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Global Volterra Series for Nonlinear Meromorphic Systems on Complex Manifolds

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Abstract

A global Volterra series is obtained for a meromorphic system defined on a complex manifold C. Local bilinearizations are pieced together by a fibre bundle giving rise to a 'twisted' bilinear system on C.

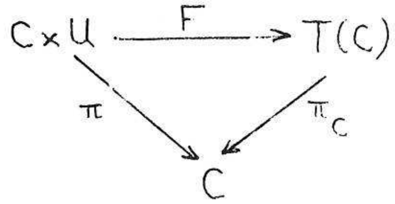
1. Introduction

Volterra series have been widely applied in the study of nonlinear systems, see, for example, Volterra, 1958, Brockett, 1976, Lesiak and Krener, 1978, Crouch, 1981, and Banks, 1985. The existence of bilinear representations of nonlinear systems has also been extensively investigated since the application of Carleman linearization to linear analytic systems by Brockett, 1976. Generalizations have been developed by Krener, 1975 and Lo, 1975 where the global linearization of systems of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))\, u(t)$$

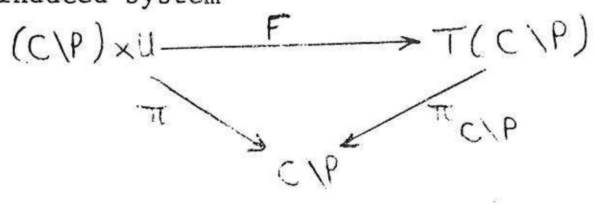
is discussed.

In this paper we shall consider a general nonlinear system



(1.1)

where C is a complex manifold and F is a meromorphic map, i.e. there exists an analytic subset $P \subseteq C$ of codimension 1 in C such that $C \setminus P$ is an open submanifold of C and the induced system

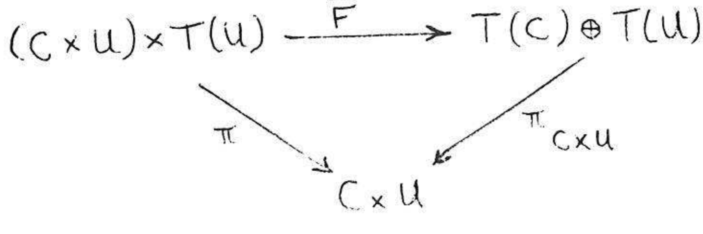


is holomorphic. In the first part we assume that $C = \mathbb{C}^n$ and $U = \mathbb{C}^m$ and that the system has a global representation

$$\dot{z} = f(z,u)/g(z,u)$$

and obtain a bilinearization for this system.

In the second part we shall replace the system (1.1) by system



which can be represented locally by an equation of the form

$$\dot{z} = f(z,u)/z,u)$$

$$\dot{u} = v$$

We can then use the local theory of the first part to derive a global Volterra series by defining an exponential map for a certain fibre bundle.

2. Notation and Terminology

We shall denote a goneric local system on $\mathbb{C}^n$ by

$$\dot{z} = f(z,u)/g(z,u)$$

where $z \in \mathbb{C}^n$, $u \in \mathbb{C}^m$ and f and g are analytic functions. If $i = (i_1, \ldots, i_n)$, $j = (j_1, \ldots, j_m)$ are multi-indices, with $i_k, j_\ell \in \mathbb{N}$, we shall write

$$z^i u^j = z_1^{i_1} \ldots z_n^{i_n} u_1^{j_1} \ldots u_m^{j_m}.$$

In particular, 1_k will denote the multi-index which has a 1 in the k^{th} place and zero elsewhere. The dimension of the vector 1_k will be clear from the context. $\mathbb{P}^n(\mathbb{C})$ will denote the n-dimensional complex projective space and T(C) will denote the tangent bundle of a complex manifold C with corresponding projection π_C. D(C) denotes the set of meromorphic vector yields on the complex manifold C and we shall use the theory of fibre bundles in the formulation of Kobayashi and Nomizu, 1963.

