Universities of Leeds, Sheffield and York
http://eprints.whiterose.ac.uk/

Formal Baire space in constructive set theory

Giovanni Curi and Michael Rathjen*

Introduction

Constructive topology is generally based on the notion of *locale*, or *formal space* (see [10, 9, 8], and [22, pg. 378], for an explanation of why this is the case). Algebraically, locales are particular kinds of lattices that, like other familiar algebraic structures, can be presented using the method of *generators and relations*, cf. e.g. [23]. Equivalently, they may be described using *covering systems* [9, 13]. Set-theoretically, 'generators and relations' and covering systems can be regarded as *inductive definitions*. Classical or intuitionistic fully impredicative systems, such as intuitionistic Zermelo-Fraenkel set theory, IZF [2], or the intuitionistic theory of a topos [11], are sufficiently strong to ensure that such inductive definitions do give rise to a locale or formal space. This continues to hold in (generalized) predicative systems as for example the constructive set theory CZF augmented by the weak regular extension axiom wREA (where the covering systems give rise to so-called inductively generated formal spaces, [1]). However, albeit being much weaker than classical set theory ZF, the system CZF + wREA is considerably stronger than CZF. As it turns out, CZF + wREA is a subsystem of classical set theory ZF plus the axiom of choice AC, but not of ZF alone (cf. [17]). Naturally, this lends itself to the question of what can be proved in the absence of wREA.

In this note we show that working in CZF alone, a covering system may fail to define a formal space already in a familiar case. It is easy to see that CZF can prove that, e.g., the covering systems used to present formal Cantor space **C**, and the formal real line **R**, do define formal spaces; this is essentially because the associated inductive definition is a finitary one for **C**, and can be replaced by a finitary one, plus an application of restricted Separation, for **R** (as apparently first noted by T. Coquand, see [7, Section 6] for more details). There has been for some

time the expectation that the same does not hold for formal Baire space **B**. The main result in this note, Theorem 3.4, will confirm this expectation.

This result, in conjunction with [3, Proposition 3.10] (or [4, B.4]), also answers in the negative the question, asked in [3], whether CZF proves that the Brouwer (or constructive) ordinals form a set (although this could in fact also be inferred by previous results, see Section 3). Independently, this was also shown in [4, Corollary B.5].[1]

A corollary of Theorem 3.4 is moreover that the full subcategory FSp_i of the category FSp of formal spaces defined by the inductively generated formal spaces [1, 5], fails to have infinitary products in CZF, according at least to the received construction. The notion of inductively generated formal space was introduced in [5] to make it possible to predicatively perform basic constructions on formal spaces as that of the product formal space; these constructions indeed do not appear to be possible for general formal spaces without recourse to some strong impredicative principle [5]. In CZF augmented by the weak regular extension axiom wREA, the category FSp_i of inductively generated formal spaces can instead be proved to have infinitary products (and more generally, all limits). Exploiting the isomorphism of the product $\prod_{n \in \mathbb{N}} \mathbb{N}$ with **B**, one shows that this need no longer be the case in CZF alone.

Although CZF does not prove that **B** is a formal space, it does prove that **B** is an *imaginary locale* [7]. More generally, every covering system defines an imaginary locale in CZF. Imaginary locales give rise to a category ImLoc that extends the category FSp of formal spaces, and that has all limits (in particular all products) already assuming a fragment of CZF. As the categories FSp and FSp_i, ImLoc is equivalent to the ordinary category of locales in a fully impredicative system as classical set theory.

1   Constructive set theory and inductive definitions

The language of *Constructive Zermelo-Fraenkel Set Theory*, CZF, is the same as that of Zermelo-Fraenkel Set Theory, ZF, with $\in$ as the only non-logical symbol. CZF is based on intuitionistic predicate logic with equality, and has the following axioms and axiom schemes:

1. Extensionality: $\forall a \forall b (\forall y (y \in a \leftrightarrow y \in b) \rightarrow a = b)$.

2. Pair: $\forall a \forall b \exists x \forall y (y \in x \leftrightarrow y = a \lor y = b)$.

---

[1] At the end of this paper there is a post scriptum explaining the relationship between some of the findings in [3], [4], and the present paper.

3. Union: $\forall a \exists x \forall y (y \in x \leftrightarrow (\exists z \in a)(y \in z))$.

4. Restricted Separation scheme:

$$\forall a \exists x \forall y (y \in x \leftrightarrow y \in a \wedge \phi(y)),$$

for ϕ a restricted formula. A formula ϕ is *restricted* if the quantifiers that occur in it are of the form $\forall x \in b$, $\exists x \in c$.

