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LINEAR STRUCTURES IN BILINEAR SYSTEMS

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1. INTRODUCTION

In this paper we study the bilinear system in R^n

$$\underline{\dot{x}}(t) = A\underline{x}(t) + \sum_{i=1}^{m} u_i(t)\, N_i\underline{x}(t) \tag{1.1}$$

with bounded, measurable controls and identify conditions on $A, N_1, N_2, \ldots, N_m$ and the initial state $\underline{x}_o$ under which solutions $\underline{x}(t)$ lie in a proper cone in R^n with vertex $\{0\}$. The results have a strong connection with the classical result that the linear system $\underline{\dot{x}}(t) = A\underline{x}(t)$, $\underline{x}(o) = \underline{x}_o$, in R^n has positive solutions $\underline{x}(t) \geq 0$, $t \geq 0$, (in the sense that $x_i(t) \geq 0$, $t \geq 0$, $1 \leq i \leq n$) if $A = D+E$ where D is diagonal, and E has zero diagonal entries and positive off-diagonal entries. This is proved by writing

$$\underline{x}(t) = e^{Dt}\underline{x}_o + \int_o^t e^{D(t-s)}\, E\, \underline{x}(s)\, ds \quad , \qquad t \geq 0 \tag{1.2}$$

and noting that the solution of any integral equation

$$\underline{X}(t) = \underline{W}_o(t) + \int_o^t W_1(t,s)\, \underline{X}(s) ds \quad , \qquad t \geq 0 \tag{1.3}$$

has a unique positive solution $\underline{X}(t) \geq 0$ if $\underline{W}_o(t) \geq 0$ and $W_1(t,s) \geq 0$ for all $t \geq 0$, $s \geq 0$.

Lie algebraic techniques have been extensively used (see, for example, [1]) in the study of bilinear controllability problems, but it appears that more straightforward techniques have been neglected. In particular, little use has been made of the natural relationship of the bilinear system (1.1) to the induced linear system

$$\underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{v}(t) \quad , \qquad \underline{y}(t) = C\underline{x}(t) \tag{1.4}$$

where

$$B = [B_1, B_2, \ldots, B_m] \quad , \qquad C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_m \end{pmatrix} \tag{1.5}$$

are constructed by full-rank factorization of N_i of the form $N_i = B_iC_i$ with $B_i(n \times k_i)$, $C_i(k_i \times n)$, $1 \leq i \leq m$. Such a factorization has been shown by P d'Alessendro et al [2] and Crouch [3] to relate strongly to realization theory for bilinear systems.

No attempt is made to be exhaustive but the results obtained indicate the possibilities inherent in the approach and will hopefully prompt further work in the area of relating bilinear phenomena to properties of the underlying linear system and its transfer function.

2. SUFFICIENT CONDITIONS FOR CONE CONTAINMENT

Let the linear system (1.4) be induced by the bilinear system (1.1) by factorization of N_i in the manner indicated above, and introduce the matrices

$$\underline{H}_o(t,k) \triangleq Ce^{(A-kBC)t}\underline{x}_o \tag{2.1}$$

$$H(t,k) \triangleq Ce^{(A-kBC)t}B \tag{2.2}$$

parameterized by the real scalar k. Let

$$\Lambda(u,k) \triangleq \text{block diag}\,\{I_{k_j}(u_j+k)\}_{1\leq j\leq m} \tag{2.3}$$

and write (1.1) in the form

$$\dot{\underline{x}}(t) = (A - k\sum_{i=1}^{m} B_iC_i)\underline{x}(t) + \sum_{i=1}^{m} (u_i(t)+k)B_iC_i\underline{x}(t)$$

$$\underline{x}(o) = \underline{x}_o \tag{2.4}$$

to obtain the fundamental identity

$$C\underline{x}(t) = \underline{H}_o(t,k) + \int_o^t H(t-s,k)\Lambda(u(s),k)C\underline{x}(s)ds \tag{2.5}$$

Theorem 1: All solutions $\underline{x}(t)$ corresponding to the initial state x_o of the bilinear system (1.1) lie in the cone $\{\underline{x} : \underline{x}\in R^n,\ C\underline{x} \geq 0\}$ if there exists a real number k^* such that, for all $k\geq k^*$ and $t\geq 0$,

$$\underline{H}_o(t,k) \geq 0 \quad , \quad H(t,k) \geq 0 \tag{2.6}$$

Proof: Fix a time $T>0$ and controls $u(t)$ on $[0,T]$. There exists $k'\geq 0$ such that $\Lambda(u(t),k')$ has only positive entries on $[0,T]$. Now take $k\geq\max(k',k^*)$ and note that (2.5) is a special case of (1.3) indicating that $C\underline{x}(t)\geq 0$, $t\in[0,T]$. Since T and u(t) were arbitrary, we obtain the desired result.

