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TENSOR OPERATORS AND LIMIT CYCLES IN NONLINEAR SYSTEMS

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Abstract

theory of tensor operators is used to obtain criteria for the
e nce a nonexistence of limit cycles in nonlinear systems.

Keywo es, nonlinear systems, tensor operators.

1. Introduction

The theory of limit cycle behaviour in nonlinear systems has, of course, a long history and is difficult because it is essentially a global phenomenon. Many results from the qualitative theory of differential equations exist (see, for example, Lefschetz, 1977, Hirsch and Smale, 1971) and the describing function method is well-known in control theory (Gelb and Vender Velde, 1968, Mees and Bergen, 1975). In this paper we consider the problem by replacing the original nonlinear system by a linear infinite-dimensional tensor - valued differential equation and derive a condition on the spectrum of the tensor operator for the existence of a limit cycle. This condition turns out to be the intuitively appealing result that the spectrum should intersect the imaginary axis. Using a generalisation of the Gersgorin circle theorem for tensors, we shall obtain a sufficient condition for the non-existence of a limit cycle in polynomial systems. This will also provide an upper bound on the frequency of oscillations in systems which do have limit cycles.

The linearisation technique used here has been applied for the study of the control of nonlinear systems as in Takata, 1979, Banks, 1985a, Banks and Ashtiani, 1985, and is a promising technique for the simplification of nonlinear systems in general.

We shall begin by introducing elementary tensor theory and then provide a criterion for the existence of limit cycles in analytic nonlinear systems. We shall then consider the spectrum of a tensor operator and generalise Gersgorin's theorem which leads to a sufficient condition for the non-existence of limit cycles. Finally we shall show how to apply this to a general polynomial system.

2. Tensors and Tensor Operators

We shall use the standard theory of tensors in the finite and infinite-dimensional Hilbert tensor product spaces $\mathbb{R}^k \otimes \mathbb{R}^k \otimes \ldots \otimes \mathbb{R}^k$ and $\ell^2 \otimes \ldots \otimes \ell^2$ (n factors in each case, for some n). This theory can be found in Greub (1978) or Takesaki (1979), for exmaple. Tensors will be used exclusively in component form in the standard bases of the above spaces. Hence we shall write, for $\Phi \in \mathbb{R}^k \otimes \ldots \otimes \mathbb{R}^k$ or $\Phi \in \ell^2 \otimes \ldots \otimes \ell^2$,

$$\Phi = \sum_{i_n} \ldots \sum_{i_n} \Phi_{i_1 \ldots i_1} e_{i_1} \otimes \ldots \otimes e_{i_n}$$

where $\{e_i\}$ is the standard basis of $\mathbb{R}^k$ or ℓ^2 and the sum ranges from 1 to k or 1 to ∞, respectively. In future we shall omit reference to the basis elements $e_{i_1} \otimes \ldots \otimes e_{i_n}$ and simply write $\Phi_{i_1 \ldots i_n}$ for the tensor Φ, the standard basis being assumed.

We shall also need tensor operators on the spaces $\mathbb{R}^k \otimes \ldots \otimes \mathbb{R}^k$ and $\ell^2 \otimes \ldots \otimes \ell^2$ Thus, if $A \in \mathcal{L}(\mathbb{R}^k \otimes \ldots \otimes \mathbb{R}^k)$ or $A \in \mathcal{L}(\ell^2 \otimes \ldots \otimes \ell^2)$ (where $\mathcal{L}(H)$ denotes the space of bounded linear operators on H) then A operates as follows:

$$(A\Phi)_{i_1 \ldots i_n} = \sum_{j_1} \ldots \sum_{j_n} A^{j_1 \ldots j_n}_{i_1 \ldots i_n} \Phi_{j_1 \ldots j_n}$$

where $A^{j_1 \ldots j_n}_{i_1 \ldots i_n}$ is the $(j_1, \ldots, j_n, i_1, \ldots, i_n)^{th}$ 'component' of A . This merely generalises matrix operations of the form $\Psi = A\Phi$, where

$$\Psi_i = \sum_j A^j_i \Phi_j$$

and, for convenience, we have written the usual matrix coefficient A_{ij} as A^j_i. Note finally that, as a linear operator, we can define the spectrum of $A \in \mathcal{L}(\mathbb{R}^k \otimes \ldots \otimes \mathbb{R}^k)$ (or $\mathcal{L}(\ell^2 \otimes \ldots \otimes \ell^2)$ in the usual way . Thus, λ is an eigenvalue of A, with eigenvector Φ if

$$A\Phi = \lambda\Phi$$

which holds if

$$\sum_{j_i, \ldots, j_n} A^{j_1 \ldots j_n}_{i_1 \ldots i_n} \Phi_{j_1 \ldots j_n} = \lambda \delta^{j_1}_{i_1} \ldots \delta^{j_n}_{i_n} \Phi_{j_i \ldots j_n}.$$