3. Differential Equations on Complex Manifolds

Consider first the example

$$\dot{z} = z^p , \tag{3.1}$$

for any integer p. This equation is defined as an analytic system on $\mathbb{C}\setminus\{0\}$ with a removable singularity at $z = 0$ if $p \geq 0$. The vector yield defined by $f(z)=z^p$ is meromorphic on $\mathbb{C}$ with a pole of order $-p$ at $z=0$ if $p<0$. For technical reasons it is desirable to extend this equation to a meromorphic equation of the compact space $\mathbb{P}^1(\mathbb{C})$. To do this let $w = 1/z$. Then we obtain the equation

$$\dot{w} = -\frac{1}{w^{p-2}} , \tag{3.2}$$

which is defined on the w-space $\mathbb{C}\backslash\{0\}$. The vector field $g(w) = -1/w^{p-2}$ is again meromorphic on $\mathbb{C}$ and so the equations (3.1), (3.2) together form the local representations of an equation

$$\dot{v} = F(v) \tag{3.3}$$

where $F(v)$ is a meromorphic function on $\mathbb{P}^1(\mathbb{C})$. More precisely, if $X(\mathbb{P}^1(\mathbb{C}))$ denotes the set of meromorphic vector yields on $\mathbb{P}^1(\mathbb{C})$, these equations (3.1) and (3.2) together represent an element of $X(\mathbb{P}^1(\mathbb{C}))$.

Consider next the equation

$$\begin{aligned} \dot{z}_1 &= \frac{1}{z_1^1 z_2^q} \\ \dot{z}_2 &= \frac{1}{z_1^r z_2^s} \end{aligned} \tag{3.4}$$

defined on $\mathbb{C}^2$. In the variables $(w_1 = 1/z, z_2)$ and $(z_1, w_2 = 1/z_z)$ this equation takes the respective forms

$$\dot{w}_1 = -\frac{1}{w_1^{p+2} z_2^q}$$

$$\dot{z}_2 = \frac{w_1^r}{z_2^s}$$

and

$$\dot{z}_1 = \frac{w_2^q}{z_1^p}$$

$$\dot{w}_2 = \frac{1}{z_1^r w_2^{s+2}}$$

Since these coordinate systems cover $\mathbb{P}^2(\mathbb{C})$ we see that the equation (3.4) defined in the affine space $\mathbb{C}^2$ can be completed to a meromorphic vector field in $X(\mathbb{P}^2(\mathbb{C}))$. Note, however, that the singularities now occur on projective subvarieties of of $\mathbb{P}^2(\mathbb{C})$ of codimension 1, and not just isolated points. (Indeed, by Hartog's theorem an analytic function defined on $\mathbb{P}^2(\mathbb{C})\backslash\{p\}$, for any point $p \in \mathbb{P}^2(\mathbb{C})$, has an analytic extension to $\mathbb{P}^2(\mathbb{C})$.)

In general, we shall consider differential equations defined on $\mathbb{P}^n(\mathbb{C})$ by a meromorphic vector field in $X(\mathbb{P}^n(\mathbb{C}))$ which is analytic except on a union of projective subvarieties of $\mathbb{P}^n(\mathbb{C})$ of codimension 1. Then locally such an equation is given by

$$\dot{z}_i = \frac{f_i(z)}{g_i(z)}, \quad 1\leq i\leq n \tag{3.5}$$

where f_i and g_i are analytic functions of $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Theorem 3.1 Given a meromorphic differential equation on $\mathbb{P}^n(\mathbb{C})$ we may represent it by a finite number of local systems of the form

$$\dot{z}_k = \sum_{i_1=-\infty}^{\infty} \cdots \sum_{i_n=-\infty}^{\infty} a^k_{i_1\ldots i_n} z_1^{i_1}\ldots z_n^{i_n}, \quad 1\leq k\leq n \tag{3.6}$$

