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ESTIMATING HIGHER ORDER SPECTRA

Dr. S. A. Billings, BEng,PhD,CEng,MIEE,AFIMA,MInstMC

K. M. Tsang, BEng

Department of Control Engineering
University of Sheffield
Mappin Street, Sheffield S1 3JD

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Dr S A Billings, BEng, PhD, CEng, MIEE, AFIMA, MInstMC
K M Tsang, BEng

Department of Control Engineering
University of Sheffield
Sheffield, UK

ABSTRACT

New parametric algorithms for estimating the frequency response for linear systems and generalised frequency response functions for nonlinear systems are described.

NOMENCLATURE

d	time delay
ℓ	process model order
m	noise model order
n	degree of nonlinearity
s	Laplace operator
T	sampling interval
e(k)	coloured noise
u(k)	input
x(k)	noisefree process output
y(k)	measured process output
ξ(k)	white noise

INTRODUCTION

The frequency domain or spectral analysis of linear systems is now well established and finds wide application in all branches of science and engineering. Traditionally spectral densities have been estimated using FFT algorithms and window functions although recently new parametric methods of estimating power spectra using autoregressive models have been introduced. Parametric methods for estimating the cross-spectra and frequency response function of a linear system appear to have been largely neglected however and new methods of achieving this objective are discussed in the present paper and shown to offer a significant improvement compared with the classical approach.

Spectral analysis of nonlinear systems has been a neglected area of study largely because of the inherent difficulty in estimating generalised frequency response functions by extending the FFT/windowing algorithms to work in many dimensions. However, by estimating the coefficients in a NARMAX model[1] (Nonlinear AutoRegressive Moving Average model with eXogenous inputs) representation of the nonlinear system any order of generalised transfer function can be computed directly. The evolution of the linear frequency response as a function of input operating point and many other properties can also be readily computed to aid the interpretation of the influence of the nonlinearities on the system frequency content. Several examples are included to illustrate the advantages of these new algorithms.

PARAMETRIC SPECTRAL ESTIMATION FOR LINEAR SYSTEMS

Although the traditional approach to spectral analysis which uses an FFT based algorithm coupled with windowing functions and smoothing procedures is widely used the autoregressive method of estimating power spectra has become well established and in many cases yields improved estimates[2]. The analysis of linear systems however requires the computation of cross-spectral densities in the expression

$$S_{uy}(j\omega) = H_1(\omega)S_{uu}(\omega)+S_{ue}(j\omega) \qquad (1)$$

in order to compute the system frequency response and parametric estimators for this type of analysis has rarely been considered[3]. This is surprising because the development of such an estimator is relatively straightforward.

Consider a system which can be represented by the pulse transfer function model

$$x(k) = z^{-d}\frac{A(z^{-1})}{B(z^{-1})}u(k) \qquad (2)$$

where
$$A(z^{-1}) = 1+a_1z^{-1}+\ldots+a_nz^{-\ell}$$
$$B(z^{-1}) = b_1z^{-1}+\ldots+b_nz^{-\ell}$$

$z = e^{sT}$. In practice the measured system output y(k) consists of x(k) plus noise

$$y(k) = x(k)+e(k) \qquad (3)$$

where e(k) is described by an AutoRegressive Moving Average (ARMA) model

$$e(k) = \frac{D(z^{-1})}{C(z^{-1})}z(k) \qquad (4)$$

driven by discrete white noise $\xi(k)$ where

$$D(z^{-1}) = 1+d_1z^{-1}+\ldots+d_mz^{-m}$$
$$C(z^{-1}) = 1+c_1z^{-1}+\ldots+c_mz^{-m}$$

and the roots of $C(z^{-1})$ lie within the unit circle

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in the z-plane. Substituting eqn's (2) and (4) into (3) yields the combined process and noise model

$$y(k) = \frac{z^{-d}A(z^{-1})}{B(z^{-1})} u(k) + \frac{D(z^{-1})}{C(z^{-1})} \xi(k) \qquad (5)$$