The 'diagonal' elements of A are the elements $A^{i_1 \ldots i_n}_{i_1 \ldots i_n}$.

3. A Limit Cycle Criterion

Consider the nonlinear system

$$\dot{x} = f(x) \tag{3.1}$$

where $x \in \mathbb{R}^n$ and f is (real) analytic. Define the functions

$$\phi_{i_1 \ldots i_n} = x_1^{i_1} \ldots x_n^{i_n} .$$

Then,

$$\dot{\phi}_{i_1 \ldots i_n} = \sum_{k=1}^{n} i_k x_1^{i_1} \ldots x_k^{i_k - 1} \ldots x_n^{i_n} \dot{x}_k$$

$$= \sum_{k=1}^{n} i_k x_1^{i_1} \ldots x_k^{i_k - 1} \ldots x_n^{i_n} \sum_{j_1, \ldots, j_n = o}^{\infty} \alpha^k_{j_1 \ldots j_n} x_1^{j_1} \ldots x_n^{j_n}$$

where

$$f_k(x) = \sum_{j_1, \ldots, j_n = o}^{\infty} \alpha^k_{j_1 \ldots j_n} x_1^{j_1} \ldots x_n^{j_n}$$

is the Taylor expansion of f_k for some constants $\alpha^k_{j_1 \ldots j_n}$. Hence,

$$\dot{\phi}_{i_1 \ldots i_n} = \sum_{k=1}^{n} \sum_{j_1, \ldots, j_n = o}^{\infty} i_k \alpha^k_{j_1 \ldots j_n} x_1^{i_1 + j_1} \ldots x_k^{i_k + j_k - 1} \ldots x_n^{i_n + j_n} .$$

$$= \sum_{j_1, \ldots, j_n = o}^{\infty} A^{j_1 \ldots j_n}_{i_1 \ldots i_n} x_1^{j_1} \ldots x_n^{j_n} , \tag{3.2}$$

where

$$A^{j_1 \ldots j_n}_{i_1 \ldots i_n} = \sum_{k=1}^{n} i_k \alpha^k_{j_1 - i_1, \ldots, j_k - i_k + 1, \ldots, j_n - i_n} \tag{3.3}$$

and $\alpha^k_{j_1 \ldots j_n}$ is defined to be zero if any subscript $j_\ell < o$. Now introduce the tensor Φ with components $\phi_{i_1 \ldots i_n}$ and the tensor operator A defined by

$$(A\Phi)_{i_1 \ldots i_n} = \sum_{j_1, \ldots, j_n = o}^{\infty} A^{j_1 \ldots j_n}_{i_1 \ldots i_n} \Phi_{j_1 \ldots j_n} . \tag{3.4}$$

Then the system (3.2) may be written as a tensorial differential equation

$$\dot{\Phi} = A\Phi . \tag{3.5}$$

Suppose now that we restrict the initial values xo of (3.1) to be contained in some bounded set $R_o = \{x_o : ||x_o|| < r_o\}$ and let T>o be some given time. We shall assume that the solutions of (3.1) exist for time T for initial conditions in R_o and belong to a bounded set $R = \{x : ||x|| < r\}$ where $r \geq r_o$. (If (3.1) has finite escape time τ then we must have $\tau > T$.) By changing coordinates to y=x/r we can assume that the system (3.1) has the set $\{x : ||x|| < 1\}$ as the 'region of interest' (i.e. the set containing any limit cycles, equilibria etc.) Since $||x|| < 1$ we have $\Phi \in \ell^2 \otimes \ldots \otimes \ell^2$; i.e. the state space of (3.5) can be taken to be the tensor product of n copies of ℓ^2. Let N denote the nonlinear subset of $\ell^2 \otimes \ldots \otimes \ell^2$ consisting of all tensors of the form of $\Phi_{i_1 \ldots i_n} = x_1^{i_1} \ldots x_n^{i_n}$. Then it is shown in Banks, 1985b, that A is a bounded operator on N† and we can define e^{At}. Moreover, e^{At} extends to a semigroup on the linear space spanned by N.

