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On the Generalization of the Lyapunov Equation to Nonlinear Systems

by

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Abstract

The Lyapunov equation for the characterization of the stability of linear systems is generalized to nonlinear systems.

Keywords

Lyapunov equation, nonlinear systems, Carleman linearization.

1. Introduction

The stability of linear system is, as is well-known, equivalent to the existence of a positive-definite, symmetric solution of the Lyapunov equation

$$PA + A^T P = -I ,$$

and then a Lyapunov function can be written directly in the form (1.1) $V(x) = \langle x, Px \rangle$. Such a simple characterization of stability for nonlinear systems is hardly to be expected, but it turns out that by using the technique of Carleman linearization (Carleman, 1932) one can obtain a direct extension of (1.1) to a nonlinear system. The method was developed by Brockett (1978) and has been applied in a variety of situations by Baillieul (1981), Banks (1986) and Loparo and Blankenship (1978). In the latter paper the Carleman linearization technique is applied to the estimation of the domain of attraction of nonlinear systems. The essential difference, here, however, is that we do not use a lexicographic ordering of the Taylor monomials involved in the expansion of the solution of the original equation. We leave these functions in the form of a tensor which allows us to generalize (1.1) directly.

2. Notation

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}^n$ be the set of n-tuples of such numbers. A typical element of $\mathbb{N}^n$ will be written $i = (i_1, \ldots, i_n)$ and, in particular, $1(k)$ will denote the n-tuple with 1 in the k'th place and zero elsewhere.

We shall use the theory of tensors as outlined in Banks and Yew (1985); any tensor Φ being written in terms of its components in the standard basis as

$$\Phi = (\phi_{i_1 \ldots i_n}) = (\phi_i) .$$

3. Tensor-Valued Differential Equations

Consider the differential equation

$$\dot{x} = f(x) \quad , \; x(o) = x_o \varepsilon \mathbb{R}^n \qquad (3.1)$$

where f is assumed to be a real analytic function with $f(o) = 0$. Any other equilibrium point can be considered similarly by a change of variables. (Of course, the analyticity of f guarantees that the equilibrium points are isolated.)

Introduce the Taylor monomials

$$\phi_i(x) = \phi_{i_1 \dots i_n}(x) = x_1^{i_1} \dots x_n^{i_n} = x^i, \quad 0 \leq i_j < \infty, \quad 1 \leq j \leq n$$

Then we have

$$\dot{\phi}_i(x) = \dot{\phi}_{i_1 \dots i_n}(x) = \sum_{k=1}^{n} i_k x^{(i-1(k))} \dot{x}_k$$

$$= \sum_{k=1}^{n} i_k x^{(i-1(k))} f_k(x) \tag{3.2}$$

Writing $f_k(x)$ in a Taylor series about the equilibrium point $(x=0)$, we have

$$f_k(x) = \sum_{i \varepsilon I} \alpha_i^k x^i, \quad 1 \leq k \leq n \tag{3.3}$$

for some constants α_i^k. Hence, from (3.2) and (3.3),

$$\dot{\phi}_i(x) = \sum_{k=1}^{n} i_k x^{(i-1(k))} \sum_{j \varepsilon I} \alpha_j^k x^j$$

$$= \sum_{j \varepsilon I} \sum_{k=1}^{n} i_k x^{(i-1(k)+j)} \alpha_j^k$$

$$= \sum_{j \varepsilon I} \sum_{k=1}^{n} i_k \alpha_{j-i+1(k)}^k x^j \quad ,$$

where we define $\alpha_\ell^k = 0$ if $\ell < 0$.

Hence,

$$\dot{\phi}_i = \sum_{j \varepsilon I} a_i^j \phi_j$$

where

$$a_i^j = \sum_{k=1}^{n} i_k \alpha_{j-i+1(k)}^k \quad .$$

Defining the tensor operator A by

$$(A\Phi)_i = \sum_{j\varepsilon I} a_i^j \phi_j$$

we can write the system (3.1) in the form

$$\dot{\Phi} = A\Phi \ . \tag{3.4}$$

Note that $\Phi_o = x_1^o \ldots x_n^o = 1$ and so the equilibrium point of this linear system corresponding to the origin for (3.1) is not the zero tensor. However, if we define

$$\Psi = \Phi - \underline{1}$$

where $\underline{1}$ is the tensor with

$$\underline{1}_o = 1 \ , \ \underline{1}_i = 0 \quad \text{for } i \neq 0 \ ,$$

then we have

$$\dot{\Psi} = A\Psi$$

Since f(o) = 0, i.e. Ψ satisfies the same equation as Φ and has equilibrium point $\Psi = 0$.