Numerous parameter estimation algorithms are available which provide estimates of the process parameters a_i, b_i, $i = 1,2,\ldots \ell$ given measurements of $y(k)$ and $u(k)$. Although the algorithms differ in the detail of implementation they can almost all be classified as prediction error methods and are well documented in the literature[4]. Estimates of the noise model parameters d_i, c_i, $i = 1,2\ldots m$ are often provided either to ensure unbiased process parameter estimates or for use in the design of stochastic controllers. Often the model orders ℓ, m and time delay d are unknown a priori but numerous methods are available for determining these[4].

If unbiased estimates of the parameters a_i, b_i, $i = 1,2,\ldots \ell$ are available the system frequency response can be obtained by substituting $z = e^{j\omega T}$ in eqn (2). This is analogous to the autoregressive methods of estimating power spectra and yields an alternative to the classical approach based on eqn (1). Notice that estimates based on eqn (1) involve averaging, windowing and smoothing with the assumption that $S_{ue}(j\omega)$ will tend to zero for a large enough sample provided u(t) and the noise e(t) are uncorrelated. In contrast to this the parametric approach using eqn (5) is based on estimation in the time domain and allows the estimation of the system frequency response without any noise. The noise, the right hand term in eqn (5) is discarded, not averaged out, before estimating the frequency response using eqn (2) and it might be anticipated that this will result in smoother frequency response estimates particularly for short record lengths.

To illustrate these ideas the system S_1

$$y(k) = \frac{z^{-1}+0.5z^{-2}}{1-1.5z^{-1}+0.7z^{-2}} u(k) + \frac{1+0.3z^{-1}}{1+0.6z^{-1}} \xi(k) \qquad (6)$$

was simulated to generate 500 data pairs y(k), u(k), k = 1,2...500 where u(k) was a sixth order prbs and $\xi(k)$ was random signal uniformly distributed between -1.0 and 1.0.

The parametric spectral estimates were obtained by using a generalised least squares algorithm which for a second order process model, zero time delay and tenth order noise model yielded the process model

$$x(k) = \frac{1.012z^{-1}+0.4884z^{-2}}{1-1.498z^{-1}+0.0698z^{-2}} u(k) \qquad (7)$$

which is very close to the actual system eqn (6). The choice of model order and time delay were identified using loss function analysis, pole-zero cancellation, autocorrelation of the residuals and cross-correlation between the input and residual tests[4]. The system frequency response was computed by substituting $z = e^{j\omega T}$, T = sampling interval, into eqn (7).

Estimates of the spectra using eqn (1) were obtained by padding the 500 data pairs with twelve zeros, applying a Hamming window and smoothing the estimates.

The frequency response function for the system of eqn (6) was also estimated using the forward/backward autoregressive spectral estimation method of Marple[5]. This is a somewhat unfair comparison because by necessity this estimate was computed by estimating the power spectra of y(k) and u(k) and using the expression

$$S_{yy}(\omega) = |H(\omega)|^2 S_{uu}(\omega) \qquad (8)$$

The order of autoregressive models which were fitted to the input and output series were selected as zero and two respectively using the AIC criterion[4,5].

The estimates are shown in Fig.1 superimposed on the theoretical frequency response function. Notice that the generalised least squares (GLS) estimate is virtually coincident with the theoretical gain for all frequencies whereas the FFT based estimate becomes ragged for higher frequencies. The autoregressive forward-backward estimate (ARFB) based on eqn (8) is considerably in error. Numerous simulations have consistently produced similar results; the parametric GLS estimate tends to be much more accurate for higher frequencies and this becomes more evident for lower S/N ratios and shorter record lengths. This can be illustrated by considering a spring-mass-damper system S_2 described by the equation

$$m\ddot{x}(t)+C\dot{x}(t)+kx(t) = u(t) \qquad (9)$$

where m = 0.0025kg, C = 0.15 Ns/m, k = 1 N/s. This system was simulated on an analogue computer with an input prbs excitation of 5th order and 16.65 mS bit interval. Coloured noise was added to the output to give a S/N ratio of 7.7dB and the signals were sampled at 5.2mS to yield 1000 data pairs.