We come now to a simple characterisation of limit cycles for the system (3.1) in terms of the linear infinite-dimensional system (3.5).

<u>Theorem 3.1</u> The system (3.1) has a limit cycle if, and only if the corresponding tensor operator A in (3.5) has an imaginary eivenvalue ($\neq 0$) with an eigenvector in the space N.

<u>Proof</u> Suppose first that (3.1) has a limit cycle and let x_0 be any point on the limit cycle. Form the tensor Φ^0 with components

$$\Phi^0_{i_1 \ldots i_n} = x_{0_1}^{i_1} \ldots x_{0_n}^{i_n}$$

Let T be the period of the limit cycle. Then

$$e^{AT} \Phi^0 = \Phi^0$$

This means that 1 is an eigenvalue of e^{AT} with the eigenvector $\Phi^0 \in N$. By the spectral mapping theorem (Yosida, 1974 or Banks, 1983) the spectrum of e^{AT} equals the exponential of the spectral values of A. Hence

† i.e. $||A|| = \sup_{\substack{x \in N \\ x \neq 0}} \frac{||Ax||}{||x||}$. Note that A may not be bounded on the linear space spanned by N.

$$e^{\lambda T} = 1$$

where $\lambda \varepsilon sp(A)$, and so λ is imaginary.

The converse is simply a reversal of this argument. □

Remarks

(a) If the conditions of the lemma hold, the period T of the oscillation is given by

$$T = \inf \{ |\lambda| : \lambda = i\mu \varepsilon sp_N(A), \mu \neq 0 \}$$

where $sp_N(A)$ is the set of eigenvalues of A with eigenvectors in N.

(b) The eigenvector Φ^0 must belong to N. If we have $e^{AT}\Phi = \Phi$ for some $\Phi \varepsilon \ell^2 \otimes \ldots \otimes \ell^2$, then a limit cycle does not necessarily exist, since Φ may not be of the form $x_1^{i_1} \ldots x_n^{i_n}$ for some x. The fact that $\Phi^0 \varepsilon N$ reflects the nonlinearity of the original problem.

The main problem with this approach, of course, is the computation of the spectrum of a tensor operator which is unlikely to be any easier than any existing method. It will be seen, however, that we can use the ideas in a negative way; that is to show non-existence of limit cycles. Thus we have

Corollary 3.2. If the spectrum of A does not intersect the imaginary axis, then the system (3.1) cannot have a limit cycle. □

We shall now truncate the system operator A so that each index is restricted to the range 0 to k. Thus we write A_k for the tensor operator with components $A_{i_1 \ldots i_n}^{j_1 \ldots j_n}$ where $0 \leq i_\ell, \ldots j_\ell \leq k$. Using the results of Banks 1985b it is easy to show that

$$||A_k|| \to ||A||$$

as $k \to \infty$. It follows that if the spectra of σ_k of the operators A_k are all bounded away from the imaginary axis, then so is the spectrum of A. To obtain a non-existence criterion from corollary 3.2 we shall prove a generalisation of Gersgorin's circle theorem for tensor operators in the next section.

4. Gersgorin's Theorem for Tensors.

First recall that λ is an eigenvalue of the tensor operator A, with eigenvector , if

$$\sum_{j_1,\dots,j_n=0}^{k} A_{i_1\dots i_n}^{j_1\dots j_n}\,\Phi_{j_1\dots j_n} = \lambda\,\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\dots\delta_{i_n}^{j_n}\,\Phi_{j_1\dots j_n}$$

where the tensors are assumed to be a rank n and dimension k+1. We shall consider the rank-2 case before discussing the general situation. Given a rank 2 tensor $A_{i_1 i_2}^{j_1 j_2}$ of dimension k+1, we can organise it into a matrix as follows