Proof We shall prove the result for n=2; the general case is similar. If $p\in\mathbb{P}^2(\mathbb{C})$ then let $z:U\to\mathbb{C}^2$ be a coordinate system mapping a neighbourhood U of p into a neighbourhood W of 0 in $\mathbb{C}^2$. In these coordinates the differential equation may be expressed in the form (3.5) where $g_i(z) = g_i(z_1,z_2)$ is analytic and not identically zero. Consider f_1/g_1 and drop the subscript 1 for simplicity. If $g(0) \neq 0$ then the representation (3.6) is clear. If $g(0) = 0$ then we may suppose that there exists an integer k>o such that

$$g(0,z_2)/z_2^k$$

is analytic and $\neq$ o at z_2=o. By the Weierstrass preparation theorem (Hörmander, 1966) we can write g uniquely in the form g=hW when h and W are analytic in a neighbourhood of 0, $h(0) \neq 0$ and W is a Weierstrass polynomial

$$W(z_1, z_2) = z_2^k + \sum_{j=o}^{k-1} a_j(z_1)z_2^j$$

where each a_j is analytic in a neighbourhood of 0 with $a_j(0)=0$. Hence the equation has the form

$$\dot{z} = \frac{f(z)}{h(z)W(z)}$$

in a neighbourhood of 0. Since f and h are analytic and $h(0) \neq 0$, f/h has an expansion of the required form and we need to consider only the term $1/W(z)$. If all the a_j in W are identically zero then the result is true and so we suppose that some of the functions a_j are not identically zero. For simplicity of exposition we assume that none of the a_j are identically zero- the contrary case is similar. Then we can write

$$a_j(z_1) = z_1^{\ell_j}\alpha_j(z_1)$$

where $\alpha_j(0) \neq 0$ and $\ell_j > 0$. Now

$$\frac{1}{W(z)} = \frac{1}{z_2^k + v_1}$$

where

$$v_1 = \sum_{j=0}^{k-1} a_j(z_1)\, z_2^j.$$

Hence, if $|z_2^k| > |v_1|$ we can expand $1/W(z)$ in the form

$$\frac{1}{z_2^k}\left(\sum_{j=0}^{\infty}\left(\frac{v_1}{z_2^k}\right)^j c_j\right)$$

for some coefficients c_j and since v_1 is analytic the result is true in the region $|z_2^k| > |v_1|$. If $|v_1| > |z_2^k|$, then we can expand $1/W(z)$ in the form

$$\frac{1}{v_2^k}\left(\sum_{j=0}^{\infty}\left(\frac{z_2^k}{v_1}\right)^j \bar{c}_j\right)$$

for some new coefficients $\bar{c}_j$. Now consider

$$\frac{1}{v_1} = \frac{1}{a_{k-1}(z_1)z_2^{k-1} + \sum\limits_{j=0}^{k-2} a_j(z_1)z_2^j}$$

$$= \frac{1}{z_1^{\ell_{k-1}} z_2^{k-1}\alpha_j(z_1) + \sum\limits_{j=0}^{k-2} a_j(z_1)z_2^j}$$

and since α_j is analytic and $\alpha_j(0) \neq 0$ we can continue as before. Eventually we have to consider a term of the form

$$\frac{1}{v_{k-1}} = \frac{1}{a_1(z_1)z_2 + a_0(z_1)}$$

$$= \frac{1}{z_1^{\ell_1} z_2\alpha_1 + z_1^{\ell_0}\alpha_0}$$

and the result follows from the compactness of $\mathbb{P}^2(\mathbb{C})$. □

It follows from theorem 3.1 that we may consider systems which have a finite number of representations of the form (3.6) in some neighbourhood of any point p in $\mathbb{P}^n(\mathbb{C})$ whose domains of definition partition the neighbourhood minus a union of some n-1 dimensional submanifolds through p .

4. Local Theory of Meromorphic Systems

As we have seen above we may write a meromorphic system

$$\dot{z} = f(z,u) \ , \ z\varepsilon\mathbb{C}^n, \ u\varepsilon\mathbb{C}^m \tag{4.1}$$

in a neighbourhood of $(0,0)\varepsilon\mathbb{C}^{n+m}$ in the form of a finite number of representations

$$\dot{z}_k = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{ij}^k z^i u^j \ , \ 1\leq k\leq n \tag{4.2}$$

where $i=(i_1,\ldots, i_n)$, $j=(j_1,\ldots,j_m)$. We shall restrict the controls to be C^1, in which case we may regard u as a state and $v=\dot{u}$ as the control by adding the equation

$$\dot{u} = v \tag{4.3}$$

to (4.2). Our object is to reduce (4.2) and (4.3) to an infinite-dimensional bilinear system, which we can do by introducing the functions