5. Subset Collection scheme:

$$\forall a \forall b \exists c \forall u ((\forall x \in a)(\exists y \in b)\phi(x,y,u) \ \rightarrow$$
$$(\exists d \in c)((\forall x \in a)(\exists y \in d)\phi(x,y,u) \wedge (\forall y \in d)(\exists x \in a)\phi(x,y,u))).$$

6. Strong Collection scheme:

$$\forall a ((\forall x \in a)\exists y \phi(x,y) \ \rightarrow$$
$$\exists b ((\forall x \in a)(\exists y \in b)\phi(x,y) \wedge (\forall y \in b)(\exists x \in a)\phi(x,y))).$$

7. Infinity: $\exists a (\exists x \in a \wedge (\forall x \in a)(\exists y \in a)x \in y)$.

8. Set Induction scheme: $\forall a ((\forall x \in a)\phi(x) \rightarrow \phi(a)) \rightarrow \forall a \phi(a)$.

See [2] for further information on CZF and related systems. We shall denote by CZF$^-$ the system obtained from CZF by leaving out the Subset Collection scheme. Note that from Subset Collection one proves that the class of functions b^a from a set a to a set b is a set, i.e., Myhill's Exponentiation Axiom. Intuitionistic Zermelo-Fraenkel set theory based on collection, IZF, has the same theorems as CZF extended by the unrestricted Separation Scheme and the Powerset Axiom. Moreover, the theory obtained from CZF by adding the Law of Excluded Middle has the same theorems as ZF.

As in classical set theory, we make use of class notation and terminology [2]. The set $\mathbb{N}$ of natural numbers is the unique set x such that

$$\forall u [u \in x \leftrightarrow (u = \emptyset \wedge (\exists v \in x)(u = v \cup \{v\}))].$$

A major role in constructive set theory is played by inductive definitions. An *inductive definition* is any class Φ of pairs. A class A is $\Phi-closed$ if:

$$(a, X) \in \Phi, \text{ and } X \subseteq A \text{ implies } a \in A.$$

The following theorem is called *the class inductive definition theorem* [2].

Theorem 1.1 (CZF⁻). *Given any class* Φ, *there exists a least* $\Phi-$*closed class* $I(\Phi)$, *the class inductively defined by* Φ.

Given any inductive definition Φ and any class U, there exists a smallest class containing U which is closed under Φ. This class will be denoted by $I(\Phi, U)$. Note that $I(\Phi, U)$ is the class inductively defined by $\Phi' = \Phi \cup (U \times \{\emptyset\})$, i.e., $I(\Phi, U) = I(\Phi')$. Given a set S, we say that Φ is an inductive definition on S if $\Phi \subseteq S \times \mathsf{Pow}(S)$, with $\mathsf{Pow}(S)$ the *class* of subsets of S. An inductive definition Φ is *finitary* if, whenever $(a, X) \in \Phi$, there exists a surjective function $f : n \to X$ for some $n \in \mathbb{N}$. Φ is *infinitary* if it is not finitary.

As is shown in Section 3, even when Φ is a set, $I(\Phi)$ need not be a set in CZF. For this reason, CZF is often extended with the *Regular Extension Axiom*, REA.

> REA: every set is the subset of a regular set.

A set c is *regular* if it is transitive, inhabited, and for any $u \in c$ and any set $R \subseteq u \times c$, if $(\forall x \in u)(\exists y)\langle x, y \rangle \in R$, then there is a set $v \in c$ such that

$$(\forall x \in u)(\exists y \in v)((x, y) \in R) \quad \wedge \quad (\forall y \in v)(\exists x \in u)((x, y) \in R). \quad (1)$$

c is said to be *weakly regular* if in the above definition of regularity the second conjunct in (1) is omitted. The weak regular extension axiom, wREA, is the statement that every set is the subset of a weakly regular set.
In CZF + wREA, the following theorem can be proved.

Theorem 1.2 (CZF + wREA). *If* Φ *is a set, then* $I(\Phi)$ *is a set.*

The foregoing result holds in more generality for inductive definitions that are bounded (see [2]).

The theory CZF has has the same strength as classical Kripke-Platek set theory or the theory one non-iterated inductive definitions ID_1 by [15, Theorem 4.14]. It is therefore much weaker than Π_1^1-comprehension. The strength of CZF + REA and CZF + wREA is the same as that of the subsystem of second order arithmetic with Δ_2^1-comprehension and Bar induction (see [15, Theorem 5.12] and [16, Theorem 4.7]). Thus it is much stronger than CZF, but still very weak compared to ZF. ZF+AC proves REA whereas wREA (and a fortiori REA) is not provable in ZF alone by [17, Corollary 7.1].

