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Investigations of subsystems of second order arithmetic and set theory in strength between Π^1_1-$\mathbf{CA}$ and Δ^1_2-$\mathbf{CA} + \mathbf{BI}$: Part I

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Abstract

This paper is the first of a series of two. It contains proof–theoretic investigations on subtheories of second order arithmetic and set theory. Among the principles on which these theories are based one finds autonomously iterated positive and monotone inductive definitions, Π^1_1 transfinite recursion, Δ^1_2 transfinite recursion, transfinitely iterated Π^1_1 dependent choices, extended Bar rules for provably definable well-orderings as well as their set-theoretic counterparts which are based on extensions of Kripke-Platek set theory. This first part introduces all the principles and theories. It provides lower bounds for their strength measured in terms of the amount of transfinite induction they achieve to prove. In other words, it determines lower bounds for their proof-theoretic ordinals which are expressed by means of ordinal representation systems. The second part of the paper will be concerned with ordinal analysis. It will show that the lower bounds established in the present paper are indeed sharp, thereby providing the proof-theoretic ordinals. All the results were obtained more then 20 years ago (in German) in the author's PhD thesis [43] but have never been published before, though the thesis received a review (MR 91m#03062). I think it is high time it got published.

1 Introduction

To set the stage for the following, a very brief history of ordinal-theoretic proof theory from the time after Gentzen's death until the early 1980s reads as follows: In the 1950's proof theory flourished in the hands of Schütte. In [57] he introduced an infinitary system for first order number theory with the so-called ω-rule, which had already been proposed by Hilbert [23]. Ordinals were assigned as lengths to derivations and via cut-elimination he re-obtained Gentzen's ordinal analysis for number theory in a particularly transparent way. Further, Schütte extended his approach to systems of ramified analysis and brought this technique to perfection in his monograph "Beweistheorie" [58]. Independently, in 1964 Feferman [13] and Schütte [59], [60] determined the ordinal bound Γ_0 for theories of autonomous ramified progressions.

A major breakthrough was made by Takeuti in 1967, who for the first time obtained an ordinal analysis of a strong fragment of second order arithmetic. In [67] he gave an ordinal analysis of Π^1_1 comprehension, extended in 1973 to Δ^1_2 comprehension in [68]. For this Takeuti returned to Gentzen's method of assigning ordinals (ordinal diagrams, to be precise) to purported derivations of the empty sequent (inconsistency).

The next wave of results, which concerned theories of iterated inductive definitions, were obtained by Buchholz, Pohlers, and Sieg in the late 1970's (see [10]). Takeuti's methods of reducing derivations of the empty sequent ("the inconsistency") were extremely difficult to follow, and therefore a more perspicuous treatment was to be hoped for. Since the use of the infinitary ω-rule had greatly facilitated the ordinal analysis of number theory, new infinitary rules were sought. In 1977 (see [5]) Buchholz introduced such rules, dubbed Ω-rules to stress the analogy. They led to a proof-theoretic treatment of a wide variety of systems, as exemplified in the monograph [11] by Buchholz and Schütte. Yet simpler infinitary rules were put forward a few years later by Pohlers, leading to the *method of local predicativity*, which proved to be a very versatile tool (see [40, 41, 42]). With the work of Jäger and Pohlers (see [28, 29, 33]) the forum of ordinal analysis then switched from the realm of second-order arithmetic to set theory, shaping what is now called *admissible proof theory*, after the models of *Kripke-Platek set theory*, **KP**. Their work culminated in the analysis of the system with Δ^1_2 comprehension plus Bar induction, (BI), [33]. In essence, admissible proof theory is a gathering of cut-elimination techniques for infinitary calculi of ramified set theory with Σ and/or Π_2 reflection rules[1] that lend itself to ordinal analyses of theories of the form **KP**+ *"there are x many admissibles"* or **KP**+ *"there are many admissibles"*. By way of illustration, the subsystem of analysis with Δ^1_2 comprehension and Bar induction can be couched in such terms, for it is naturally interpretable in the set theory $\mathbf{KPi} := \mathbf{KP} + \forall y \exists z (y \in z \wedge z \textit{ is admissible})$ (cf. [33]).