A generalised least squares routine was used to estimate the process model given by

$$x(k) = \frac{0.005186z^{-1}+0.006153z^{-2}}{1-1.645z^{-1}+0.6577z^{-2}} u(k)$$

The FFT based estimate was computed by padding the data with 24 zeros, applying a Hamming window and smoothing over a record length of 15. The autoregressive forward-backward estimate was computed using eqn (8) with autoregressive model orders of two and four for the output and input processes respectively. Estimates of the gain plot computed using the GLS, FFT and ARFB algorithms are compared with the theoretical result in Fig.2. Inspection of Fig.2 confirms the conclusions of the previous example.

PARAMETRIC SPECTRAL ESTIMATES FOR NONLINEAR SYSTEMS

The application of linear spectral estimation procedures to data generated from nonlinear systems can introduce significant errors. For example, estimation of the frequency response function for a system represented by the model

$$x(k) = ax(k-1) + bu(k-1) + cu^2(k-1) \quad (10)$$

where u(k) is a signal whose third order moments are zero (eg zero mean Gaussian or sine wave inputs) yields

$$\hat{H}_1(\omega) = \frac{S_{ux}(j\omega)}{S_{uu}(\omega)} = \frac{b}{e^{j\omega}-a} \quad (11)$$

The estimate of the frequency response function $\hat{H}_1(\omega)$ is completely independent of c the nonlinear term in eqn (10). This and other examples which can readily be constructed demonstrates the limitations of linear methods applied to nonlinear systems.

Efforts to resolve these problems have to date concentrated on the functional series approach and higher order spectra[6]. Although a considerable body of theory has been developed for these approaches the computation involves the design of multidimensional window functions, produces results which are input dependent and is limited by the enormous computational effort required. Consequently only a handful of papers describe the practical computation of the bispectrum and none describe the computation of higher order spectra or generalised frequency transfer functions.

The traditional description for nonlinear systems has been based on the Volterra series model

$$x(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int h_n(\tau_1,\tau_2,\ldots\tau_n) \prod_{i=1}^{n} u(t-\tau_i)d\tau_i \quad (12)$$

where $h_n(\tau_1,\ldots\tau_n)$ is the n'th order Volterra kernel which can be visualised as a nonlinear impulse response of order n. The Fourier transform of the Volterra kernels which are called generalized transfer functions are defined by

$$H_n(\omega_1,\omega_2,\ldots\omega_n) = \int_{}^{\infty}\cdots\int h_n(\tau_1,\ldots\tau_n) \exp[-j(\omega_1\tau_1+\ldots\omega_n\tau_n)]d\tau_1\ldots d\tau_n \quad (13)$$

If the system under investigation were linear only the first term, the standard convolution integral, would exist in eqn (12) and eqn (13) would yield $H_1(\omega)$ which is the linear frequency response function of eqn (1). When the system under study is nonlinear there is no single function which characterizes the frequency response behaviour and generalized frequency response functions of the form of eqn (13) must be evaluated up to order n the degree of nonlinearity. Several methods have been proposed for estimating the generalized transfer functions but all of these are based on multidimensional FFT / window algorithms and involve the computation of polyspectra[6]. Unfortunately, all of the algorithms require an excessive length of data, work only on the premise of unrealistic assumptions regarding the system under study or require a special input excitation, and produce results which are input dependent.