$$\begin{bmatrix}
A^{00}_{00} & A^{10}_{00} & A^{20}_{00}\cdots A^{k0}_{00} & A^{01}_{00} & A^{11}_{00} & A^{21}_{00}\cdots & A^{k1}_{00} & \cdots & A^{0k}_{00} & \cdots & A^{kk}_{00} \\
A^{00}_{10} & A^{10}_{10} & A^{20}_{00}\cdots A^{k0}_{10} & A^{01}_{00} & A^{11}_{00} & A^{21}_{10}\cdots & A^{k1}_{10} & \cdots & A^{0k}_{10} & \cdots & A^{kk}_{10} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A^{00}_{k0} & A^{10}_{k0} & \cdots A^{k0}_{k0} & A^{01}_{k0} & \cdots & \cdots & A^{k1}_{k0} & \cdots & A^{0k}_{k0} & \cdots & A^{kk}_{k0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A^{00}_{0k} & A^{10}_{0k} & \cdots A^{k0}_{0k} & A^{01}_{0k} & \cdots & \cdots & A^{k1}_{0k} & \cdots & A^{0k}_{0k} & \cdots & A^{kk}_{0k} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A^{00}_{kk} & A^{10}_{kk} & \cdots A^{k0}_{kk} & A^{01}_{kk} & \cdots & \cdots & A^{k1}_{kk} & \cdots & A^{0k}_{kk} & \cdots & A^{kk}_{kk}
\end{bmatrix}$$

This corresponds to the ordering of $\Phi_{j_1\dots j_n}$ in the rank 1 array

$$(\Phi_{00\dots0}\ \ \Phi_{10..0}\cdots\ \Phi_{k0..0}\ \ \Phi_{01..0}\ \ \Phi_{110..0}\cdots\ \Phi_{k10..0}\cdots\cdots\Phi_{kk...k}).$$

Let $\bar{A}$ denote the matrix associated with the tensor A as above. Then the eigenvalues of $\bar{A}$ are clearly the same as the eigenvalues of A. Hence, by the standard (matrix) Gersgorin theorem we have

Theorem 4.1 Let $A_{i_1 i_2}^{j_1 j_2}$ be a rank-2 tensor of dimension (k+1). Define the numbers $\mathcal{A}_{pq}^{pq}$ by

$$\mathcal{A}_{pq}^{pq} = \sum_{m,n=o}^{k} |A_{pq}^{mn}| - |A_{pq}^{pq}| \ , \quad o \leq p, \ q \leq k$$

Then the eigen values of A are contained in the union of the interiors of the circles with centres A_{pq}^{pq} and radii $\mathcal{A}_{pq}^{pq}$.

Remark. A similar result hold for 'columns'.

The general result now follows easily:

Corollary 4.2. If $A_{i_1 \ldots i_n}^{j_1 \ldots j_n}$ is a rank-n tensor of dimension (k+1) and we define

$$\mathcal{A}_{p_1 \ldots p_n}^{p_1 \ldots p_n} = \sum_{m_1, \ldots, m_n = 0}^{k} \left| A_{p_1 \ldots p_n}^{m_1 \ldots m_n} \right| - \left| A_{p_1 \ldots p_n}^{p_1 \ldots p_n} \right| \ , \quad 0 \leq p_1, \ldots, p_n \leq k,$$

then the eigenvalues of A are contained in the union of the interiors of the $(k+1)^n$ circles with centres $A_{p_1 \ldots p_n}^{p_1 \ldots p_n}$ and radii $\mathcal{A}_{p_1 \ldots p_n}^{p_1 \ldots p_n}$. □

From the representation (3.2), (3.3) of the nonlinear system (3.1) we therefore obtain the following sufficient condition for the nonexistence of a limit cycle:

Corollary 4.3. The system (3.1) will have no limit cycles if

$$2 \left| \sum_{\ell=1}^{n} p_\ell \alpha_{0, \ldots, 1, \ldots, 0}^{\ell} \right| > \sum_{m_1, \ldots, m_n = 0}^{\infty} \left| \sum_{\ell=1}^{n} p_\ell \alpha_{m_1 - p_1, \ldots, m_\ell - p_\ell + 1, \ldots, m_n - p_n}^{\ell} \right| \qquad (4.1)$$

for all $p_1, \ldots, p_n \in \{0,1,2,\}$, provided the sum on the right exists.