The investigations of this paper focus, as far as subsystems of second order arithmetic are concerned, on theories whose strength strictly lies in between that of Δ^1_2-**CA** and Δ^1_2-**CA**+(BI). Δ^1_2-**CA** is actually not much stronger than Π^1_1-**CA**, the difference being that the latter theory allows one to carry out iterated hyperjumps of length $< \omega$ while the former allows one to carry out iterated hyperjumps of length $< \varepsilon_0$. The jump from Δ^1_2-**CA** to Δ^1_2-**CA** + (BI) is indeed enormous. By comparison, even the ascent from Π^1_1-**CA** to Δ^1_2-**CA** + BR (with BR referring to the Bar rule) is rather benign. To get an appreciation for the difference one might also point out that all hitherto investigated subsystems of second order arithmetic in the range from Π^1_1-$\mathbf{CA}_0$ to Δ^1_2-**CA** + BR can be reduced (as far as strength is concerned) to first order theories of iterated inductive definitions. The theories investigated here are beyond that level. Among the principles on which these theories are based one finds autonomously iterated positive and monotone inductive definitions, Π^1_1 transfinite recursion, Δ^1_2 transfinite recursion, transfinitetely iterated Π^1_1 dependent choices, extended Bar rules for provably definable well-orderings as well as their set-theoretic counterparts which are based on extensions of Kripke-Platek set theory. This first part introduces all the principles and theories. It provides lower bounds for their strength measured by the amount of provable transfinite induction. In other words, it determines lower bounds for their proof-theoretic ordinals which are expressed by means of ordinal representation systems. The second part of the paper will be concerned with ordinal analysis. It will show that the lower bounds established in the present paper are indeed sharp, thereby providing the proof-theoretic ordinals. All the results were obtained more then 20 years ago (in German) in the author's PhD thesis [43] but have never been published before, though the thesis received a review (MR 91m#03062). I always thought that the results in my thesis were worth publishing

[1] Recall that the salient feature of admissible sets is that they are models of Δ_0 collection and that Δ_0 collection is equivalent to Σ reflection on the basis of the other axioms of **KP** (see [3]). Furthermore, admissible sets of the form L_α also satisfy Π_2 reflection.

but in the past I never seemed to have enough time to sit down for six weeks and type the entire PhD thesis again. The thesis was produced by the now obsolete word processing system "Signum" and it was also written in German. Over the past 20 years or so academic life has changed in that time, e.g. for doing research, has become a luxury good. I would like to thank Andreas Weiermann for nudging me again and again to publish it.

Zueignung

Den kreatürlichen Freunden Bobby, Honky Tonk, Schnuffi und Marlene gewidmet.

Outline of the paper

In the following I give a brief outline of the contents of this paper. It is roughly divided into two chapters. The first chapter, entitled "THEORIES", introduces the background and presents all the principles and theories to be considered. It also establishes interrelationships between various theories. The second chapter, entitled "WELL-ORDERING PROOFS", introduces an ordinal representation system and establishes lower bounds for the proof-theoretic ordinals of most of the theories considered.

Section 2 carefully defines the basic theory of arithmetical comprehension, $\mathbf{ACA}_0$, which forms the basis for all subsystems of second order arithmetic, and also the basic set theory **BT** which forms the basis of all set theories. While such attention to detail will not matter that much for the present paper it will certainly be of importance to its sequel which features proof analyses of infinitary calculi. Section 3 introduces second order theories of iterated inductive definitions. Systems investigated in the literature before used to be first order theories with the inductively defined sets being captured via additional predicates and iterations restricted to arithmetical well-orderings. Going to second order theories allows one to formalize iterations along arbitrary well-orderings and also to address the more general scenario of monotone inductive definitions. Section 4 compares the theories of the foregoing section with theories of transfinite Π^1_1 comprehension. In section 5 it is shown that theories of iterated inductive definitions can be canonically translated into set theories of iterated admissibility. This translation exploits the structure theory of Σ_+-inductive definitions on admissible sets originating in Gandy's Theorem (cf. [3, VI]). Section 6 features iterations based on stronger operations such as Δ^1_2 comprehension and Σ^1_2 dependent choices. Section 7 deals with their set-theoretic counterparts which are to be found in certain forms of Σ recursion.

In order to approach the strength of $\Delta^1_2\text{-}\mathbf{CA} + (\mathrm{BI})$ it is natural to restrict the schema (BI) to specific syntactic complexity classes of formulae, ($\mathcal{F}$-BI). An alternative consists in directing the attention to the well-ordering over which transfinite induction is allowed in that one requires them to be provably well-ordered or parameter-free. This will be the topic of section 8. Particular rules and schemata considered include the rule $\mathrm{BR}(\text{impl-}\Sigma^1_2)$ and the schema $\mathrm{BI}(\text{impl-}\Sigma^1_2)$:

$$(\mathrm{BR}(\text{impl-}\Sigma^1_2)) \quad \frac{\exists! X\,(\mathrm{WO}(X) \,\wedge\, G[X])}{\forall X\,(\mathrm{WO}(X) \,\wedge\, G[X] \to \mathrm{TI}(X, H))}$$

where $G[U]$ is a Σ^1_2 formula (without additional parameters), $H(a)$ is an arbitrary $\mathcal{L}_2$ formula, $\mathrm{WO}(X)$ expresses that X is a well-ordering, and $\mathrm{TI}(X, H)$ expresses the instance of tranfinite induction along X with the formula $H(a)$.