$$\phi_{ij} = z^i u^j .$$

Then, following the technique of Carleman linearization

we have

$$\dot{\phi}_{ij} = \sum_{k=1}^{n} i_k z^{i-1_k} u^j \dot{z}_k + \sum_{\ell=1}^{m} j_\ell z^i u^{j-1_\ell} \dot{u}_\ell .$$

$$= \sum_{k=1}^{n} i_k \sum_{i'=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} a_{i'j'}^k \cdot z^{i+i'-1_k} u^{j+j'} + \sum_{\ell=1}^{m} j_\ell z^i u^{j-1_\ell} v_\ell .$$

$$= \sum_{i'=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \sum_{k=1}^{n} i_k a_{i'-i+1_k, j'-j}^k \cdot z^{i'} u^{j'} + \sum_{\ell=1}^{m} j_\ell z^i u^{j-1_\ell} v_\ell$$

$$= \sum_{i'=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \alpha_{ij}^{i'j'} \phi_{i'j'} + \sum_{i'=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \beta_{ij,\ell}^{i'j'} \cdot \phi_{i'j'} v_\ell .$$

where

$$\alpha_{ij}^{i'j'} = \sum_{k=1}^{n} i_k a_{i'-i+1_k, j'-j}^k \ , \ \beta_{ij,\ell}^{i'j'} = \sum_{\ell=1}^{m} j_\ell \delta_{i \ j-1_\ell}^{i'j'} .$$

In the latter expression we define

$$\delta_{k\ell}^{ij} = \delta_{k_1}^{i_1} \delta_{k_2}^{i_2} \ldots \delta_{k_n}^{i_n} \delta_{\ell_1}^{j_1} \ldots \delta_{\ell_m}^{j_m} .$$

Defining the tensor operators A and B by

$$(A\Phi)_{ij} = \sum_{i'=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \alpha_{ij}^{i'j'} \phi_{i'j'}$$

and

$$(B_\mu \Phi)_{ij} = \sum_{i'=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \beta_{ij,\mu}^{i'j'} \phi_{i'j'}, \quad 1 \le \mu \le m$$

we have

$$\dot{\Phi} = A\Phi + \sum_{\mu=1}^{m} v_\mu B_\mu \Phi \qquad (4.4)$$

where

$$(\Phi)_{ij} = \phi_{ij} .$$

In order to consider the theory of equations of the form (4.4), we must put a suitable norm on the space of tensors. First let ℓ^1 denote the Banach space of doubly infinite sequences $\alpha = \{\alpha_n\}_{-\infty<n<\infty}$ with the norm

$$||\alpha||_{\ell^1} = \sum_{n=-\infty}^{\infty} |\alpha_n| .$$

Then we define ℓ_e^1 to be the space of sequences $\alpha = \{\alpha_n\}_{-\infty<n<\infty}$ such that the sequence $(\ldots,\alpha_{-2}/2!,\alpha_{-1}/1!,\alpha_o,\alpha_1/1!,\alpha_2/2!,\ldots)$ belongs to ℓ^1. Define a norm on ℓ_e^1 by

$$||\alpha||_e = \sum_{n=-\infty}^{\infty} \frac{|\alpha_n|}{|n|!} , \quad \alpha \in \ell_e^1 .$$

Next introduce the space

$$\mathcal{L}_n = \otimes_n \ell_e^1 ,$$

i.e. the algebraic tensor produce of n copies of ℓ_e^1, and let $||.||$ be any cross norm on $\mathcal{L}_n$ (see Taksaki, 1979). Thus, for any tensor $\Phi \in \mathcal{L}_n$ of the form

$$\Phi = (\phi_{i_1 \ldots i_n}) = (\alpha_{1i_1} \alpha_{2i_2} \ldots \alpha_{ni_n}) = \alpha_1 \otimes \ldots \otimes \alpha_n$$

where $\alpha_k = (\ldots,\alpha_{k,-2},\alpha_{k,-1},\alpha_{k,0},\alpha_{k1},\alpha_{k2},\ldots) \in \ell_e^1$ we have

$$||\Phi|| = \prod_{k=1}^{n} ||\alpha_k||_e .$$

Lerma 4.1. For any vector $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$, with $z_1 \neq 0$ for $i=1,\ldots,n$ the tensor $\Phi = (z_1^{i_1},\ldots,z_n^{i_n})$ belongs to $\otimes_n \ell_e^1$ and we have