Many of these difficulties can be avoided by developing a parametric approach for estimating frequency response characteristics for nonlinear systems. The recent introduction of the NARMAX model[1] (Nonlinear AutoRegressive Moving Average model with eXogenous inputs)

$$y(t) = F\{y(t-1),\ldots y(t-n_y)u(t-d),\ldots u(t-d-n_u+1), e(t-1)\ldots e(t-n_\varepsilon)\} + e(t) \quad (14)$$

where $F\{\cdot\}$ is some nonlinear function provides an alternative representation for nonlinear systems. Difference equation models arise naturally from physio-chemical laws and yield expressions which map past inputs and outputs into the present system output. This considerably reduces the computational burden and excessive parameter set associated with the functional series approach, eqn (12) which maps only past inputs into the present output. Notice that discrete-time bilinear systems cannot approximate to all nonlinear systems. Conditions for the existence of the NARMAX model have been derived[1] and these show that they represent a broad class of nonlinear systems. Methods of detecting nonlinearity in data, detecting the structure of the model or which nonlinear terms to include, estimating the system parameters and validating the models obtained have been derived[1,7,8]. These results have recently been extended to include a new orthogonal estimation algorithm for the NARMAX model[9]. This algorithm allows each coefficient in the model to be estimated recursively and quite independently of the other terms in the model because of the orthogonal property which holds for any input and automatically shows the contribution that each term makes to the output variance. This provides an alternative to the prediction error stepwise regression algorithm[7].

The system described by eqn (10) is for example in the form of a NARMAX model. Estimation of the parameters (a,b,c) completely characterizes the process, the presence of a nonlinear term is obvious and $H_1(\omega)$ and the higher order generalized transfer functions $H_n(\omega_1,\omega_2,\ldots\omega_n)$, n = 2,3... can be estimated directly for any input. This method is therefore a direct extension of the linear results presented above to the nonlinear case and is best illustrated by examples.

Consider the Hammerstein model S_3 illustrated in Fig.3 where the noise e(t) was N(0,0.08) and 1000 data pairs were considered. In the early stages of any identification procedure it is important to establish if the process under test exhibits nonlinear characteristics which will warrant a nonlinear model. This can readily be achieved using a simple correlation test[8]. If an input u(t)+b, b ≠ 0 is applied to the process where the third order moments of u(t) are zero and all even order moments exist (a sine wave, Gaussian or ternary sequence would for example satisfy these properties) then the process is linear iff[8]

$$\phi_{y'y'^2}(\tau) = E[(y(k)-\bar{y})(y(k)-\bar{y})^2] = 0 \,\forall\tau \quad (15)$$

$\phi_{y'y'^2}(\tau)$ for the system of Fig.3 computed for the input N(1,1) is illustrated in Fig.4 and clearly shows that the system is highly nonlinear.

From Fig.3 the true system model can be expressed as a NARMAX model

$$x(k) = 0.9x(k-1)+0.5u(k-1)+u^2(k-1)$$

$$y(k) = x(k)+e(k) \quad (16)$$

One thousand data pairs were generated for an input excitation N(0,1.0) and a NARMAX model with first order dynamics and second degree nonlinearity defined by

$$\begin{aligned} y(k) = {} & \alpha_c+\alpha_1 y(k-1)+\alpha_2 y^2(k-1)+\alpha_3 u(k-1) \\ & +\alpha_4 u^2(k-1)+\alpha_5 u(k-1)y(k-1)+e(k) \\ & +\beta_1 e(k-1)+\beta_2 e(k-2)+\beta_3 e^2(k-1)+\beta_4 e^2(k-2) \\ & +\beta_5 e(k-1)e(k-2) \end{aligned} \quad (17)$$

was postulated as a model to represent the system. The orthogonal estimation algorithm[9] produced the following estimated model

$$\begin{aligned} y(k) = {} & 0.001582+0.8997y(k-1)+0.5007u(k-1) \\ & +1.001u^2(k-1)+e(k)-0.8746e(k-1) \end{aligned} \quad (18)$$

Estimation using the prediction error/stepwise regression algorithm[7] produced virtually identical results. Notice that in estimating eqn (18) the algorithm has both detected the significant terms in eqn (7) and estimated the unknown coefficients associated with them. The algorithm estimated that the terms in eqn (18) contributed 99.954% to the variation in y(k) thus indicating why several of the possible terms in eqn (17) were omitted from the final model eqn (18). A comparison of the estimates in eqn (18) with the true system model eqn (16) demonstrates the effectiveness of this algorithm.