$$(\mathrm{BI}(\text{impl-}\Sigma^1_2))\quad \exists! X\,(\mathrm{WO}(X)\,\wedge\, G[X]) \to \forall X\,(\mathrm{WO}(X)\,\wedge\, G[X] \to \mathrm{TI}(X,H))$$

where $G[U]$ is a Σ^1_2 formula (without additional parameters) and $H(a)$ is an arbitrary $\mathcal{L}_2$ formula.

The rule BR(impl-Σ^1_2) is, on the basis of Δ^1_2-**CA**, much stronger than the rule BR whereas BR(impl-Σ^1_2) is still much weaker than (BI). The difference in strength between (BI) and BR(impl-Σ^1_2) is of course owed to the fact that the first is a rule while the second is a schema. But one can say something more illuminative about it. As it turns out, BR(impl-Σ^1_2) and BI(impl-Σ^1_2) are of the same strength (on the basis of Δ^1_2-**CA**), in actuality the theories Δ^1_2-**CA**+BR(impl-Σ^1_2) and Δ^1_2-**CA** + BI(impl-Σ^1_2) prove the same Π^1_1 statements. Thus the main difference between BR(impl-Σ^1_2) and (BI) is to be found in the premiss of BI(impl-Σ^1_2) requiring the well-ordering to be describable via a Σ^1_2 formula without parameters.

Section 8 also considers set-theoretic versions of (BR(impl-Σ^1_2) and (BI(impl-Σ^1_2)) which can be viewed as formal counterparts of the notion of a good Σ_1 definition of an ordinal/set known from the theory of admissible sets (cf. [3, II.5.13]).

With the next section we enter the second chapter of this paper. Sections 9 and 10 develop an ordinal representation system OT(Φ) which will be sufficient unto the task of expressing the proof-theoretic ordinals of all the foregoing theories.

Section 11 introduces the technical basis for well-ordering proofs. By a well-ordering proof in a given theory T we mean a proof formalizable in T which shows that a certain ordinal representation system (or a subset of it) is well-ordered. The notion of a *distinguished set* (of ordinals) (in German: ausgezeichnete Menge) will be central to carrying out well ordering proofs in the various subtheories of second order arithmetic introduced in earlier sections. A theory of distinguished sets developed for this purpose emerged in the works of Buchholz and Pohlers [4, 6, 7].

The remaining sections 12-15 are concerned with well-ordering proofs for most of the theories featuring in this paper. The lower bounds for the proof-theoretic ordinals of theories established in this article turn out to be sharp. Proofs of upper bounds, though, will be dealt with in the second part of this paper which will be devoted to ordinal analysis. The final section of this paper provides a list of all theories and their proof-theoretic ordinals.

I. THEORIES

2 The formal set-up

This section introduces the languages of second order arithmetic and set theory with the natural numbers as urelements. Moreover, a collection of theories, comprehension and induction principles formalized in these languages will be introduced. Our presentation of second order arithmetic is equivalent to those found in the standard literature (e.g. [10, 63]). The same applies to set theory with urelements, where we follow the standard reference [3]. Slight deviations are of a purely technical nature, one pecularity being that we define formulae in such a way that negations occur only in front of prime formulae, another being that function symbols will be avoided. Instead, we axiomatize number theory by means of relation symbols representing their graphs.

2.1 The language $\mathcal{L}_2$

The vocabulary of $\mathcal{L}_2$ consists of free number variables $a_0, a_1, a_2, \ldots$, bound number variables $x_0, x_1, x_2, \ldots$, free set variables $U_0, U_1, U_2, \ldots$, bound set variables $X_0, X_1, X_2, \ldots$, the logical constants $\neg, \wedge, \vee, \forall, \exists$, the constants (numerals) $\bar{n}$ for each $n \in \mathbb{N}$, a 1-place relation symbol P, three 2-place relation symbols $\in, \equiv$, SUC and two 3-place relation symbols ADD, MULT. In addition, $\mathcal{L}_2$ has auxiliary symbols such as parentheses and commas. The intended interpretation of these symbols is the following:

1. Number variables range over natural numbers while set variables range over sets of natural numbers.
2. The constant $\bar{n}$ denotes the nth natural number.
3. P stands for an arbitrary set of natural numbers.
4. $\in$ denotes the elementhood relation between natural numbers and sets of natural numbers.
5. $\equiv$ denotes the identity relation between natural numbers.
6. SUC, ADD, and MULT denote the graphs of the numerical functions $n \mapsto n+1$, $(n,m) \mapsto n+m$, and $(n,m) \mapsto n \cdot m$, respectively.