Remark. The sum on the right of (4.1) is finite if f is a polynomial function, since only a finite number of the Taylor polynomials have non-zero coefficients.

In the next section we shall present some examples to illustrate the application of corollary 4.3. For the method to be useful we must be able to compute the sum in (4.1) effectively. We shall show that this is possible in many cases.

5. Examples.

1. Consider the two dimensional system with polynomial right hand side:

$$\begin{aligned}\dot{x}_1 &= \sum_{i=0}^{\ell_1} \sum_{j=0}^{m_1} \alpha_{ij} x_1^i x_2^j \\ \dot{x}_2 &= \sum_{i=0}^{\ell_2} \sum_{j=0}^{m_2} \beta_{ij} x_1^i x_2^j\end{aligned} \qquad (5.1)$$

Then if $_{\lambda\mu} = x_1^\lambda x_2^\mu$, we have

$$\dot{\phi}_{\lambda\mu} = \lambda x_1^{\lambda-1} \dot{x}_1 x_2^{\mu} + \mu x_1^{\lambda} x_2^{\mu-1} \dot{x}_2$$

$$= \lambda \sum_{i=0}^{\ell_1} \sum_{j=0}^{m_1} \alpha_{ij} x_1^{i+\lambda-1} x_2^{j+\mu} + \mu \sum_{i=0}^{\ell_2} \sum_{j=0}^{m_2} \beta_{ij} x_1^{i+\lambda} x_2^{\mu+j-1}$$

$$= \lambda \sum_{i=1}^{\ell_1} \sum_{j=1}^{m_1} \alpha_{ij} \phi_{i+\lambda-1,\ j+\mu} + \mu \sum_{i=0}^{\ell_2} \sum_{j=0}^{m_2} \beta_{ij} \phi_{i+\lambda,\ \mu+j-1}$$

$$= \left(\lambda \sum_{i=0}^{\ell_1} \sum_{j=0}^{m_1} \alpha_{ij} \delta^p_{i+\lambda-1} \delta^q_{j+\mu} + \mu \sum_{i=0}^{\ell_2} \sum_{j=0}^{m_2} \beta_{ij} \delta^p_{i+\lambda} \delta^q_{\mu+j-1} \right) \phi_{pq}$$

$$= \sum_{p,q=o}^{\infty} A^{pq}_{\lambda\mu} \phi_{pq}$$

where

$$A^{pq}_{\lambda\mu} = \lambda \sum_{i=0}^{\ell_1} \sum_{j=0}^{m_1} \alpha_{ij} \delta^p_{i+\lambda-1} \delta^q_{j+\mu} + \mu \sum_{i=0}^{\ell_2} \sum_{j=0}^{m_2} \beta_{ij} \delta^p_{i+\lambda} \delta^q_{\mu+j-1}.$$

Now we have

$$|A^{pq}_{pq}| = \left| p \sum_{i=0}^{\ell_1} \sum_{j=0}^{m_1} \alpha_{ij} \delta^p_{i+p-1} \delta^q_{j+q} + q \sum_{i=0}^{\ell_2} \sum_{j=0}^{m_2} \beta_{ij} \delta^p_{i+p} \delta^q_{q+j-1} \right|$$

$$= |p\alpha_{10} + q\,\beta_{01}| \,,$$

and

$$\sum_{\lambda,\mu=0}^{\infty} |A^{\lambda\mu}_{pq}| = \sum_{\lambda,\mu=0}^{\infty} \left| p \sum_{i=0}^{\ell_1} \sum_{j=0}^{m_1} \alpha_{ij} \delta^\lambda_{i+p-1} \delta^\mu_{j+q} + q \sum_{i=0}^{\ell_2} \sum_{j=0}^{m_2} \beta_{ij} \delta^\lambda_{i+p} \delta^\mu_{q+j-1} \right|$$

$$= \sum_{\lambda,\mu=0}^{\infty} | p\, \alpha_{\lambda-1+1,\mu-q} + q\, \beta_{\lambda-p,\mu-q+1} | ,$$

where we take $\alpha_{k\ell}=0$ or $\beta_{k\ell}=0$ if k and ℓ are outside the respective ranges 0 to ℓ_1, 0 to m_1 or 0 to ℓ_2, 0 to m_2. If we put

$$\ell = \max \{\ell_1, \ell_2\}, \quad m = \max \{m_1, m_2\}$$

then we have

$$\sum_{\lambda,\mu=0}^{\infty} |A_{pq}^{\lambda\mu}| = \sum_{\lambda=p-1}^{p+\ell} \sum_{\mu=q-1}^{q+m} |p\alpha_{\lambda-p+1,\, \mu-q} + q\beta_{\lambda-p,\, \mu-q+1}|$$

$$= \sum_{\lambda=-1}^{\ell} \sum_{\mu=-1}^{m} |p\alpha_{\lambda+1,\mu} + q\beta_{\lambda,\mu+1}|$$