Sometimes one considers extensions of CZF by constructively acceptable choice principles, such as the principle of countable choice:

> AC_ω : for every class A, if $R \subseteq \mathbb{N} \times A$ satisfies $(\forall n \in \mathbb{N})(\exists a \in A)R(n, a)$ then there exists $f : \mathbb{N} \to A$ such that $f \subseteq R$.

2   Constructive locale theory

Unless stated otherwise we will be working in CZF^-. The notion of *locale* [11, 8, 9] provides the concept of topological space adopted in intuitionistic fully impredicative systems such as topos logic (Higher-order Heyting arithmetic). In the absence of the Powerset Axiom, however, as for instance in constructive generalized predicative systems, this notion splits into inequivalent concepts.

A *preordered set* is a pair $(S, \leq)$ with S and $\leq$ sets, and $\leq$ a reflexive and transitive relation. For U a subset, or a subclass, of S, $\downarrow U$ abbreviates $\{a \in S : (\exists b \in U)a \leq b\}$. We also use $U \downarrow V$ for $\downarrow U \cap \downarrow V$.

A *generalized covering system* on a preordered set $(S, \leq)$ is an inductive definition Φ on S such that, for all (a, X) in Φ,

1. $X \subseteq \downarrow \{a\}$,

2. if $b \leq a$ then there is $(b, Y) \in \Phi$ with $Y \subseteq \downarrow X$.

An *imaginary locale* is a structure of the form

$$X \equiv (S, \leq, \Phi),$$

with Φ a generalized covering system on the preordered set $(S, \leq)$. The set S is called the *base* of X.

Given an imaginary locale $X \equiv (S, \leq, \Phi)$, we let $\Phi_\leq$ denote the class of pairs $\Phi \cup \{(b, \{a\}) \mid b \leq a\}$. As $\Phi_\leq$ is an inductive definition, given any *subclass* U of S, by Theorem 1.1, there exists (in CZF^-)

$$A(U) \equiv I(\Phi_\leq, U),$$

i.e. the smallest class containing U closed under $\Phi_\leq$.

Theorem 2.1 (CZF^-). *For every $a, b \in S$, and for all subclasses U, V of S, the following hold:*

0. $\downarrow \{a\} \subseteq A(\{a\})$,

1. $U \subseteq A(U)$,

2. $U \subseteq A(V)$ implies $A(U) \subseteq A(V)$,

3. $A(U) \cap A(V) \subseteq A(U \downarrow V)$.

See [7] for the proof of this result and further information on imaginary locales. Equipped with a suitable notion of *continuous function*, imaginary locales form the (superlarge) category ImLoc.

Two full subcategories of this category had been considered earlier as possible counterpart of the category of locales in constructive (generalized) predicative settings. Let FSp be the full subcategory of ImLoc given by those imaginary locales $X \equiv (S, \leq, \Phi)$ which satisfy

(A-smallness)   for every $U \in \mathsf{Pow}(S)$, $A(U)$ *is a set*,

and let FSp$_i$ be the full subcategory of ImLoc given by those $X \equiv (S, \leq, \Phi)$ which satisfy the smallness condition above, and are such that

(Φ-smallness) Φ *is a set*,

i.e., such that Φ is an ordinary *covering system*. FSp and FSp$_i$ are respectively (equivalent to) the category of *formal spaces* and the category of *inductively generated formal spaces* [1, 5].

Formal spaces are generally presented in terms of a covering relation on a preordered set. Given a (class-)relation $\lhd \subseteq S \times \mathsf{Pow}(S)$, let the *saturation* of a subset U of S be defined as the class $A(U) = \{a \in S : a \lhd U\}$, where we write $a \lhd U$ for $\lhd(a, U)$. Then, by definition, $\lhd$ is a *covering relation* if, with the class $A(U)$ thus re-defined, the A-smallness condition is satisfied, and the conditions in Theorem 2.1 are satisfied for every $U, V \in Pow(S)$.

One passes from one definition of formal space to the other by associating to an imaginary locale $(S, \leq, \Phi)$ satisfying A-smallness, the structure

$$(S, \leq, \lhd),$$

where $a \lhd U \iff a \in I(\Phi_{\leq}, U)$; in the other direction, given a covering relation $\lhd$ on $(S, \leq)$, one obtains a generalized covering system $(S, \leq, \Phi)$ satisfying A-smallness by letting

$$\Phi \equiv \{(a, U) \mid a \lhd U \ \& \ U \subseteq \downarrow a\}.$$

Note that the same correspondence exists more generally between imaginary locales and covering relations that are not required to satisfy the A-smallness condition.