$$||\Phi|| = \prod_{k=1}^{n} (\exp(|z_k|) + \exp(\frac{1}{|z_k|}) - 1) .$$

Proof. $$||(z_1^{i_1} \ldots z_n^{i_n})|| = \prod_{k=1}^{n} ||(z_k^{i_k})||_e$$

$$= \prod_{k=1}^{n} \left\{ \sum_{\ell=-\infty}^{\infty} \frac{|z_k|^\ell}{|\ell|!} \right\}$$

$$= \prod_{k=1}^{n} \left((\exp(|z_k|) + \exp\left(\frac{1}{|z_k|}\right) - 1 \right). \quad \square$$

Now introduce the nonlinear subspace $\mathcal{L}_n^T$ of $\mathcal{L}_n$ consisting of all tensors of the form $\Phi=(z^i u^j)$.

Theorem 4.2 We have

$$||A\Phi|| \leq C_1 \max\left\{ \frac{1}{|z_1|^2}, \ldots, \frac{1}{|z_n|^2} \right\} \sum_{k=1}^{n} |f_k(z,u)| \; ||\Phi|| \qquad (4.5)$$

for all $\Phi \varepsilon \mathcal{L}_{n+m}^T$, where

$$C_1 = \max(r(z_1), \ldots, r(z_n), 1)$$

where
$$r(z) = \frac{|z|^2 \exp|z|}{\exp(z)-1}$$

and

$$||B\Phi|| \triangleq \sum_{i=1}^{m} ||B_i \Phi|| \leq m\, C_2 \max\left\{ \frac{1}{|u_1|^2}, \ldots, \frac{1}{|u_m|^2} \right\} ||\Phi|| \qquad (4.6)$$

where

$$C_2 = \max((u_1), \ldots, r(u_m), 1).$$

Proof. To prove (4.5) note that

$$(A\Phi)_{ij} = \sum_{k=1}^{n} i_k z^{(i-1_k)} u^j f_k(z,u).$$

Consider the term

$$(A_1 \Phi)_{ij} \triangleq i_1 z_1^{i_1-1} z_2^{i_2} \ldots z_n^{i_n} u_1^{j_1} \ldots u_m^{j_m} f_1(z,u).$$

We have

$$||A_1 \Phi|| = || i_1 z_1^{i_1-1} ||_e \prod_{k=2}^{n} ||(z_k^{i_k})||_e \prod_{k=1}^{m} ||(u_k^{j_k})||_e \; |f_1(z,u)|$$

$$= \sum_{i_1=-\infty}^{\infty} \frac{|i_1| |z_1|^{i_1-1}}{|i_1|!} \prod_{k=2}^{n} (\exp|z_k| + \exp \frac{1}{|z_k|} - 1) \prod_{k=1}^{m} \cdot (\exp|u_k| + \exp \frac{1}{|u_k|} - 1)$$

However,

$$\sum_{i_1=-\infty}^{\infty} \frac{|i_1| |z_1|^{i_1-1}}{|i_1|!} = \sum_{i_1=1}^{\infty} \frac{|z_1|^{-i_1-1}}{(i_1-1)!} + \sum_{i_1=0}^{\infty} \frac{|z_1|^{i}}{i_1!}$$

$$= |z_1|^{-2} \exp(|z_1|^{-1}) + \exp|z_1|$$

$$\leq C_1 \frac{1}{|z_1|^2} (\exp|z_1| + \exp \frac{1}{|z_1|} - 1)$$

Hence

$$||A_1 \Phi|| \leq C_1 \frac{1}{|z_1|^2} ||\Phi|| \; |f_1(z,u)|$$

by lemma 4.1.(4.5) now follows easily and (4.6) is similar. $\square$

Corollary 4.3 $e^{At}\Phi$ exists for all t and for all $\Phi\varepsilon\mathcal{L}^T_{n+m}$ and we have

$$||e^{At}\Phi|| \leq \exp(\omega(z,u)t)||\Phi||$$

where

$$\omega(z,u) = C_1\max\left\{\frac{1}{|z_1|^2},\ldots,\frac{1}{|z_n|^2}\right\}\sum_{k=1}^{n}|f_k(z,u)|. \square$$

