{k_j}\ldots-\alpha^1_{k_2})$$

$$\text{and} \quad \sum_{\ell\geq o} V_{1k_1\ell} E_\ell(s)/E_{k_1}(s)$$

are independent of k_1, $k_1 \geq o$, we can obtain another characterization of a bilinear realization. For this, consider $\hat{v}(s) = \sum_{\ell\geq o} V_{1k_1\ell} E_\ell(s)/E_{k_1}(s)$ and

$$\tilde{V}_{k_2\ldots k_j}(s,\sigma^1_{k_2},\ldots,\sigma^1_{k_j}) = \sum_{\ell\geq o} V_{jk_1\ldots k_j\ell}\ E_\ell(s)/E^1_{k_j}\ldots E^1_{k_2}.E_1(s -\alpha^1_{k_j}-\ldots-\alpha^1_{k_2}),\ j\geq 2 \quad (5.27)$$

For fixed s, say $\bar{s}$, as $k_2,\ldots,k_j$ ranges from 0 to ∞, $\tilde{v}_{k_2\ldots k_j}(\bar{s},\alpha^1_{k_2},\ldots,\alpha^1_{k_j})$ constitute an infinite sequence of complex numbers in $\bar{C} \simeq P^1(C)$ (the Riemann sphere). Therefore it converges to $\tilde{v}_j(\bar{s},s_2,\ldots,s_j)$. Here $s_p = \lim_{k_p\to\infty} \alpha_{k_p} \in \bar{C}$.

Corollary:

A necessary and sufficient condition for a sequence $\{V_{jk_1\ldots k_j\ell}\}$ $j\geq 1$, $k_1\ldots k_j\ell\geq o$ of numbers to be the 'generalized frequency response' of a bilinear system with zero initial condition (with respect to a given basis $\{e_k\}_{k\geq o}$ of $L^2[o,\infty]$) is that

(i) $\hat{v}(s)$ is a strictly rational function

(ii) $\tilde{v}_j(s,s'_3-s'_2,\ldots,s'_j-s'_{j-1}, s-s'_j)$

is a strictly proper rational recognizable function in $s, s'_2, \ldots, s'_j$ such that

$$\tilde{v}_j(s,s'_3-s'_2,\ldots,s'_j-s'_{j-1},s-s'_j) =$$

$$= G_1(s)G_2(s'_j)\ldots G_2(s'_3)\;G_3(s'_2),\; j\geq 2 \tag{5.28}$$

where $G_1(s)$, $G_2(s)$, $G_3(s)$ are matrices with dimensions respectively 1xm, mxm, mx1.

Proof:

(i) immediate

(ii) We have

$$\tilde{v}_{k_2\ldots k_j}(s,\alpha^1_{k_2},\ldots,\alpha^1_{k_j}) = G_1(s)\;G_2(s-\alpha^1_{k_j}).$$
$$\ldots G_2(s-\alpha^1_{k_j}-\ldots-\alpha^1_{k_3})\;G_3(s-\alpha^1_{k_j}-\ldots-\alpha^1_{k_2}) \quad \text{for } j\geq 2.$$

For fixed s, say $\bar{s}$, as $k_2,\ldots,k_j$ ranges from o to ∞, the right and left side constitute sequences in $\bar{C}$, therefore they converge respectively to,

$$\tilde{v}_j(\bar{s},s_2,\ldots,s_j) \text{ and } G_1(\bar{s})G_2(\bar{s}-s_j)\ldots G_2(\bar{s}-s_j-\ldots-s_3)\cdot$$
$$\cdot G_3(\bar{s}-s_j-\ldots-s_2)$$

hence

$$\tilde{v}_j(\bar{s},s_2,\ldots,s_j) = G_1(\bar{s})G_2(\bar{s}-s_j)\ldots G_2(\bar{s}-s_j-\ldots-s_3)\cdot G_3(\bar{s}-s_j-\ldots-s_2) \tag{5.29}$$

since $\bar{s}$ was arbitrary,

$$\tilde{v}_j(s,s_2,\dots,s_j) = G_1(s)\, G_2(s-s_j)\dots G_2(s-s_j-\dots-s_3)\,.\,G_3(s-s_j-\dots-s_2) \tag{5.30}$$

Therefore we have (5.28).

Remark 3:

Theorem 2 cannot, however, be extended to general non-linear system as the one given in (3.1) and (3.2) since we assumed the initial condition to be zero, whereas in (3.5) the initial tensor ϕ_o is never the zero tensor even if the initial condition in the original system (3.1) and (3.2) is zero.