Assuming the full Separation scheme Sep, the categories ImLoc and FSp coincide, since the A-smallness condition is always satisfied. On the other hand, even

in CZF + Sep, ImLoc is not the same as FSp_i, as there are formal spaces of various types that cannot be inductively generated in this system [6].

In CZF, every formal space X has an associated *set-generated class-frame* $Sat(X)$, see [1]; the carrier of $Sat(X)$ is given by the class $\{A(U) \mid U \in \mathsf{Pow}(S)\}$ of saturated subsets of X (meets and joins are given by $U \wedge V \equiv U \downarrow V$, and $\bigvee_{i \in I} U_i \equiv A(\bigcup_{i \in I} U_i)$). Note that if X is instead an imaginary locale, $Sat(X)$ cannot be constructed in a generalized predicative setting, as $A(U)$ may fail to be a set for some U.

With the full Separation scheme and the Powerset axiom available, as e.g., in IZF, $Sat(X)$ is an ordinary frame (locale); in such a system, the categories ImLoc, FSp, FSp_i (coincide, and) are all equivalent to the category of locales.

The concept of formal space was the first to be introduced, in [21]. The reason that led to consider the stronger notion of inductively generated formal space is that it does not appear to be possible to carry out, in a generalized predicative setting, standard basic constructions for general formal spaces, such as that of the product space [5]. The category FSp_i has been shown to have all products and equalizers (hence all limits) [23, 14]. However, this only holds in CZF + REA, and we shall see in the next section that the given construction of products may in fact fail to yield, in CZF, an inductively generated formal space from inductively generated formal spaces. Moreover, as already recalled, the restriction to FSp_i rules out several types of formal spaces of interest [6]. By contrast, the category ImLoc is complete (has all limits) already over CZF$^-$ [7], and, as seen, is an extension of the category of formal spaces which in a fully impredicative system as IZF is still equivalent to the category of locales.

We conclude this section noting that, in the absence of REA, an imaginary locale satisfying Φ-smallness need not satisfy A-smallness (with REA it does, recall Theorem 1.2). The next section presents an example of such a phenomenon. Imaginary locales of this kind determine a full subcategory of ImLoc, called the category of *geometric locales*; this category is itself complete in CZF$^-$ [7].

3 Formal Baire space

Recall that Baire space is the set $\mathbb{N}^{\mathbb{N}}$, endowed with the product topology. Its point-free version, formal Baire space, is defined as follows. Let $\mathbb{N}^*$ be the set of finite sequences of natural numbers; *formal Baire space* $\mathbf{B}$ is the generalized covering system

$$\mathbf{B} \equiv (\mathbb{N}^*, \leq, \Phi^{\mathbf{B}}),$$

where, for $s, t \in \mathbb{N}^*$, $s \leq t$ if and only if t is an initial segment of s, and

$$\Phi^{\mathbf{B}} = \{(s, \{s * \langle n \rangle \mid n \in \mathbb{N}\}) \mid s \in \mathbb{N}^*\}.$$

$\mathbf{B}$ is then an imaginary locale in CZF^-. Note that $\Phi^{\mathbf{B}}$ is an *infinitary* inductive definition. The class $\Phi^{\mathbf{B}}$ is a set in CZF^-, so that it is a set in $\mathrm{CZF} + \mathrm{REA}$. By Theorem 1.2, then, $A_{\mathbf{B}}(U) \equiv I(\Phi^{\mathbf{B}}_{\leq}, U)$ is a set for every $U \in \mathsf{Pow}(\mathbb{N}^*)$. Thus:

Proposition 3.1. $\mathrm{CZF} + \mathrm{REA}$ *proves that $\mathbf{B}$ is a formal space.*

Recall that a *point* of a formal space (or, more generally, of an imaginary locale) $X \equiv (S, \leq, \Phi^X)$ is a inhabited subset α of S such that
   (i) for every $a, b \in \alpha$ there is $c \in \{a\} \downarrow \{b\}$ with $c \in \alpha$,
   (ii) for every $a \in S$, $U \in \mathsf{Pow}(S)$, if $a \in \alpha$ and $a \in I(\Phi^X_{\leq}, U)$ then there is $b \in U$ such that $b \in \alpha$.
Points of $\mathbf{B}$ may be identified with infinite sequences of non-negative integers. For $a \in S$, let $ext(a)$ denote the class of points 'in a', i.e. the class of points to which a belongs. Then, $X \equiv (S, \leq, \Phi^X)$ is *spatial* whenever, for every $a \in S$ and $U \in \mathsf{Pow}(S)$, one has $a \in I(\Phi^X_{\leq}, U)$ if and only if $ext(a) \subseteq \bigcup_{b \in U} ext(b)$.