The *terms* of $\mathcal{L}_2$ are the free number variables and the constants $\bar{n}$. As syntactical we also use a, b, c, d, e for free number variables, R, S, U, V for free set variables, u, v, w, x, y, z, i, j for bound number variables, W, X, Y, Z for bound set variables, r, s, t for terms, and A, B, C, D, F, G, H for formulae of $\mathcal{L}_2$. If E is an expression, $\tau_1, \ldots, \tau_n$ are distinct primitive symbols and $\sigma_1, \ldots, \sigma_n$ are arbitrary expressions, then by $E(\tau_1, \ldots, \tau_n \mid \sigma_1, \ldots, \sigma_n)$ we mean the expression obtained from A by writing σ_i in place of τ_i at each occurrence of τ_i. If A is a formula of the form $B(\tau_1, \ldots, \tau_n \mid \sigma_1, \ldots, \sigma_n)$ then this fact will also be expressed (less accurately) by writing B as $B(\tau_1, \ldots, \tau_n)$ and A as $B(\sigma_1, \ldots, \sigma_n)$.

Definition 2.1 The *atomic formulae* of $\mathcal{L}_2$ are of the form $(s \equiv t)$, $(s \in U)$, $\mathrm{SUC}(s,t)$, $\mathrm{P}(s)$, $\mathrm{ADD}(s,t,r)$, and $\mathrm{MULT}(s,t,r)$.

The $\mathcal{L}_2$-*formulae* are defined inductively as follows: If A is an atomic formula then A and $\neg A$ are $\mathcal{L}_2$-formulae. If A and B are $\mathcal{L}_2$-formulae then so are $(A \wedge B)$ and $(A \vee B)$. If $F(a)$ is an $\mathcal{L}_2$-formula in which the bound number variable x does not occur then $\forall x\, F(x)$ and $\exists x\, F(x)$ are $\mathcal{L}_2$-formulae. If $G(U)$ is an $\mathcal{L}_2$-formula in which the bound set variable X does not occur then $\forall X\, G(X)$ and $\exists X\, G(X)$ are $\mathcal{L}_2$-formulae.

The *negation*, $\neg A$, of a non–atomic formula A is defined to be the formula obtained from A by

(i) putting $\neg$ in front any atomic subformula,
(ii) replacing $\wedge, \vee, \forall x, \exists x, \forall X, \exists X$ by $\vee, \wedge, \exists x, \forall x, \exists X, \forall X$, respectively, and
(iii) dropping double negations.

As usual, $(A \rightarrow B)$ abbreviates $(\neg A \vee B)$ and $(A \leftrightarrow B)$ stands for $((A \rightarrow B) \wedge (B \rightarrow A))$. Outer most parentheses will usually be dropped. We write $s \neq t$ for $\neg(s \equiv t)$ and $s \notin U$ for $\neg(s \in U)$. To avoid parenthesis we also adopt the conventions that $\neg$ binds more strongly than the other connectives and that $\wedge, \vee$ bind more strongly than $\rightarrow$ and $\leftrightarrow$.

We also use the following abbreviations with $Q \in \{\forall, \exists\}$:
$Qx_1 \ldots x_n\, F(x_1, \ldots, x_n) \ := \ Q\, x_1 \ldots Q\, x_n\, F(x_1, \ldots, x_n)$,
$Q\, X_1 \ldots, X_n\, F(X_1, \ldots, X_n) \ := \ QX_1 \ldots QX_n\, F(X_1, \ldots, X_n)$,
and $\forall x \exists! y\, H(x, y) \ := \ \forall x \exists y\, H(x, y) \ \wedge \ \forall xyz\, (H(x, y) \ \wedge \ H(x, z) \rightarrow y \equiv z)$.

Definition 2.2 The formula class Π^1_0 (as well as Σ^1_0) consists of all arithmetical $\mathcal{L}_2$-formulae, i.e., all formulae which do not contain set quantifiers.

If $F(U)$ is a Σ^1_n-formula (Π^1_n-formula) then $\forall X\, F(X)$ ($\exists X\, F(X)$) is a Π^1_{n+1} (Σ^1_{n+1}) formula.