The generalized transfer functions $H_n(\omega_1,\ldots\omega_n)$ defined in eqn (13) can now be computed using the probing or harmonic input method[10]. Probing eqn (18) with a single exponential $u(k) = e^{j\omega kT}$ ignoring the almost zero constant term all the noise terms and setting T = 1 yields

$$\begin{aligned} H_1(\omega)e^{j\omega k} = {} & 0.8997H_1(\omega)e^{j\omega(k-1)}+0.5007e^{j\omega(k-1)} \\ & +1.00[e^{j\omega(k-1)}]^2 \end{aligned} \quad (19)$$

Equating coefficients of $e^{j\omega k}$ yields the first order frequency response function

$$H_1(\omega) = \frac{0.5007}{e^{j\omega}-0.8997} \quad (20)$$

Probing with the input $u(k) = e^{j\omega_1 k}+e^{j\omega_2 k}$ yields, in a similar manner, the second order generalized transfer function

$$H_2(\omega_1,\omega_2) = \frac{1.001}{e^{j(\omega_1+\omega_2)}-0.8997} \quad (21)$$

It can easily be shown that for this system all the higher order functions $H_n(\omega_1,\omega_2\ldots\omega_n)$ $n\geq 2$ are zero. The gain and phase plots associated with the estimates $H_1(\omega)$ and $H_2(\omega_1,\omega_2)$ are illustrated in Figs. 5 and 6 respectively.

By linearizing the estimated model eqn (18) about different input operating points a plot showing the gain or phase versus frequency versus operating point can be constructed as illustrated in Fig.7.

As a second example consider a nonlinear electronic circuit S_4 which can be represented by the equation

$$\frac{dv(t)}{dt} + 5v(t) + 0.8v^2(t) = 5i(t) \quad (22)$$

This system was simulated on an analogue computer and 1000 data pairs were recorded by sampling the system at 16 ms in response to a Gaussian noise input of 5 Hz bandwidth. A NARMAX model representation of the form

$$\begin{aligned} v(k) = F^2\{ & v(k-1),v(k-2),v(k-3),i(k-1),i(k-2),i(k-3), \\ & e(k-1),e(k-2),e(k-3),e(k-4),e(k-5)\}+e(k) \end{aligned} \quad (23)$$

was postulated where $F^2\{\cdot\}$ indicates a second order polynomial expansion of the terms within the brackets. The number of possible terms in the model is therefore very large. Using the orthogonal estimation algorithm most of these terms were found to be redundant and the final model fitted was

$$\begin{aligned} v(k) = {} & 0.001092+0.58v(k-1)+0.221v(k-2) \\ & +0.08455v(k-3)+0.1498i(k-1)-0.0637i(k-2) \\ & +0.02687i(k-3)-0.01824v^2(k-1)+e(k) \\ & +0.164e(k-1)-0.0523e(k-5) \end{aligned} \quad (24)$$

The first and second order generalized transfer functions eqn (13) were then computed from eqn (24) to yield

$$\hat{H}_1(\omega) = \frac{0.1498e^{-j\omega T}-0.0637e^{-2j\omega T}+0.02687e^{-3j\omega T}}{1-0.58e^{-j\omega T}-0.221e^{-2j\omega T}-0.08455e^{-3j\omega T}}$$

$$\begin{aligned} \hat{H}_2(\omega_1,\omega_2) = {} & (-0.0182e^{-j(\omega_1+\omega_2)T} H_1(\omega_1)H_1(\omega_2))/ \\ & (1-0.58e^{-j(\omega_1+\omega_2)T}-0.221e^{-2j(\omega_1+\omega_2)T}-\ldots \\ & 0.0845e^{-3j(\omega_1+\omega_2)T}) \end{aligned} \quad (25)$$

where T = 0.016 secs.