Note that the principle of Monotone Bar Induction BI_M is exactly the statement that $\mathbf{B}$ is spatial [8]. Moreover, spatiality of $\mathbf{B}$ implies the spatiality of formal Cantor space and of the formal Real Unit Interval; spatiality of these formal spaces is in turn respectively equivalent to the Fan Theorem, and to compactness of the real unit interval (see [8], or [3]).

It is well-known that the compactness of the Real Unit Interval (and hence Monotone Bar Induction) is inconsistent with Church Thesis, so that the spatiality of the above-mentioned formal spaces is independent from the systems we are considering.

Contrary to what happens with the Fan Theorem, FT, adding decidable Bar Induction BI_D (which is a consequence of monotone Bar Induction, BI_M) to CZF has a marked effect.

Theorem 3.2. ([20, Corollary 4.8],[19, Theorem 9.10(i)])

   (i) $\mathrm{CZF} + \mathrm{BI}_D$ *proves the 1-consistency of* CZF.

   (ii) CZF *and* $\mathrm{CZF} + \mathrm{FT}$ *have the same proof-theoretic strength.*

On the other hand, BI_M has no effect on the proof-theoretic strength in the presence of REA.

Theorem 3.3. [19, Theorem 9.10(ii)] $\mathrm{CZF} + \mathrm{REA}$ *and* $\mathrm{CZF} + \mathrm{REA} + \mathrm{DC} + \mathrm{BI}_M$ *have the same proof-theoretic strength.*

Formal Cantor Space (defined as formal Baire space, but with $\mathbb{N}^*$ replaced everywhere by $\{0,1\}^*$, and with $\mathbb{N}$ replaced by $\{0,1\}$ in the covering system), and the formal Real Unit Interval involve finitary inductive definitions, and can be proved formal spaces already over CZF^- (the covering system for the formal Real Unit Interval is in fact given by an infinitary inductive definition, but this can be seen to have the same effect of a finitary one plus an application of the restricted Separation scheme, cf. [7, Section 6]).

It has been an open question for some time whether CZF alone proves that $\mathbf{B}$ is a formal space.

Theorem 3.4. $\mathrm{CZF} + \mathrm{AC}_\omega$ *does not prove that, for every* $U \in \mathbf{Pow}(\mathbb{N}^*)$, $I(\Phi^{\mathbf{B}}_{\leq}, U)$ *is a set. The unprovability result obtains even if one adds the Dependent Choices Axiom,* DC *(cf. [2]), and the Presentation Axiom (cf. [2]),* PA*, to CZF.*

Proof. We plan to show, using the axioms of CZF, that from the assertion

$$\forall U \subseteq \mathbb{N}^* \; I(\Phi^{\mathbf{B}}_{\leq}, U) \text{ is a set} \tag{2}$$

it follows that the well-founded part, $\mathrm{WF}(\prec)$, of every decidable ordering $\prec$ on $\mathbb{N}$ is a set. Here decidability means that $\forall n, m \in \mathbb{N} \, (n \prec m \lor \neg \, n \prec m)$ and by an ordering we mean any transitive and irreflexive binary relation (which is also a set). Recall that $\mathrm{WF}(\prec)$ is the smallest class X such that for all $n \in \mathbb{N}$,

$$\forall m \in \mathbb{N} \, (m \prec n \to m \in X) \quad \text{implies} \quad n \in X. \tag{3}$$

If one then takes $\prec$ to be the ordering which represents the so-called Bachmann-Howard ordinal, it follows from [18, 4.13, 4.14] that (2) implies the 1-consistency of CZF (actually the uniform reflection principle for CZF and more), and therefore, in light of [15, Theorem 4.14], also the 1-consistency of $\mathrm{CZF} + \mathrm{AC}_\omega + \mathrm{DC} + \mathrm{PA}$. As a result, (2) is not provable in $\mathrm{CZF} + \mathrm{AC}_\omega + \mathrm{DC} + \mathrm{PA}$.

It remains to show that, assuming (2), $\mathrm{WF}(\prec)$ is a set. Define U to be the subset of $\mathbb{N}^*$ consisting of all sequences $\langle n_1, \ldots, n_r \rangle$ with $r > 1$ such that $\neg \, n_j \prec n_i$ for some $1 \leq i < j \leq r$. Observe that $\forall s \in \mathbb{N}^* \, (s \in U \lor s \notin U)$ owing to the decidability of $\prec$.