Next we would like to define an inner product on the tensor space $\mathcal{L}_n$ so that it becomes a Hilbert space. Let ℓ_e^2 denote the space of sequences $(\alpha_o, \alpha_1, \ldots)$ such that the sequence $(\alpha_o, \alpha_1/1!, \alpha_2/2!, \ldots)$ belongs to ℓ^2. Then we define an inner product on ℓ_e^2 by

$$\langle \alpha, \beta \rangle_e = \sum_{n=o}^{\infty} \frac{\alpha_n \beta_n}{(n!)^2} \quad , \quad \alpha, \beta \varepsilon \ell_e^2 \ .$$

Then ℓ_e^2 is a Hilbert space such that the map

$$E : \ell_e^2 \to \ell^2$$

defined by

$$E(\alpha) = (\alpha_n/n!) \quad , \quad \alpha \varepsilon \ell_e^2$$

is an isometric isomorphism. We can define an inner product on $\mathcal{L}_n = \bigotimes_n \ell_e^2$ by

$$\langle \Phi, \Psi \rangle = \prod_{k=1}^{n} \langle \alpha_k, \beta_k \rangle_e \tag{3.6}$$

where Φ, Ψ are the simple tensors

$$\Phi = \alpha_1 \otimes \cdots \otimes \alpha_n \quad , \quad \Psi = \beta_1 \otimes \cdots \otimes \beta_n \quad , \quad \alpha_i, \beta_j \in \ell_e^2$$

and by linear extension to $\mathcal{L}_n$.

Lemma 3.1 For any tensor Φ of the form $(x_1^{i_1} \ldots x_n^{i_n}) = (x^i)$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we have

$$\|\Phi\| \leq \exp(\tfrac{1}{2}\textstyle\sum_{k=1}^n x_k^2) = \exp(\tfrac{1}{2}\|x\|^2)$$

Proof

$$\|\Phi\| = \prod_{k=1}^n \|(x_k^{i_k})\|_e$$

$$= \prod_{k=1}^n \left\{ \sum_{\ell=o}^{\infty} \frac{x_k^{2\ell}}{(\ell!)^2} \right\}^{\frac{1}{2}}$$

$$\leq \prod_{k=1}^n \left\{ \sum_{\ell=o}^{\infty} \frac{x_k^{2\ell}}{\ell\,!} \right\}^{\frac{1}{2}}$$

$$= \exp\{\tfrac{1}{2}\textstyle\sum_{k=1}^n x_k^2\} \quad . \quad \square$$

Theorem 3.2 On the space $\mathcal{L}_n^T$ of tensors of the form $\Phi = (x^i)$ we have

$$\|A\Phi\| \leq \sum_{k=1}^n |f_k(x)| \cdot \|\Phi\|.$$

Proof We have

$$(A\Phi)_i = \sum_{k=1}^n i_k x^{(i-1(k))} f_k(x) \quad .$$

Consider the term

$$(A_1\Phi)_i \triangleq i_1 x_1^{i_1-1} x_2^{i_2} \ldots x_n^{i_n} f_1(x) \quad .$$

Then,

$$\|A_1\Phi\| = \|i_1 x_1^{i_1-1}\|_e \prod_{k=2}^n \|(x_k^{i_k})\|_e |f_1(x)| \quad .$$

$$= \left(\sum_{\ell=1}^{\infty} \frac{\ell^2 (x_1^2)^{\ell-1}}{(\ell!)^2} \right)^{\frac{1}{2}} \prod_{k=2}^n \|(x_k^{i_k})\|_e |f_1(x)|$$

$$= \prod_{k=1}^{n} \| (x_k^{i_k}) \|_e |f_1(x)|$$

$$= \| \Phi \| |f_1(x)|$$

The result now follows easily. □

Corollary 3.3 $\quad \| A^{\ell}\Phi \| \leq (\sum_{k=1}^{n} |f_k(x)|)^{\ell} \|\Phi\|$. □