A similar result can be obtained however for bilinear systems when the initial state is arbitrary. The proof follows the same lines as the one given for the zero initial condition and shall be omitted. One remark can be made, however; there is a term involving t in the integrand, one can take it out and consider the Laplace transform of the product of two time functions, that is the frequency convolution of their Laplace transforms.

For simplicity we state the theorem for the case B = o. The general case can be obtained immediately by combining the following theorem with theorem 2:

Theorem 3:

A necessary and sufficient condition for a sequence $\{W_{jk_1\dots k_j\ell}\}_{j\geq 1,\, k_1\dots k_j\, \ell\geq o}$ of numbers, to be the 'generalized frequency response' of a bilinear system with non-zero initial condition (with respect to a given basis $\{e_k\}_{k\geq o}$ of $L^2[o,\infty]$) is that there exist four matrices $G_o'(s)$, $G_1(s)$, $G_2(s)$, $G_3'(s)$ with dimensions respectively 1x1, 1xm, mxm, mx1 of strictly proper rational functions such that

$$\text{(i)} \quad \sum_{\ell\geq o} W_{o\ell}\, E_\ell(s) = G_o'(s) \tag{5.31}$$

(ii) $$\sum_{\ell\geq o} W_{jk_1\ldots k_j\ell}E_\ell(s) = \sum_{\ell_{k_j}=1}^{r_{k_j}} \cdots \sum_{\ell_{k_1}=1}^{r_{k_1}} E_{k_j}^{\ell_{k_j}} \cdots E_{k_1}^{\ell_{k_1}}$$

$$.G_1(s)\, G_2(s-\alpha_{k_1}^{\ell_{k_1}}) \;\ldots\; G_2(s-\alpha_{k_1}^{\ell_{k_1}}-\ldots-\alpha_{k_{j-1}}^{\ell_{k_{j-1}}}).$$

$$G_3'(s-\alpha_{k_1}^{\ell_{k_1}}-\ldots-\alpha_{k_j}^{\ell_{k_j}}) \tag{5.32}$$

5.C: Non-linear Systems:

Now, we can state a corollary for the general nonlinear system (3.1) and (3.2).

Corollary:

A necessary condition for a sequence $\{W_{ki_1\ldots i_k j}^{\ell_{1,i_1}\ldots\ell_{i_1,i_1}\ldots\ell_{1i_k}\ldots\ell_{i_k i_k}}\}$ of numbers to be the 'generalized frequency response' of a general system of the form (3.1) and (3.2) (with respect to a given basis $\{e_k\}_{k\geq o}$ of $L^2[o,\infty])$ is that there exist a tensor $G_o(s)$ and three sequences of infinite k-dimensional tensors $G_{1k}(s)$, $G_{2k}(s)$, $G_{3k}(s)$ whose components are analytic at infinity such that the following conditions hold for all indices:

$$\sum_{j\geq o} W_{oj}\, E_j(s) = G_o(s)$$

$$\sum_{j\geq o} W_{ki_1\ldots i_k j}^{\ell_{1i_1}\ldots\ell_{i_1 i_1}\ldots\ell_{1i_k}\ldots\ell_{i_k i_k}}\, E_j(s) =$$

$$= \sum_{j_1\geq o} \cdots \sum_{j_k\geq o} V_{i_1 j_1}^{\ell_{1i_1}\ldots\ell_{i_1 i_1}} \cdots V_{i_k j_k}^{\ell_{1i_k}\ldots\ell_{i_k i_k}}$$

$$\sum_{p_k=1}^{r_{k,j_k}} \cdots \sum_{p_1=1}^{r_{1,j_1}} E_{j_k}^{p_k} \cdots E_{j_1}^{p_1}\; G_{1k}^{k_1\ldots k_j}(s)\; G_{2k}^{k_1\ldots k_j}(s-\alpha_{j_1}^{p_1})\,.$$

$$\ldots G_{2k}^{k_1\ldots k_j}(s-\alpha_{j_1}^{p_1}-\ldots-\alpha_{j_{k-1}}^{p_{k-1}})\; G_{3k}^{k_1\ldots k_j}(s-\alpha_{j_1}^{p_1}-\ldots-\alpha_{j_k}^{p_k})$$

6. Conclusions:

In this paper we have introduced a generalized frequency response for a nonlinear system in terms of the response of the system to a set of standard inputs which form a basis of $L^2[o,\infty]$. The theory has been illustrated in the linear, bilinear and general nonlinear cases. A realization theory has been presented in terms of the multidimensional Laplace transform. The method may be developed in a variety of ways and, in particular, we are now considering the application of these ideas in the design of nonlinear filters.

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