Let $s \in \mathbb{N}^*$. We say that n *is in* s if s is of the form $s = \langle n_1, \ldots, n_r \rangle$ with $r \geq 1$ and $n = n_i$ for some $1 \leq i \leq r$. Another way of saying that n is in s is that $s = s_1 * \langle n \rangle * s_2$ for some $s_1, s_2 \in \mathbb{N}^*$. Let $V \subseteq \mathbb{N}$ be the class defined as follows:

$$n \in V \quad \text{iff} \quad \forall s \in \mathbb{N}^* \, (n \text{ in } s \to s \in I(\Phi^{\mathbf{B}}_{\leq}, U)). \tag{4}$$

Claim 1: $\mathrm{WF}(\prec) \subseteq V$. To show this, assume $n \in \mathbb{N}$ and $m \in V$ for all $m \prec n$. Suppose n is in s. For an arbitrary $k \in \mathbb{N}$ we then have $s * \langle k \rangle \in I(\Phi^{\mathbf{B}}_{\leq}, U)$, for

$\neg k \prec n$ implies $s * \langle k \rangle \in U$ and hence $s * \langle k \rangle \in I(\Phi_{\leq}^{\mathbf{B}}, U)$, whereas $k \prec n$ entails $s * \langle k \rangle \in I(\Phi_{\leq}^{\mathbf{B}}, U)$ by assumption. Thus $s \in I(\Phi_{\leq}^{\overline{\mathbf{B}}}, U)$ owing to its inductive definition. Whence $n \in V$.

$\langle \rangle$ denotes the empty sequence of $\mathbb{N}^*$. Let

$$Y = \{ s \in \mathbb{N}^* \mid s \in U \ \vee \ \exists m \in \mathrm{WF}(\prec) \, (m \text{ in } s) \ \vee \ \forall m \, m \in \mathrm{WF}(\prec) \}.$$

Claim 2: $I(\Phi_{\leq}^{\mathbf{B}}, U) \subseteq Y$. By definition of Y, we have $U \subseteq Y$ and whenever $s \in Y$ and $t \in \overline{\mathbb{N}}^*$ then $s * t \in Y$. To confirm the claim it thus suffices to show that $s * \langle n \rangle \in Y$ for all $n \in \mathbb{N}$ implies $s \in Y$. So assume $s * \langle n \rangle \in Y$ for all $n \in \mathbb{N}$. $s \in U$ implies $s \in Y$. If $s = \langle \rangle$, then $\langle n \rangle \in Y$ for all n. Thus $s = \langle \rangle$ implies that for all n, $n \in \mathrm{WF}(\prec)$ or $\forall m \, m \in \mathrm{WF}(\prec)$, and therefore $n \in \mathrm{WF}(\prec)$, yielding $s \in Y$.

Henceforth we may assume that $s \notin U$ and $s \neq \langle \rangle$. In particular $s = t * \langle k \rangle$ for some $t \in \mathbb{N}^*$, $k \in \mathbb{N}$, and the components of s are arranged in $\prec$-descending order. Let $n \prec k$. We then have $s * \langle n \rangle = t * \langle k \rangle * \langle n \rangle \notin U$ and hence there is an $l \in \mathrm{WF}(\prec)$ such that l is in $s * \langle n \rangle$ or else $\forall m \, m \in \mathrm{WF}(\prec)$. Thus $n \prec l \ \vee \ n = l$ for some $l \in \mathrm{WF}(\prec)$ or $\forall m \, m \in \mathrm{WF}(\prec)$, yielding $n \in \mathrm{WF}(\prec)$. In consequence, $\forall n \prec k \, n \in \mathrm{WF}(\prec)$, thus $k \in \mathrm{WF}(\prec)$, and hence $s \in Y$. This finishes the proof of Claim 2.

As $n \in V$ implies $\langle n \rangle \in I(\Phi_{\leq}^{\mathbf{B}}, U)$, we deduce with the help of Claim 2 that $\langle n \rangle \in Y$. Hence $n \in \mathrm{WF}(\prec)$ or $\forall m \, m \in \mathrm{WF}(\prec)$, whence $n \in \mathrm{WF}(\prec)$. Thus $V \subseteq \mathrm{WF}(\prec)$, and in view of Claim 1, we have $V = \mathrm{WF}(\prec)$. If $I(\Phi_{\leq}^{\mathbf{B}}, U)$ were a set, V would be a set, too. Hence (2) entails that $\mathrm{WF}(\prec)$ is a set. $\quad\square$

Corollary 3.5. $\mathrm{CZF} + \mathrm{AC}_\omega + \mathrm{DC} + \mathrm{PA}$ *does not prove that the imaginary locale* **B** *is a formal space.*

This in particular answers the question in footnote 2 of [3]. Note that **B** is in fact a geometric locale, since $\Phi^{\mathbf{B}}$ is a set in CZF^-.