Corollary 3.4 $\quad e^{At}\Phi$ exists for all t and for all $\Phi\varepsilon\mathcal{L}_n^T$, and satisfies

$$\| e^{At}\Phi \| \leq \exp\{ (\sum_{k=1}^{n} |f_k(x)|)t \} \| \Phi \| . \quad □$$

Hence the system

$$\dot{\Phi} = A\Phi$$

is soluble in $\mathcal{L}_n^T$ with the norm introduced above and if $\Phi_o = (x_o^i)$ for some $x_o \varepsilon \mathbb{R}^n$, then the solution $\Phi(t)$ satisfies

$$\|\Phi(t)\| \leq \exp\{(\sum_{k=1}^{n} |f_k(x_o)|)t\} \|\Phi_o\| \tag{3.7}$$

Note, however, that (3.7) cannot be extended to span $(\mathcal{L}_n^T)$ so that e^{At} is not a linear semigroup, although it is a nonlinear semigroup on $\mathcal{L}_n^T$.

We now come to our main result. Consider the system (3.1) and the tensor operator A associated with it as above. Before stating the theorem let us note first that the tensor Lyapunov equation

$$A^TP + PA = -I \tag{3.8}$$

does not generally have a solution. In fact it does not even have a 'weak' solution P in the sense that

$$< (A^TP + PA)\Psi , \Psi> = -<\Psi ,\Psi> \quad , \Psi\varepsilon\mathcal{L}_n^T - \underline{1} \ .$$

However, it may have an <u>ε-approximate</u> weak solution, i.e. a positive definite symmetric tensor P_ε for each $\varepsilon > o$ such that

$$<(A^T P_\varepsilon + P_\varepsilon A)\underline{\Psi}, \underline{\Psi}> = -<\underline{\Psi}, \underline{\Psi}> + \varepsilon \ , \quad \underline{\Psi} \varepsilon \mathcal{L}_n^T - \underline{1} \ . \tag{3.9}$$

Theorem 3.5 If there exists a strictly positive definite, symmetric tensor P_ε, with lower bound independent of ε for each bounded $\varepsilon > o$, such that the Lyapunov equation (3.8) has an ε-approximate weak solution (i.e. (3.9) holds) then the system (3.1) is asymptotically stable at the origin.

Conversely, if the system (3.1) is (globally) asymptotically stable at the origin then if any solution $x(t;x_o)$ of (3.1) satisfies

$$\| x(t;x_o) \|^2 \leq \log(1+C/t^\alpha) \ , \tag{3.10}$$

for some constant C and for sufficiently large t, then there exists an ε-approximate weak solution of (3.8).

Proof Suppose that an ε-approximate solution P_ε exists and that the system (3.1) us not asymptotically stable. Then there exists $\delta > o$ such that $\| \psi(t_i) \|^2 > \delta$ for some sequence $t_i \to \infty$, where $\underline{\Psi}(t) = (x^i(t)) - \underline{1} \varepsilon \mathcal{L}_n^T$ is the tensor associated with the solution x(t). Let $\varepsilon < \delta$ and consider the function

$$V = <\underline{\Psi}, P_\varepsilon \underline{\Psi}> \ , \quad (\underline{\Psi} + \underline{1}) \varepsilon \mathcal{L}_n^T \ .$$

Then

$$\begin{aligned} \dot{V} &= <A\underline{\Psi}(t), P_\varepsilon \underline{\Psi}(t)> + <\underline{\Psi}(t), P_\varepsilon A\underline{\Psi}(t)> \\ &= -<\underline{\Psi}, \underline{\Psi}> + \varepsilon \ . \end{aligned}$$

Since $P_{\varepsilon/2}$ is strictly positive definite we can write

$$<\underline{\Psi}, P_\varepsilon \underline{\Psi}> \geq \alpha \ \|\underline{\Psi}\|^2 \ ,$$

for some α , independent of ε. Hence,

$$\frac{\alpha d}{dt} \| \underline{\Psi}(t) \|^2 \leq - \|\underline{\Psi}(t)\|^2 + \varepsilon$$

$$< \ 0$$

if $\| \underline{\Psi}(t) \|^2 > \varepsilon$. This is a contradiction .