It follows from corollary 4.3 that the system

$$\dot{\Phi} = A\Phi$$

is soluble in $\mathcal{L}^T_{n+m}$ and if $\Phi_o=(z_o^i u_o^j)$ for some $z_o\varepsilon\mathbb{C}^n$, $u_o\varepsilon\mathbb{C}^m$ (4.7)

we have

$$||\Phi(t)|| \leq \exp(\omega(z_o,u_o)t)||\Phi_o||.$$

We can now define the Volterra series solution for the equation (4.4) in the usual way:

$$\Phi(t) = w_o(t)+\sum_{\gamma=1}^{\infty}\int_o^t\ldots\int_o^t w_\gamma(t,\sigma_1,\ldots,\sigma_\gamma)v(\sigma_1)\otimes\ldots\otimes v(\sigma_\gamma)d\sigma_1\ldots d\sigma_\gamma \quad (4.8)$$

where

$$w_o(t) = e^{At}\Phi_o$$

$$w_\gamma(t,\sigma_1,\ldots,\sigma_\gamma) = e^{A(t-\sigma_1)}Be^{A(\sigma_1-\sigma_2)}B\ldots Be^{A\sigma_\gamma}\Phi_o, \text{ for } t\geq\sigma_1\geq\ldots\geq\sigma_\gamma$$

$$= 0 \qquad \text{otherwise}$$

and

$$e^{A(t-\sigma_1)}Be^{A(\sigma_1-\sigma_2)}B\ldots Be^{A\sigma_\gamma}\Phi_o(v(\sigma_1)\otimes\ldots\otimes v(\sigma_2))$$

$$= e^{A(t-\sigma_1)}v(\sigma_1)\,Be^{A(\sigma_1-\sigma_2)}v(\sigma_2)B\ldots v(\sigma_\gamma)Be^{A\sigma_\gamma}\Phi_o.$$

Then we have

Theorem 4.4 The Volterra series in (4.8) converges (in $\mathcal{L}_{n+m}$) and is the unique solution of equation (4.4), provided $v(t)\varepsilon L^\infty(o,\infty)$ and $x_0,u_0\neq 0$.

Proof. The only nontrivial part to prove is the convergence of the Volterra series. Let

$$||v||_\infty = \max_{i=1,..,m}\ \operatorname*{ess\,sup}_{s\in[o,\infty)}|v_i(s)|.$$

Then

$$||\Phi(t)||\leq||w_o(t)|| + \sum_{\gamma=1}^{\infty}||v||_\infty^\gamma\int_o^t\ldots\int_o^{\sigma_{\gamma-2}}\int_o^{\sigma_{\gamma-1}}||w_\gamma(t,\sigma_1,\ldots,\sigma_\gamma)||\cdot d\sigma_1\ldots d\sigma_\gamma.$$

Hence by corollary 4.3 and (4.6) we have

$$||w_\gamma(t,\sigma_1,\ldots,\sigma_\gamma)||\leq \eta^\gamma\exp(w(z_o,u_o)t)||\Phi_o||,$$

where

$$\eta = mC_2 \max\{|u_{o1}|^{-2}, \ldots, |u_{om}|^{-2}\},$$

and $\Phi_o = (x_o^i u_o^j)$. Hence,

$$||\Phi(t)|| \leq ||w_o(t)|| + \sum_{\gamma=1}^{\infty} ||v||_\infty^\gamma \eta^\gamma \int_o^t \cdots \int_o^{\sigma_{\gamma-1}} \exp(w(z_o,u_o)t)\, ||\Phi_o||\, d\sigma_1 \cdots d\sigma_\gamma$$

$$= ||w_o(t)|| + \sum_{\gamma=1}^{\infty} ||v||_\infty^\gamma \eta^\gamma \exp(w(z_o,u_o)t)\, ||\Phi_o||\, t^\gamma/\gamma!$$