Remark 3.6. Direct calculations show that the construction of products for inductively generated formal spaces [23] gives $\mathbf{B} \cong \prod_{n \in \mathbb{N}} \mathbb{N}$, where $\mathbb{N}$ is the discrete formal space of the natural numbers, which is trivially inductively generated in CZF. It is easy to see that the assumption that the imaginary locale $\prod_{n \in \mathbb{N}} \mathbb{N}$ is a formal space implies that $I(\Phi_{\leq}^{\mathbf{B}}, U)$ is a set, for U as in the proof of Theorem 3.4, so that $\prod_{n \in \mathbb{N}} \mathbb{N}$ also cannot be proved to be a formal space in $\mathrm{CZF} + \mathrm{AC}_\omega + \mathrm{DC} + \mathrm{PA}$.

This fact shows that the category of inductively generated formal spaces, which is complete in CZF + REA, is not closed for infinitary products (at least according to the received construction) in (CZF and) $\mathrm{CZF} + \mathrm{AC}_\omega + \mathrm{DC} + \mathrm{PA}$. Note that $\prod_{n \in \mathbb{N}} \mathbb{N} \cong \mathbf{B}$ *is* the product in the category of imaginary locales [7].

The above theorem also answers the question in footnote 3 of [3], whether CZF proves that the Brouwer (or constructive) Ordinals form a set. Call a relation $R \subseteq S \times \mathsf{Pow}(S)$ *set-presented* if a mapping $D : S \rightarrow \mathsf{Pow}(\mathsf{Pow}(S))$ is given satisfying $R(a, U) \iff (\exists V \in D(a)) \, V \subseteq U$, for every $a \in S, U \in \mathsf{Pow}(S)$.

Lemma 3.7 (CZF$^-$)**.** *Let* $X \equiv (S, \leq, \Phi)$ *be an imaginary locale. If* $R(a, U) \equiv a \in I(\Phi_\leq, U)$ *is set-presented, then* X *is a formal space.*

Proof. The class $I(\Phi_\leq, U)$ is a set for every $U \in \mathsf{Pow}(S)$, since $I(\Phi_\leq, U) = \{a \in S \mid (\exists V \in D(a)) \, V \subseteq U\}$, and the latter class is a set by Replacement and Restricted Separation. □

The following result, without mentioning PA, was also independently established in [4, Corollary B.5].

Corollary 3.8. CZF $+ \mathrm{AC}_\omega + \mathrm{DC} + \mathrm{PA}$ *does not prove that the Brouwer Ordinals form a set.*

Proof. The proof of [3, Proposition 3.10] or [4, Proposition B.4] shows that, in CZF $+ \mathrm{AC}_\omega$ plus the assertion that the Brouwer Ordinals form a set, the relation $R_{\mathbf{B}}(s, U) \equiv s \in I(\Phi_\leq^{\mathbf{B}}, U)$ is set-presented. But then $I(\Phi_\leq^{\mathbf{B}}, U)$ is a set for every $U \in \mathsf{Pow}(\mathbb{N}^*)$, by Lemma 3.7. So if CZF proves that the Brouwer Ordinals form a set, CZF $+ \mathrm{AC}_\omega$ proves that $\mathbf{B}$ is a formal space. □

Implicitly, it has been known for a long time that CZF $+ \mathrm{AC}_\omega + \mathrm{DC} + \mathrm{PA}$ does not prove that the Brouwer ordinals form a set, owing to [15, Theorem 4.14] and an ancient result due to Kreisel. In [12] Kreisel showed that the intuitionistic theory $\mathrm{ID}^i(\mathcal{O})$ of the Brouwer ordinals $\mathcal{O}$ is of the same proof-theoretic strength as the classical theory of positive arithmetical inductive definitions ID_1. $\mathrm{ID}^i(\mathcal{O})$ is an extension of Heyting arithmetic via a predicate for the Brouwer ordinals and axioms pertaining to $\mathcal{O}$'s inductive nature. Thus from [15, Theorem 4.14] it follows that $\mathrm{ID}^i(\mathcal{O})$ and CZF $+ \mathrm{AC}_\omega + \mathrm{DC} + \mathrm{PA}$ have the same proof-theoretic strength. But if the latter theory could prove that the Brouwer ordinals form a set, it could easily prove that $\mathrm{ID}^i(\mathcal{O})$ has a set model, and in particular the consistency of $\mathrm{ID}^i(\mathcal{O})$.