2.2 The theory $\mathbf{ACA}_0$

As a base for all theories in the language $\mathcal{L}_2$ we use the theory $\mathbf{ACA}_0$ which in addition to the usual number-theoretic axioms has the axiom schema of arithmetical comprehension and an induction axiom for sets. As we will subject these theories to proof-theoretic treatment we shall present the axiomatization of $\mathbf{ACA}_0$ in more detail than would otherwise be necessary.

Definition 2.3 The mathematical axioms of $\mathbf{ACA}_0$ are the following:

(i) Equality axioms

(G1) $\forall x\, (x \equiv x)$.

(G2) $\forall xy\, (x \equiv y \rightarrow [F(x) \leftrightarrow F(y)])$ for $F(a)$ in Π^1_0.

(G3) $\bar{n} \equiv \bar{n}$.

(G4) $\bar{n} \not\equiv \bar{m}$ if n, m are different natural numbers.

(i) Axioms for SUC, ADD, MULT.

(SUC1) $\forall x\, \exists! y\, \mathrm{SUC}(x, y)$.

(SUC2) $\forall y\, [y \equiv \bar{0} \ \vee \ \exists x\, \mathrm{SUC}(x, y)]$.

(SUC3) $\forall xyz\, (\mathrm{SUC}(x, z) \ \wedge \ \mathrm{SUC}(y, z) \rightarrow x \equiv y)$.

(SUC4) $\mathrm{SUC}(\bar{n}, \overline{n+1})$.

(SUC5) $\neg\mathrm{SUC}(\bar{n}, \bar{m})$ if $n + 1 \neq m$.

(ADD1) $\forall xy\, \exists! z\, \mathrm{ADD}(x, y, z)$.

(ADD2) $\forall x\, \mathrm{ADD}(x, \bar{0}, x)$.

(ADD3) $\forall uvwxy\, [\mathrm{ADD}(u, v, w) \ \wedge \ \mathrm{SUC}(v, x) \ \wedge \ \mathrm{SUC}(w, y) \ \rightarrow \ \mathrm{ADD}(u, x, y)]$.

(ADD4) $\mathrm{ADD}(\bar{n}, \bar{m}, \overline{n+m})$.

(ADD5) $\neg\mathrm{ADD}(\bar{n}, \bar{m}, \bar{k})$ if $n + m \neq k$.

(MULT1) $\forall xy\, \exists! z\, \mathrm{MULT}(x, y, z)$.

(MULT2) $\forall x\, \mathrm{MULT}(x, \bar{0}, \bar{0})$.

(MULT3) $\forall uvwxy\, [\mathrm{MULT}(u, v, w) \wedge \mathrm{SUC}(v, x) \wedge \mathrm{ADD}(w, u, y) \rightarrow \mathrm{MULT}(u, x, y)]$.

(MULT4) $\mathrm{MULT}(\bar{n}, \bar{m}, \overline{n \cdot m})$.

(MULT5) $\neg\mathrm{MULT}(\bar{n}, \bar{m}, \bar{k})$ if $n \cdot m \neq k$.

(iii) *Induction Axiom*

$$\text{(Ind)} \qquad \forall X\, [\bar{0} \in X \wedge \forall xy\, [\mathrm{SUC}(y, x) \wedge y \in X \rightarrow x \in X] \rightarrow \forall x\, (x \in X)].$$

(iv) *Arithmetical Comprehension*

$$\text{(ACA)} \qquad \exists X\, \forall y\, [y \in X \leftrightarrow F(y)]$$

where $F(a)$ is Π^1_0 and X does not occur in $F(a)$.

As logical rules and axioms for every theory formulated in the language of $\mathcal{L}_2$ we choose the following:

(L1) All formulae of $\mathcal{L}_2$ that are valid in propositional logic.

(L2) The number quantifier axioms $\forall x\, F(x) \rightarrow F(t)$ and $F(t) \rightarrow \exists x\, F(x)$ for every $\mathcal{L}_2$-formula $F(a)$ in which x does not occur and every term t .

(L3) The set quantifier axioms $\forall X\, H(X) \rightarrow H(U)$ and $H(U) \rightarrow \exists X\, F(X)$ for every $\mathcal{L}_2$-formula $H(V)$ in which X does not occur and set variable U.

(L4) *Modus ponens*: From A and $A \rightarrow B$ deduce B.

(L5) From $A \rightarrow F(a)$ deduce $A \rightarrow \forall x\, F(x)$ and from $F(a) \rightarrow A$ deduce $\exists x\, F(x) \rightarrow A$ providing the free number variable a does not occur in the conclusion and x does not occur in $F(a)$.

(L6) From $A \rightarrow H(U)$ deduce $A \rightarrow \forall X\, H(X)$ and from $H(U) \rightarrow A$ deduce $\exists X\, F(X) \rightarrow A$ providing the free set variable U does not occur in the conclusion and X does not occur in $F(U)$.