The gain and phase plots associated with the estimates of eqn (25) are illustrated in Figs. 8 and 9. A comparison of the estimated $\hat{H}_1(\omega)$ and $\hat{H}_2(\omega_1,\omega_2)$ with the true values, which can be computed from eqn (22), showed that they were virtually coincident.

CONCLUSIONS

New parametric methods of estimating the frequency response characteristics of both linear and nonlinear systems have been introduced. The new algorithms are easy to apply, appear to be largely insensitive to measurement noise and provide estimates that are often better than those obtained using classical FFT/windowing methods. The nonlinear algorithm permits, for the first time, the experimenter to estimate any order of generalized transfer function from short data record lengths.

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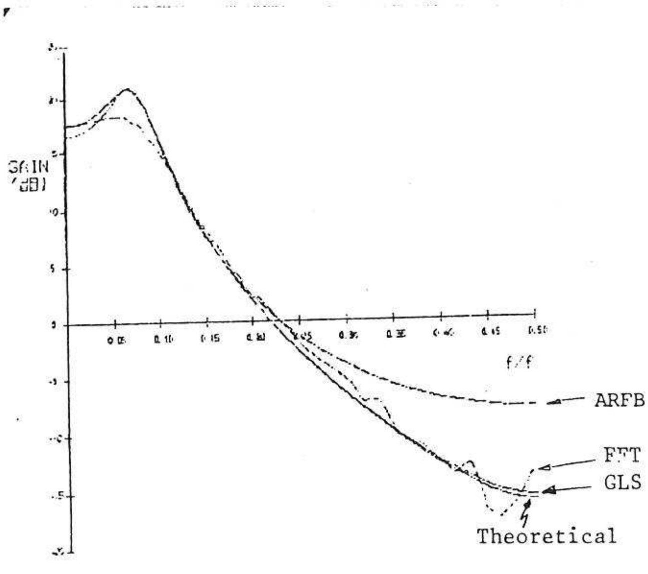