Corollary 1.1: The bilinear system is not point controllable on $R^n-\{0\}$ if there exists a real k^* such that $H(t,k)\geq 0$ for all $k\geq k^*$, for all $t\geq 0$.

Proof: Let $\underline{\alpha}\geq 0$ and $\underline{x}_o = B\underline{\alpha}$, then $\underline{H}_o(t,k) = H(t,k)\underline{\alpha}\geq 0$, $k\geq k^*$, $t\geq 0$ and the reachable set is contained in the proper cone $\{\underline{x} : C\underline{x} \geq 0\}$.

Theorem 1 has an asymptotic structure that, at first sight, is difficult to confirm. In the remainder of the paper therefore we concentrate on the single-input case (m = 1) with rank N = 1 and N = BC where B and C^T are elements of R^n. In such a situation, the induced linear system is single-input/single-output with transfer function

$$g(s) = C(sI - A)^{-1}B \tag{2.7}$$

and hence

$$\mathcal{L}\{H(t,k)\} = \frac{g(s)}{1+kg(s)} \qquad ,(2.8)$$

This 'feedback interpretation' is used in the following section to investigate conditions for the validity of (2.6).

3. TRANSFER FUNCTION ANALYSIS AND CONE CONTAINMENT

Elementary classical root-locus arguments and inspection of (2.8) indicates that:

Proposition 1: A necessary condition for (2.6) to hold is that $CB \neq 0$.

'Proof': The positivity condition on $H(t,k)$ requires that the pole-zero excess of $g(s)$ is unity to prevent oscillation of the dominant modes. This requirement is equivalent to $CB \neq 0$.

We will therefore assume for the remainder of the paper that $CB \neq 0$ and, without loss of generality, that

$$CB = 1 \qquad (3.1)$$

as the input can always be scaled to produce this condition.

The inverse system plays an important role in the analysis so we write

$$g^{-1}(s) = s + \alpha + h(s) \qquad (3.2)$$

where $h(s)$ is strictly proper and uniquely defined with impulse response

$$h(t) \triangleq \mathcal{L}^{-1}\{h(s)\} \qquad (3.3)$$

Substitution into (2.8) yields

$$\mathcal{L}\{H(t,k)\} = \frac{g_k(s)}{1+g_k(s)h(s)} \qquad (3.4)$$

(after a little manipulation) where

$$g_k(s) = \frac{1}{s+\alpha+k} \qquad (3.5)$$

Before stating the main result of this section we need the following technical lemma:

Lemma 1: Let

$$\eta_\ell(t,k) \triangleq \frac{(-\mu(k))^{\ell+1}}{\ell!} t^\ell e^{\mu t} \qquad (3.6)$$

where $\mu(k)$ is continuous and satisfies $\lim_{k\to+\infty} k^{-1}\mu(k) = -1$. Then

$$\lim_{k\to+\infty} \int_0^T \eta_\ell(t,k)f(t)dt = \begin{cases} 0 & , \ T = 0 \\ f(o) & , \ T > 0 \end{cases} \quad (3.7)$$

for any continuous function f(t) on $[0,T]$.

(Remark: $\eta_\ell(t,k)$ is, in effect, a representation of the unit delta function).

Proof: It is clear that $\eta_\ell(t,k) \geq 0$, $t \geq 0$, k large and that it converges uniformly to zero on any interval $[\delta,+\infty[$ with $\delta>0$. Also

$$\int_0^\delta \eta_\ell(t,k)dt = \frac{(-\mu)^{\ell+1}}{\ell!} \int_0^\delta t^\ell e^{\mu t} dt$$

$$= \frac{(-\mu)^{\ell+1}}{\ell!} \frac{d^\ell}{d\mu^\ell} \{ \int_0^\delta e^{\mu t} dt \}$$

$$= \frac{(-\mu)^{\ell+1}}{\ell!} \frac{d^\ell}{d\mu^\ell} \{ \mu^{-1}(e^{\mu\delta} - 1) \} \quad (3.8)$$