We write $\mathbf{T} \vdash \mathbf{A}$ when T is a theory in the language of $\mathcal{L}_2$ and A can be deduced from T using the axioms of T and any combination of the preceding axioms and rules of $\mathbf{ACA}_0$.

By **ACA** we denote the theory $\mathbf{ACA}_0$ augmented by the scheme of induction for all $\mathcal{L}_2$-formulae:

$$\text{(IND)} \qquad F(\bar{0}) \wedge \forall xy\, [\mathrm{SUC}(y, x) \wedge F(y) \rightarrow F(x)] \rightarrow \forall x\, F(x)$$

where $F(a)$ is an arbitrary formula of $\mathcal{L}_2$.

The sublanguage of $\mathcal{L}_2$ without set variables will be denoted by $\mathcal{L}_1$.

2.3 The languages $\mathcal{L}^*$ and $\mathcal{L}^*(\mathrm{M})$

$\mathcal{L}^*$ will be the language of set theory with the natural numbers as urelements. $\mathcal{L}^*$ comprises $\mathcal{L}_1$ and in addition has a constant N for the set of natural numbers, a 1-place predicate symbol Set for the class of sets, and a 1-place predicate symbol Ad for the class of admissible sets. The intended interpretations of $\mathcal{L}_1$ and $\mathcal{L}^*$ diverge with respect to the scopes of the quantifiers $\forall x$ and $\exists x$ which in the case of $\mathcal{L}^*$ are viewed as ranging over all sets and urelements. Moreover, $\mathcal{L}^*$ has also bounded quantifiers $(\forall x \in t)$ and $(\exists x \in t)$ which will be treated as quantifiers in their own right.

We will also have use for an extended language $\mathcal{L}^*(\mathrm{M})$ which has a constant M, intended to denote the smallest admissible set.

The terms of $\mathcal{L}^*$ ($\mathcal{L}^*(\mathrm{M})$) consist of the free variables and the constants $\bar{n}$ and N (and M).

The atomic formulae of $\mathcal{L}^*$ ($\mathcal{L}^*(\mathrm{M})$) consists of all strings of symbols of the forms $(s \equiv t)$, $(s \in t)$, $\mathrm{P}(t)$, $\mathrm{SUC}(s,t)$, $\mathrm{ADD}(s,t,r)$, $\mathrm{MULT}(s,t,r)$, $\mathrm{Ad}(s)$, and $\mathrm{Set}(s)$, where s, t, r are arbitrary terms of $\mathcal{L}^*$ ($\mathcal{L}^*(\mathrm{M})$).

Definition 2.4 $\mathcal{L}^*$-formulae are inductively defined as follows:

1. A and $\neg A$ are $\mathcal{L}^*$-formulae whenever A is an atomic $\mathcal{L}^*$-formula.
2. If A and B are $\mathcal{L}^*$-formulae so are $(A \wedge B)$ and $(A \vee B)$.
3. If $F(a)$ is an $\mathcal{L}^*$-formula in which x does not appear and t is an $\mathcal{L}^*$ term then $\forall x\, F(x)$, $\exists x\, F(x)$, $(\forall x \in t)\, F(x)$, and $(\exists x \in t)\, F(x)$ are $\mathcal{L}^*$-formulae.

$\mathcal{L}^*(\mathrm{M})$-formulae are defined in a similar vein. The *negation*, $\neg A$, of a formula A is defined as in Definition 2.1, but extended by the clauses $\neg(\forall x \in t)F(x) := (\exists x \in t)\neg F(x)$ and $\neg(\exists x \in t)F(x) := (\forall x \in t)\neg F(x)$ for the bounded quantifiers.

Definition 2.5 (Translating $\mathcal{L}_2$ into $\mathcal{L}^*$) Let $U_i^* := a_{2\cdot i}$, $X_i^* := x_{2\cdot i+2}$, $a_i^* := a_{2\cdot i+1}$, $x_i^* := x_{2\cdot i+1}$, and $\bar{n}^* := \bar{n}$.