Fig.1. Gain estimates for S_1

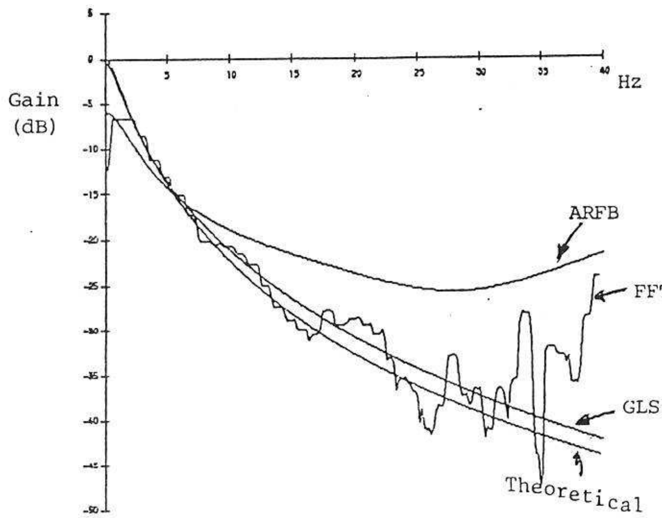

Fig.2. Gain estimates for S_2

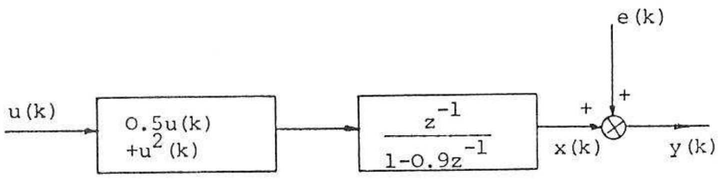

Fig.3. The system S_3

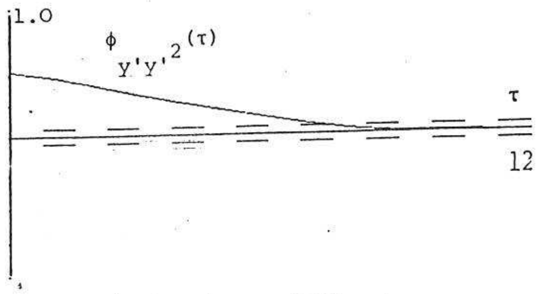

Fig.4. $\phi_{y'y'^2}$ test

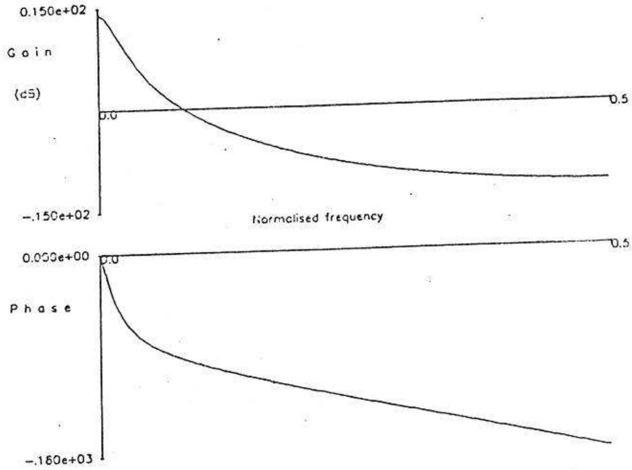

Fig.5. $\hat{H}_1(\omega)$ for S_3

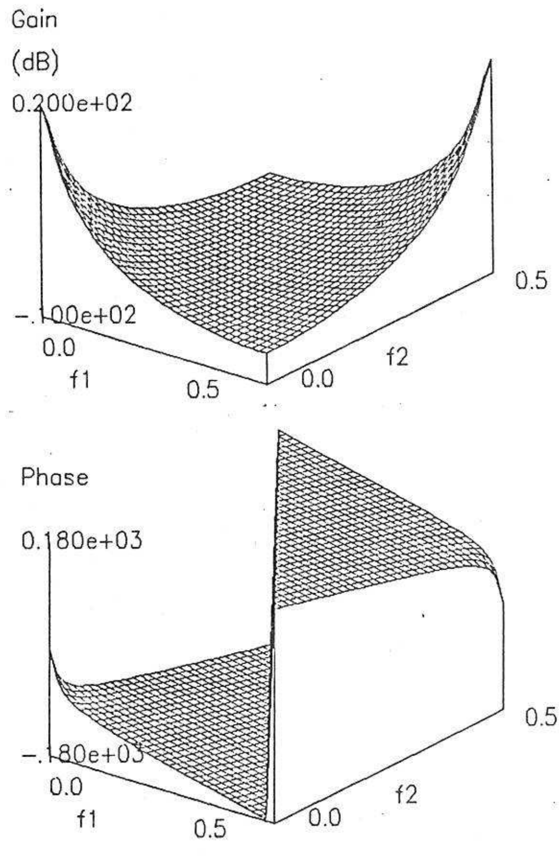