To prove the converse we shall follow the classical proof for the linear case. Thus, consider the tensor operator-valued differential equation

$$\dot{X} = A^T X + XA, \quad X(0) = I \,. \tag{3.11}$$

This equation has the unique solution

$$X(t) = e^{A^T t} e^{At} .$$

Let Ψ_o be such that $\Psi_o + \underline{1} \; \varepsilon \mathcal{L}_n^T$. Then

$$\Psi_o^T X \Psi_o = \Psi_o^T e^{A^T t} e^{At} \Psi_o .$$

However, as we have seen, $e^{At}\Psi_o$ is the solution of (3.5) with initial value Ψ_o and so $e^{At}\Psi_o$ is of the form $\{x^i(t)\} - \underline{1}$, where x(t) is the solution of (3.1) with initial condition x_o for which

$$\{x_o^i\} - \underline{1} = \Psi_o .$$

Hence,

$$\Psi_o^T X \Psi_o = \langle \Psi^T(t\ ), \Psi(t) \rangle$$

where $\Psi(t) = e^{At}\Psi_o$ and so

$$\Psi_o^T X \Psi_o = \| \Psi(t) \|^2 = \| \{x^i(t)\} \|^2 - 1$$

$$\leq \exp(\tfrac{1}{2}\| x(t) \|^2) - 1$$

By (3.11)

$$\langle X(t)\Psi_o, \psi_o \rangle - \langle \Psi_o, \Psi_o \rangle = \int_o^t \langle (A^T X(t) + X(t)A)\Psi_o, \Psi_o \rangle \, dt$$

Using (3.10) we see that

$$\langle X(t)\Psi_o, \Psi_o \rangle \leq \exp(\tfrac{1}{2}\| x(t;x_o) \|^2) - 1 \to 0$$

as $t \to \infty$, so if $\varepsilon > o$ choose $t(\varepsilon)$ so that $\langle X(t)\Psi_o, \Psi_o \rangle \leq \varepsilon$, and define

$$P_\varepsilon = \int_o^{t(\varepsilon)} X(t) \, dt \ .$$

Then,

$$\varepsilon - \langle \Psi_o, \Psi_o \rangle = \langle (A^T P_\varepsilon + P_\varepsilon A)\Psi_o, \Psi_o \rangle ,$$

so that (3.9) has an ε-approximate weak solution. P_ε is clearly symmetric and it is strictly positive definite since

$$\langle P_\varepsilon \Psi_o, \Psi_o \rangle = \int_o^{t(\varepsilon)} \{E(x(t;x_o)) - 1\} dt$$

$$\geq c \; \| x(t;x_o) \|^2$$

for some constant c, if ε is bounded above, where $E(x) = \prod_{k=1}^{n} \left\{ \sum_{\ell=0}^{\infty} \frac{x_k^{2\ell}}{(\ell!)^2} \right\}^{1/2}$.

Remarks (1) We can replace I in (3.8) by an arbitrary positive definite symmetric tensor, just as in the classical case.

(2) By using different norms on $\mathcal{L}_n^T$ it may be possible to relax the condition (3.10).

(3) If an equation has several stable equilibria then the result can be applied with $\Psi_o = \{x_o^i\}$ where x_o is in the region of attraction of a particular stable equilibrium.