To every $\mathcal{L}_2$-formula A we assign an $\mathcal{L}^*$-formula A^* as follows: Replace every free variable $\mathcal{X}$ in A by $\mathcal{X}^*$. Replace number quantifiers $\forall x \ldots x \ldots$ and $\exists x \ldots x \ldots$ by $(\forall x^* \in \mathrm{N}) \ldots x^* \ldots$ and $(\exists x^* \in \mathrm{N}) \ldots x^* \ldots$, respectively. Replace set quantifiers $\forall X \ldots X \ldots$ and $\exists X \ldots X \ldots$ by $\forall X^*[\mathrm{Set}(X^*) \wedge X^* \subseteq \mathrm{N} \to \ldots X^* \ldots]$ and $\exists X^*[\mathrm{Set}(X^*) \wedge X^* \subseteq \mathrm{N} \wedge \ldots X^* \ldots]$, respectively, where $X^* \subseteq \mathrm{N}$ stands for $\forall u\, [u \in X^* \to u \in \mathrm{N}]$. The translation $A \mapsto A^*$ provides an embedding of $\mathcal{L}_2$ into $\mathcal{L}^*$, preserving the intended interpretations. In what follows we view $\mathcal{L}_2$ as sublanguage of $\mathcal{L}^*$, formally fixed by the natural translation *.

2.4 Syntactic classifications

Definition 2.6 The Δ_0 *formulae* are the smallest collection of $\mathcal{L}^*$ formulae containing all quantifier-free formulae closed under $\neg, \wedge, \vee$ and bounded quantification. Spelled out in detail the last closure clause means that if $F(a)$ is Δ_0, t is a term and x is a bound variable not occurring in $F(a)$ then $(\exists x \in t)F(x)$ and $(\forall x \in t)F(x)$ are Δ_0.

$\mathcal{L}^*$ formulae which are Δ_0 or of the form $\exists x\, F(x)$ with $F(a)$ Δ_0 are said to be Σ_1. Dually, a formula is Π_1 if it is the negation of a Σ_1 formula.

A formula is a Σ *formula* (Π *formula*) if it belongs to the smallest collection of formulae containing the Δ_0 formulae which is closed under $\wedge, \vee$, bounded quantification, and existential (universal) quantification.

Definition 2.7 The collection of P-positive Δ_0 formulae of $\mathcal{L}^*$, $\Delta(\mathrm{P}^+)$, is inductively generated from the Δ_0 formulae in which P does not occur and all formulae $\mathrm{P}(t)$ by closing off under $\vee, \wedge, (\forall x \in s)$, and $(\exists x \in s)$.

The collection of P-positive arithmetical formulae, $\Pi^1_0(\mathrm{P}^+)$, is the collection of $\mathcal{L}_2$ formulae generated from the arithmetical formulae in which P does not occur and the atomic formulae $\mathrm{P}(t)$ by closing off under $\wedge, \vee$, and numerical quantification.

Remark 2.8 *If A is $\Pi^1_0(\mathrm{P}^+)$ then A^* is $\Delta_0(\mathrm{P}^+)$.*

Definition 2.9 For $\mathcal{L}^*$ formulae A and terms s the relativization of A to s, A^s, arises from A by restricting all unbounded quantifiers $\forall x \ldots$ and $\exists x \ldots$ to s, i.e., by replacing them with $(\forall x \in s) \ldots$ and $(\exists x \in s) \ldots$, respectively.

Note that A^s is always a Δ_0 formula.

Many mathematical and set-theoretic predicates have Δ_0 formalizations. For those that occur most frequently we introduce abbreviations:
$\mathrm{Tran}(s) := (\forall x \in s)(\forall y \in x)\,(y \in s)$.
$\mathrm{Ord}(s) := \mathrm{Tran}(s) \wedge (\forall x \in s)\mathrm{Tran}(x)$.
$\mathrm{Lim}(s) := \mathrm{Ord}(s) \wedge (\exists x \in s)(x \in s) \wedge (\forall x \in s)(\exists y \in s)(x \in y)$.
$s \subseteq t := (\forall x \in s) s \in t$.
$(s = t) := (\mathrm{Set}(s) \wedge \mathrm{Set}(t) \wedge s \subseteq t \wedge t \subseteq s) \vee (s \in \mathrm{N} \wedge t \in \mathrm{N} \wedge s \equiv t)$.
$(s = \{t, r\}) := t \in s \wedge r \in s \wedge (\forall x \in s)(x = t \vee x = r)$.
$(s = \langle t, r\rangle) := s = \{\{t, r\}, \{r\}\}$.
$\mathrm{Fun}(f) := f$ is a function.
$(\mathrm{dom}(f) = s) := f$ is a function with domain s.
$(\mathrm{rng}(f) = s) := f$ is a function with range s.
$(f(r) = t) := \mathrm{Fun}(f) \wedge \langle r, t\rangle \in f$.
$(s = \bigcup r) := (\forall x \in r)(x \subseteq s) \wedge (\forall y \in s)(\exists x \in r) y \in x$.