$$\leq \exp\{w(z_o,u_o)t + ||v||_\infty \eta t\}\, ||\Phi_o||.$$

5. Equations on Complex Manifolds.

We now consider the system

$$\begin{array}{ccc} C \times U & \xrightarrow{F} & T(C) \\ & {\scriptstyle\pi} \searrow \quad \swarrow {\scriptstyle\pi_C} & \\ & C & \end{array} \tag{5.1}$$

defined on a complex manifold C where F is a meromorphic map, i.e. there exists a set P which is a union of submanifolds of codimension 1 such that $C\backslash P$ is an open submanifold of C and the system

$$\begin{array}{ccc} (C\backslash P) & \xrightarrow{F} & T(C\backslash P) \\ & {\scriptstyle\pi} \searrow \quad \swarrow {\scriptstyle\pi_{C\backslash P}} & \\ & C\backslash P & \end{array}$$

is analytic. The system (5.1) is given locally (near $p\varepsilon C$) by an equation of the form

$$\dot{z} = f_p(z,u)/g_p(z,u)$$

where $f_p : \mathbb{C}^{n+m} \to \mathbb{C}^n$ and $g_p : \mathbb{C}^{n+m} \to \mathbb{C}^n$ are analytic.

In order to apply the theory of section 4 we reformulate the problem in the following way. We can regard a system of the form (5.1) as a map $X(.) : U \to D(C)$ where $D(C)$ is the set of meromorphic vector fields on C and

$$X(u) = F(.,u).$$

For any point $(p,u)\varepsilon C \times U$ we have $T_{(p,u)}(C \times U) \cong T_p C \oplus T_u U$. We define the system

$$Y(.) : T(U) \to D(C \times U)$$

such that, for each $Z \varepsilon T(U)$,

$$Y(Z)_{(p,u)} = (X(u)_p, v)$$

where $v \varepsilon T_u U$. Then, in a neighbourhood of (p,u) we can write Y in the form

$$\begin{aligned} \dot{z} &= f_p(z,u)/g_p(z,u) \\ \dot{u} &= v \end{aligned} \tag{5.2}$$

Note that the control space is now the gangent bundle of U rather than U. From the results of section (4) we can replace the local system (5.2) by the system

$$\dot{\Phi}_p = A_p \Phi_p + vB_p \Phi_p \tag{5.3}$$

where $vB_p = \sum_{i=1}^{m} v_i B_{i,p}$ and $\Phi_p = (z^i u^j)$.

Similar equations hold for each $p \varepsilon C$ and we must relate the systems arising from two intersecting coordinate neighbourhoods. To do this let $(\zeta,w) = g(z,u)$ be a biholomorphic coordinate transformation from (z,u)-coordinates (in a neighbourhood U) to (ζ,w)-coordinates with $(0,0) = g(0,0)$. Then we can write

$$g(z,u) = \sum_{i=o}^{\infty} \sum_{j=o}^{\infty} g^{ij} z^i u^j .$$

By theorem 3.1 we can write, for any α,β with $-\infty<\alpha,\beta<\infty$,

$$\begin{aligned} \zeta^\alpha w^\beta &= \left(\sum_{i=o}^{\infty} \sum_{j=o}^{\infty} g^{ij} z^i u^j\right)^{(\alpha,\beta)} \\ &= \sum_{i=o}^{\infty} \sum_{j=o}^{\infty} g^{ij}_{\alpha\beta}\, z^i u^j \end{aligned} \tag{5.4}$$

in a finite number of open subsets of U, for some numbers $g^{ij}_{\alpha\beta}$ (which depend on the particular open subset). Hence we can write

$$\Psi = G\,\Phi$$

where

$$\Psi = (\zeta^\alpha w^\beta) \quad , \quad \Phi = (z^i u^j)$$

and

$$G = (g^{ij}_{\alpha\beta}).$$

Since g is biholomorphic we can also write

$$\Phi = G^{-1}\Psi$$

where G^{-1} is the inverse tensor operator of G.