Hence, by (4.1), the System (5.1) will have no limit cycles if

$$2|p\alpha_{10} + q\beta_{01}| > \sum_{\lambda=-1}^{\ell} \sum_{\mu=-1}^{m} |p\alpha_{\lambda+1,\mu} + q\beta_{\lambda,\mu+1}| \qquad (5.2)$$

for all $p,q \geq 0$. In particular, if α_{10} and β_{01} have the same sign it is sufficient that

$$2|\alpha_{10}| > \sum_{\lambda=-1}^{\ell_1} \sum_{\mu=0}^{m_1} |\alpha_{\lambda+1,\mu}|$$

$$2|\beta_{01}| > \sum_{\lambda=0}^{\ell_2} \sum_{\mu=-1}^{m_2} |\beta_{\lambda,\mu+1}| \qquad (5.3)$$

$$\operatorname{sgn} \alpha_{10} = \operatorname{sgn} \beta_{01}$$

Hence, any 'diagonally dominant' system, where the polynomials satisfy (5.3) has no limit cycles.

2. For example, the system

$$\dot{x}_1 = 3x_1 + x_2^2 + x_1 x_2$$

$$\dot{x}_2 = 4x_2 + x_1^3 + x_2^4 + x_1^2 x_2$$

does not oscillate, since the 'diagonal' coefficients are 3 and 4 and the corresponding 'off-diagonal' sums are 2 and 3 respectively.

3. Even if the Gersgorin circles do intersect the imaginary axis then we can still obtain some information about on limit cycles which may exist. For example, suppose that we denote by L the intersection of the imaginary axis and the union of the Gergorin circles of the system, and assume that L is bounded, say $\ell_{max} = \sup\{|z| : z \in L\}$. Then, since the criterion for the existence of a limit cycle of period T is

$$e^{\lambda T} = 1 = e^{2n\pi i}$$

we must have

$$\lambda T = 2n\pi i$$

and so the minimum period is

$$T_{min} = \frac{2\pi}{\ell_{max}} .$$

Hence, such a system cannot have an oscillation of frequency $> \ell_{max}$. Thus, for example, the system

$$\dot{x}_1 = x_1 + x_1^2 + x_2^2$$

$$\dot{x}_2 = x_2 + x_2^3 + x_1 x_2$$

cannot oscillate at a frequency greater than $\sqrt{5}$ rad/sec.

4. Note finally the generalisation of (5.3) to the n-dimensional system

$$\dot{x}_k = \sum_{i_1=0}^{\ell_{1k}} \cdots \sum_{i_n=0}^{\ell_{nk}} \alpha^k_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}$$

is the condition

$$2\,|\alpha^k_{0\ldots \underset{\underset{k}{\uparrow}}{1} \ldots 0}| > \sum_{\lambda_1=-1}^{\ell_{1k}} \cdots \sum_{\lambda_n=-1}^{\ell_{nk}} |\alpha^k_{\lambda_1, \ldots, \lambda_k+1, \ldots, \lambda_n}|$$

for $1 \leqslant k \leqslant n$.

6. Conclusions

In this paper we have given a simple sufficient condition for the non-existence of limit cycles in polynomial systems. By associating an infinite-dimensional tensor-valued system with the original nonlinear system and generalising the Gersgorin circle theorem to tensors, we have been able to apply linear spectral theory to show that the system will oscillate if and only if the spectrum of the associated tensor operator intersects the imaginary axis.

The use of the method to predict the existence of limit cycles is much more difficult, of course. This would involve finding the spectrum of a tensor operator and showing the existence of an eigenvector of a specific form. Further research into this problem is necessary and we shall examine it in more detail in a future paper.

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