Post scriptum.   The research reported in this paper was carried out in June 2011. The question of whether CZF proves that the Brouwer ordinals form a set, raised in [3], was also answered (in the negative) by the authors of [3], Benno van den Berg and Ieke Moerdijk. They posted a new version of their paper on arxiv.org in November 2011 which in the meantime got published as [4]. In [4, Corollary B.5] they show that the Brouwer ordinals cannot be proved to be a set on the basis of CZF+DC. Theirs and our proof both hinge on proof-theoretic results from

[20]. Our non-provability result 3.4, however, is stronger than [4, Corollary A.3]. The latter shows that CZF+DC cannot prove that Baire space has a presentation whereas the former shows that this theory (or even CZF+PA) does not prove that **B** is a formal space.

References

[1] P. Aczel, "Aspects of general topology in constructive set theory", *Ann. Pure Appl. Logic*, 137, 1-3 (2006), pp. 3-29.

[2] P. Aczel, M. Rathjen "Notes on Constructive Set Theory", Mittag-Leffler Technical Report No. 40, 2000/2001.

[3] B. van den Berg, I. Moerdijk, "Derived rules for predicative set theory: an application of sheaves". Version of September 18, 2010 [http://arxiv.org/abs/1009.3553v1].

[4] B. van den Berg, I. Moerdijk, "Derived rules for predicative set theory: an application of sheaves". Annals of Pure and Applied Logic (2012), doi:10.1016/j.apal.2012.01.010

[5] T. Coquand, G. Sambin, J. Smith, S. Valentini, "Inductively generated formal topologies", *Annals of Pure and Applied Logic* 124, 1-3 (2003), pp. 71–106.

[6] G. Curi, "On some peculiar aspects of the constructive theory of point-free spaces", *Mathematical Logic Quarterly*, 56, 4 (2010), pp. 375 – 387.

[7] G. Curi, "Topological inductive definitions". *Ann. Pure Appl. Logic*, doi:10.1016/j.apal.2011.12.005.

[8] M. Fourman and R. Grayson, "Formal Spaces", in: A.S. Troelstra and D. van Dalen (eds.), *The L.E.J. Brouwer Centenary Symposium*, pp. 107–122. North Holland (1982).

[9] P. T. Johnstone, "Stone Spaces", Cambridge University Press, 1982.

[10] P. T. Johnstone, "The points of pointless topology", *Bull. Am. Math. Soc.* **8** 1, 41–53 (1983).

[11] P. T. Johnstone, "Sketches of an elephant. A topos theory compendium. II," Oxford Logic Guides 44; Oxford Science Publications. Oxford: Clarendon Press, 2002.

[12] G. Kreisel "A survey of proof theory", *J. Symbolic Logic* 33 (1968) pp. 321–388.

[13] S. MacLane, I. Moerdijk, "Sheaves in Geometry and Logic - A First Introduction to Topos Theory", Springer, 1992.

[14] E. Palmgren, "Predicativity problems in point-free topology", in: V. Stoltenberg-Hansen et al. (eds.), Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic, held in Helsinki, Finland, August 14–20, 2003, *Lecture Notes in Logic* 24, ASL, (AK Peters Ltd, 2006), pp. 221–231.

[15] M. Rathjen "The strength of some Martin–Löf type theories", Department of mathematics, Ohio State University, Preprint series (1993) 39 pages.

[16] M. Rathjen "The anti-foundation axiom in constructive set theories", in: G. Mints, R. Muskens (eds.) *Games, Logic, and Constructive Sets.* (CSLI Publications, Stanford, 2003) 87–108.

[17] M. Rathjen, R.S. Lubarsky "On the regular extension axiom and its variants", *Math. Log. Q.* 49, 5 (2003), pp. 511–518.

[18] M. Rathjen "Replacement versus collection in constructive Zermelo-Fraenkel set theory", *Ann. Pure Appl. Logic* 136 (2005) pp. 156–174.

[19] M. Rathjen, "Constructive set theory and Brouwerian principles", *J. Universal Computer Science* 11 (2005), pp. 2008–2033.

[20] M. Rathjen, "A note on bar induction in constructive set theory", *Math. Log. Q.* 52, 3 (2006), pp. 253–258.

[21] G. Sambin, "Intuitionistic formal spaces - a first communication", in: *Mathematical Logic and its Applications*, D. Skordev, ed. Plenum, (1987), pp. 187-204.

[22] A. Troelstra and D. van Dalen, "Constructivism in mathematics, an introduction, Vol. I,II", Studies in logic and the foundation of mathematics **121, 123** (Amsterdam etc., North-Holland, 1988).

[23] S. Vickers, "Some constructive roads to Tychonoff", in: L. Crosilla and P. Schuster (eds.) *From Sets and Types to Topology and Analysis: Practicable Foundations for Constructive Mathematics*, pp. 223 – 238, Oxford Logic Guides 48, OUP (2005).