Fig.6. $\hat{H}_2(\omega_1,\omega_2)$ for S_3

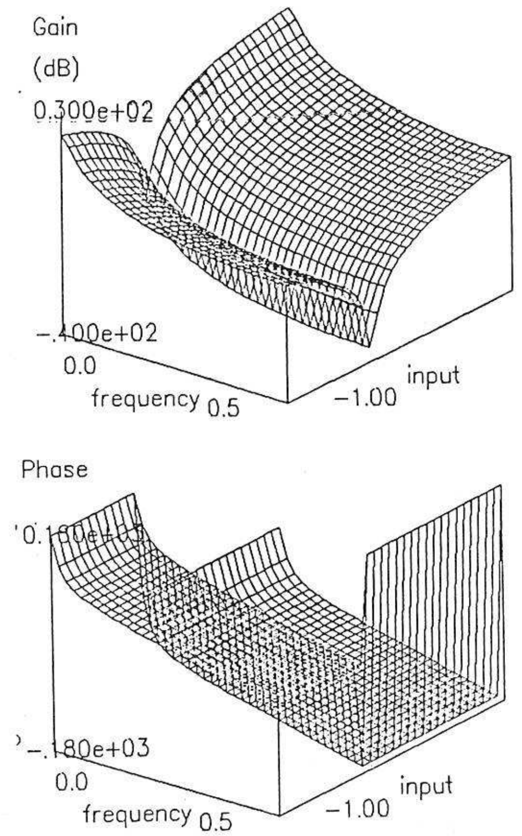

Fig.7. Evolution of linearized spectra

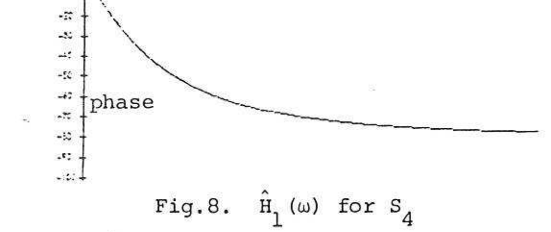

Fig.8. $\hat{H}_1(\omega)$ for S_4

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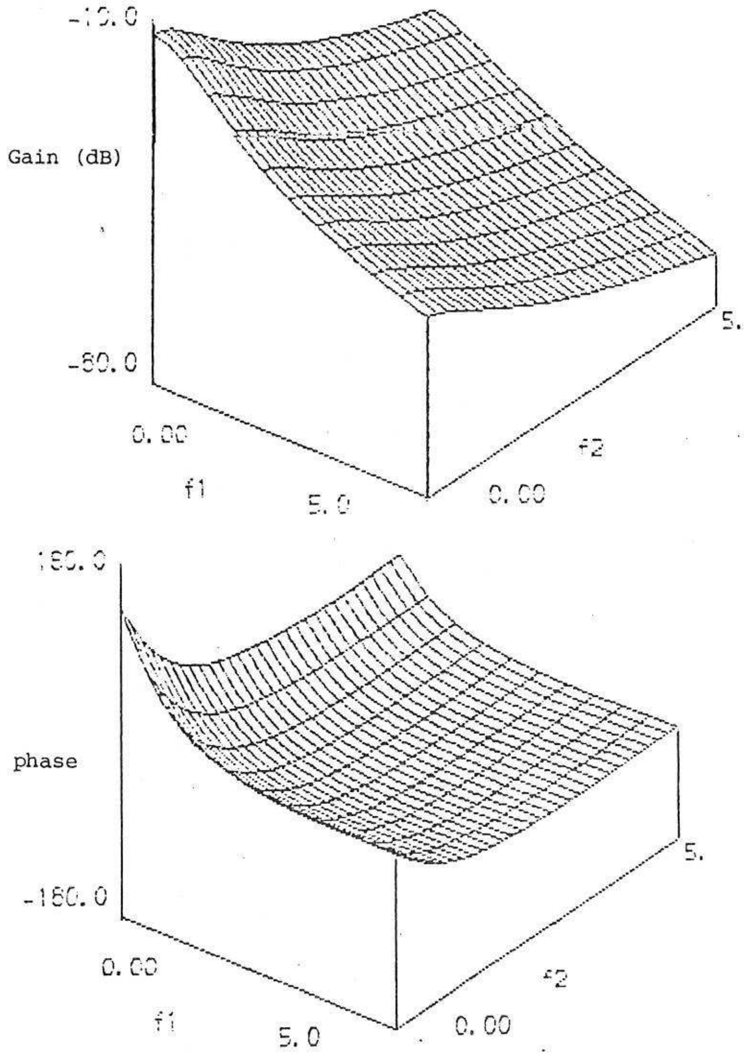

Fig.9. $\hat{H}_2(\omega_1,\omega_2)$ for S_4