Similarly, if

$$(\zeta, w) = g(z,u) \ , \ (\eta,\gamma) = h(\zeta,w)$$

then

$$(\eta,\gamma) = h \circ g(z,u)$$

and

$$\Xi = HG\Phi$$

where

$$\Xi = (\eta^{\alpha}\gamma^{\beta}), \ \Phi = (z^{i}u^{j}),$$

and H,G are defined as above. Hence the set of tensor operators of the type (5.4) is a group and operates as a transformation group on $\mathcal{L}^{T}_{n+m}$. Thus, assigning a space of tensors $\mathcal{L}^{T}_{n+m,p}$ of type $(z^{i}u^{j})$ at each point p of C (with coordinates (z,u)) then we can make $\bigcup_{p\in C}\mathcal{L}^{T}_{n+m,p}$ with projection

$$\pi : \bigcup_{p\in C} \mathcal{L}^{T}_{n+m,p} \to C$$

into a fibre bundle over C. We denote this bundle by Γ_{n+m}.

We can now define the concept of global bilinear system.

<u>Definition</u>. Let $X(.): U \to D(C)$ be a system as defined above. We shall say that m+1 sections $\mathcal{A}_1, \mathcal{B}_1, \ldots, \mathcal{B}_m$ of the fibre bundle Γ_{n+m} form a bilinear system on C if the local representation of X(.) given by

$$\dot{z} = f_p(z,u)/g_p(z,u)$$

$$\dot{u} = v$$

at p is related to the bilinear system

$$\dot{\Phi}_p = A_p\Phi_p + v_1B_{1p}\Phi_p + \ldots + v_mB_{mp}\Phi_p \qquad (5.5)$$

as above, where $\mathcal{A}_p = A_p\Phi_p$, $\mathcal{B}_{ip} = B_{ip}\Phi_p$.

The group action of the transformations of type (5.4) imply that local representations of the form (5.5) are related by

$$\dot{\Psi}_q = GA_pG^{-1}\Psi_q + v_1GB_{ip}G^{-1}\Psi_q + \ldots + v_m GB_{mp}G^{-1}\Psi_q$$

where $\Psi_q = G\Phi_p$ and G is a transformation of type (5.4) between the local coordinates (w,v) at q and (z,u) at p, where

$$w(q) = x(p) = 0, \ v(q) = u(p) = 0.$$

If a given section $\mathcal{A}$ of Γ_{n+m} belongs to a bilinear system, we can define an exponential map for this 'tensor field' by

$$(e^{\mathcal{A}t})_p = e^{A_pt}\Phi_p.$$

This is well defined since A_p is just a linear tensor operator. Moreover, under a change of coordinates G, we have

$$G(e^{A_p t})G^{-1} = e^{GA_pG^{-1}t}.$$

We can now state

Theorem 5.1. Given a nonlinear system $X(.):U\to D(C)$ on a complex manifold C, we can associate with it a Volterra series

$$\Phi(t) = w_o(t) + \sum_{\gamma=1}^{\infty} \int_o^t \cdots \int_o^t w_\gamma(t,\sigma_1,\ldots,\sigma_\gamma)v(\sigma_1)\otimes \cdots \otimes v(\sigma_\gamma)d\sigma_1\ldots d\sigma_\gamma$$

where the kernels are given locally by

$$w_\gamma(t,\sigma_1,\ldots,\sigma_\gamma;p) = e^{A_p(t-\sigma_1)} B_p e^{A_p(\sigma_1-\sigma_2)} B_p \ldots B_p e^{A_p\sigma_\gamma}\Phi_{op}$$

where $B_p = (B_{1p},\ldots,B_{mp})$. Moreover the kernels transform according to

$$w_\gamma(t,\sigma_1,\ldots,\sigma_\gamma\ q) = e^{A_q(t-\sigma_1)} B_q e^{A_q(\sigma_1-\sigma_2)} B_q \ldots B_q e^{A_q\sigma_\gamma}\Psi_{oq}$$

$$= G_e^{A_p(t-\sigma_1)} G^{-1}GB_pG^{-1}G_e^{A_p(\sigma_1-\sigma_2)} \ldots G_e^{A_p\sigma_\gamma}G^{-1}G\Phi_{op}$$

$$= Gw_\gamma(t,\sigma_1,\ldots,\sigma_\gamma;p).$$

This theorem shows, therefore, that a meromorphic system on a complex manifold has a global Volterra series expansion.

6. Conclusions

In this paper we have derived a global Volterra series for a meromorphic differential equation defined on a complex manifold. As we have seen, such a series is a 'bundle' of local Volterra series with a transformation group induced by local coordinate changes.

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