Noting however that, for $0\leq j\leq \ell$

$$\left(\frac{d^j}{d\mu^j}(\mu^{-1})\right) \left(\frac{d^{\ell-j}}{d\mu^{\ell-j}}(e^{\mu\delta} - 1)\right)$$

$$= \left(\frac{(-1)^j j!}{\mu^{j+1}}\right) \left(\delta^{\ell-j} (e^{\mu\delta} - \delta_{j\ell})\right) \quad (3.9)$$

and that $\lim_{k\to\infty} k^{-1}\mu(k) = -1$ by assumption, we obtain

$$\lim_{k\to+\infty} \int_0^\delta \eta_\ell(k,t)dt = 1 \quad (3.10)$$

directly from (3.8). Now let T>0 be arbitrary (the case of T = 0 is trivial) and denote the norm of f in $L_p(0,T)$ by $||f||_p$, $1\leq p\leq\infty$. It is clear that

$$\lim_{k\to\infty} \left|\int_\delta^T \eta_\ell(k,t)f(t)dt\right| \leq ||f||_\infty (T-\delta) \lim_{k\to\infty} \sup_{\delta\leq t\leq T} |\eta_\ell(t,k)|$$

$$= 0 \quad (3.11)$$

and hence that we need only prove (3.7) for an arbitrary $\delta>0$. By the continuity of f choose δ such that $f(o)-\varepsilon \leq f(t) \leq f(o)+\varepsilon$ for $t \in [0,\delta]$ with ε arbitrarily small. It is trivially shown that, for all large k,

$$(f(o)-\varepsilon) \int_0^\delta \eta_\ell(t,k)dt \leq \int_0^\delta \eta_\ell(t,k)f(t) \leq (f(o)+\varepsilon) \int_0^\delta \eta_\ell(t,k)dt \quad (3.12)$$

or, using (3.10) and (3.11),

$$f(o)-\varepsilon \leq \lim_{k\to\infty} \int_0^T \eta_\ell(t,k)f(t)dt \leq f(o)+\varepsilon \quad (3.13)$$

which proves the lemma as ε is arbitrary.

Theorem 2: A necessary and sufficient condition for the existence of k^* such that $H(t,k) \geq 0$ for $k > k^*$ and $t \geq 0$ is that

$$h(t) \leq 0 \quad , \qquad \forall\, t \geq 0 \tag{3.14}$$

Moreover, under this condition, if $\lim_{t\to+\infty} e^{-\beta t}h(t) = 0$ we can choose any $k^* > \|e^{-\beta t}h\|_1 - (\alpha+\beta)$.

Proof: The proof makes use of the feedback structure (3.4) by defining an operator L_k by the relation

$$L_k f \triangleq g_k * h * f \tag{3.15}$$

where * denotes convolution, and g_k is interpreted as $\mathcal{L}^{-1}\{g_k(s)\} = e^{-(k+\alpha)t}$. The domain of definition presents a minor problem as we will be using boundedness conditions to prove the result. If we concentrate on the case of $k+\alpha > 0$ and $h(t)$ stable, L_k is a mapping of $C(0,\infty)$ into itself. The first condition presents no problem as we are interested in the case of $k\to+\infty$ but the stability of $h(t)$ requires the assumption that linear system (1.4) is minimum phase. We will continue with this assumption as $H(t,k) \geq 0$, $t \geq 0$, $k \geq k^*$ iff $e^{-\beta t}H(t,k) \geq 0$, $t \geq 0$, $k \geq k^*$ and standard shift theorems for Laplace transforms indicate that this transformation is equivalent to replacing $h(t)$ by $e^{-\beta t}h(t)$. It is clear that a suitable choice of β will ensure stability of $e^{-\beta t}h(t)$!

To estimate the norm $\|L_k\|_\infty$ of L_k in $C(0,\infty)$ note that

$$\begin{aligned}
\|L_k f\|_\infty &= \sup_{t\geq o} |(h*(g_k*f))(t)| \\
&\leq \|h\|_1 \sup_{t\geq o} \left|\int_o^t e^{-(k+\alpha)t'} f(t-t')dt'\right| \\
&\leq \|h\|_1 \left(\int_o^\infty e^{-(k+\alpha)t}dt\right) \|f\|_\infty \\
&= \frac{\|h\|_1}{k+\alpha} \|f\|_\infty
\end{aligned} \tag{3.16}$$

whence

$$\|L_k\|_\infty \leq \|h\|_1 (k+\alpha)^{-1} \tag{3.17}$$

and L_k is a contraction for $k > \|h_1\| - \alpha$. Under these conditions we can write $H(t,k)$ as a uniformly convergent power series in k of the form

$$\begin{aligned}
H(t,k) &= g_k(t) - (L_k g_k)(t) + (L_k^2 g_k)(t) - \ldots \\
&= g_k(t) - (L_k g_k)(t) + (L_k^2 g_k)(t) + O(k^{-3})
\end{aligned} \tag{3.18}$$

as $\|L_k^j g_k\|_\infty \leq \|L_k\|_\infty^j \|g_k\|_\infty = \|L_k\|_\infty^j = O(k^{-j})$. A necessary condition for