4. Evaluation of the Lyapunov Function

In the last section we have seen that, given an analytic system

$$\dot{x} = f(x) \tag{4.1}$$

which has x = 0 as an asymptotically stable equilibrium point, we can construct a Lyapunov function of the form

$$V = \int_o^\infty \langle \Psi, e^{A^T t} e^{At} \Psi \rangle \, dt , \tag{4.2}$$

which we have seen exists provided the condition (3.10) holds. Since $e^{At}\Psi$ is the solution of the system with initial condition $\Psi = \{x_o^i\} - \underline{1}$, evaluating (4.2) requires us to solve the equation (4.1). Hence finding a Lyapunov function for a general system of the form can only be achieved by solving the original differential equation. Of course, the Lyapunov function (4.2) can also be expressed in the form

$$V(x_o) = \int_o^\infty \{ E(x(t;x_o)) - 1 \} dt$$

where $x(t;x_o)$ is the solution of (4.1) with initial condition $x(o) = x_o$. This is not surprising since, for any system of the form (4.1) which is asymptotically stable, we can write down the Lyapunov function

$$V(x_o) = \int_o^\infty \rho(\| x(t;x_o) \|) \, dt$$

where ρ is any function which makes the integral exist.

The advantage of the expression (4.2) is that, since the formal evaluation of $e^{A^T t}\Psi$ as a series just gives the Taylor series of the solution $x(t;x_o)$ as a function of t and x_o, is we truncate ψ and A in the obvious way to a finite dimensional tensor and tensor operator, respectively, then we can write the finite-dimensional approximation

$$\dot{\Psi}_m = A_m \Psi_m$$

to the exact equation $\dot{\Psi} = A\Psi$, for each $m \geq 1$. The solutions $e^{A_m t}\Psi_m$ to these equations clearly converge uniformly to the solution $e^{At}\Psi$ of the exact equation on compact sets since the former are just approximations of Taylor series.

If we solve the finite-dimensional Lyapunov equations

$$A_m^T P_m + P_m A_m = -I_m$$

then we obtain a sequence of Lyapunov functions

$$V_m(x) = \langle P\psi_m, \psi_m \rangle_m \tag{4.3}$$

which are valid in expanding neighbourhoods of 0 as m increases. Here, $\langle .,. \rangle_m$ is not the usual inner product in finite dimensional space, but the obvious truncation of the one introduced above on $\mathcal{L}_n^T$. Each function V_m in (4.3) is a polynomial in x the zeros of which move away from the origin to ∞ as $m \to \infty$.

Example Consider the 'time reversed' van der Pol oscillator:

$$\dot{x}_1 = -x_2+x_1^3 -x_1$$

$$\dot{x}_2 = x_1$$

Then if $\phi_{ij}(x) = x_1^i x_2^j$ we have

$$\dot{\phi}_{ij} = ix_1^{i-1} x_2^j \dot{x}_1 + jx_1^i x_2^{j-1} \dot{x}_2$$

$$= ix_1^{i-1} x_2^{j+1} \quad ix_1^{i+2} x_2^j - ix_1^i x_2^j + jx_1^{i+1} x_2^{j-1}$$

$$= -i\phi_{i-1,j+1} + i\phi_{i+2,j} -\phi_{ij} + j\phi_{i+1,j-1}$$

$$= \sum_{k,\ell} a_{ij}^{k\ell} \phi_{k\ell}$$

where

$$a_{ij}^{k\ell} = -i\delta_{i-1}^k \delta_{j+1}^\ell + i\delta_{i+2}^k \delta_j^\ell - i\delta_i^k \delta_j^\ell + j\delta_{i+1}^k \delta_{j-1}^\ell \ .$$

If $A = (a_{ij}^{k\ell})$ and $A_m = (a_{ij}^{k\ell})_{1\leq k,\ell\leq m, 1\leq ij\leq m}$, then we can obtain approximations to a Lyapunov function on a neighbourhood U_m of O by solving the tensor operator equation

$$A_m^T P_m + P_m A_m = -I_m \ . \qquad (4.4)$$

The resulting Lyapunov function $V_m = \langle P\Psi_m, \Psi_m \rangle_m$ converges to $\langle P\,\Psi, \Psi \rangle$ uniformly on each U_m and $\overset{\infty}{\underset{m=o}{U}} U_m = U$, where U is the interior of the limit cycle of the system.

5. Conclusions

In this paper we have generalized the familiar Lyapunov equation to nonlinear systems and have demonstrated a method for evaluating approximations to a

Lyapunov function for the nonlinear system. Since we have seen that this involves solving a sequence of tensorial generalizations of the well-known linear equation

$$A^T P + PA = - I$$

as defined by (4.4), efficient methods of computation of the latter equation are desirable.

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