To save us from writing too many symbols we shall adopt the following conventions. Frequently parentheses around bounded quantifiers will be dropped. In writing $A[\vec{a}]$ we intend to convey that all free variables in the formula occur in the list of variables $\vec{a}$. Boldface versions of variables are meant to stand for tuples of variables. If $\vec{x} = (x_1, \ldots, x_r)$ we write $\forall \vec{x}$, $\forall \vec{x} \in s$, $\exists \vec{x}$, and $\exists \vec{x} \in s$ for $\forall x_1 \ldots \forall x_r$, $(\forall x_1 \in s) \ldots (\forall x_r \in s)$, $\exists x_1 \ldots \exists x_r$, and $(\exists x_1 \in s) \ldots (\exists x_r \in s)$, respectively.

We also use class notations $\{x \mid F(x)\}$ as abbreviations with the usual meaning: $s \in \{x \mid F(x)\} := F(s)$, $t = \{x \mid F(x)\} := \forall z[z \in t \leftrightarrow F(z)]$, $\{x \in t \mid F(x)\} := \{x \mid x \in t \wedge F(x)\}$, etc.

Lower case Greek variables $\alpha, \beta, \gamma, \ldots$ range over ordinals. The letters f, g, h will be reserved for functions, i.e., $\forall f \ldots$ and $\exists f \ldots$ stand for $\forall f(\mathrm{Fun}(f) \to \ldots)$ and $\exists f(\mathrm{Fun}(f) \wedge \ldots)$, respectively. We write $\alpha < \beta$ instead of $\alpha \in \beta$.

2.5 A base system for set theory

We fix a formal theory **BT** to serve as a base system for all our set theories. The language of **BT** is $\mathcal{L}^*$.

Definition 2.10 The axioms of **BT** come in four groups.

Logical Axioms

1. Every propositional tautology is an axiom.
2. $\forall x F(x) \rightarrow F(s)$.
3. $F(s) \rightarrow \exists x F(x)$.
4. $(\forall x \in t)F(x) \rightarrow (s \in t \rightarrow F(s))$.
5. $(s \in t \wedge F(s)) \rightarrow (\exists x \in t)F(x)$.

Ontological Axioms

(O1) $s = t \rightarrow [F(s) \leftrightarrow F(t)]$ for $F(a)$ in Δ_0.

(O2) $\mathrm{Set}(t) \rightarrow t \notin \mathrm{N}$.

(O3) $\bar{n} \in \mathrm{N}$ for all $n \in \mathbb{N}$.

(O4) $s \in t \rightarrow \mathrm{Set}(t)$.

(O5) $\mathrm{R}(s_1, \ldots, s_k) \rightarrow s_1 \in \mathrm{N} \wedge \ldots \wedge s_k \in \mathrm{N}$ for $\mathrm{R} \in \{\equiv, \mathrm{SUC}, \mathrm{ADD}, \mathrm{MULT}, \mathrm{P}\}$ and k being the arity of R.

(O6) $\mathrm{Ad}(s) \rightarrow \mathrm{N} \in s \wedge \mathrm{Tran}(s)$.

(O7) $\mathrm{Ad}(s) \wedge \mathrm{Ad}(t) \rightarrow s \in t \vee s = t \vee t \in s$.

(O8) $\mathrm{Ad}(s) \rightarrow \forall x \in s \forall y \in s \exists z \in s\, (x \in z \vee y \in z)$.

(O9) $\mathrm{Ad}(s) \rightarrow \forall x \in s \exists y \in s\, (y = \bigcup x)$.

(O10) $\mathrm{Ad}(s) \rightarrow \forall \vec{u} \in s \forall x \in s\, \exists y \in s\, [\mathrm{Set}(y) \wedge \forall z \in s(z \in y \leftrightarrow z \in x \wedge A[z, x, \vec{u}])]$ for all Δ_0 formulae $A[a, b, \vec{c}]$.

(O11) $\mathrm{Ad}(s) \rightarrow \forall \vec{u} \in s \forall x \in s\, (\forall y \in x\, \exists z \in s\, B[y, z, x, \vec{u}] \rightarrow \exists w \in s\, \forall y \in x\, \exists z \in w\, B[y, z, x, \vec{u}])$ for all Δ_0 formulae $B[a, b, c, \vec{d}]$.

The axioms (O8)–(O11) assert that every admissible set is a model of pairing, union, Δ_0 separation and Δ_0 collection, respectively. (O7) asserts that admissible sets are linearly ordered with respect to $\in$.

Arithmetical Axioms.

All *-translations of the equality axioms and the axioms pertaining to SUC, ADD, and MULT of Definition 2.3 (i) and (ii).

Set Existence Axioms.

(Pairing) $\exists z\, s \in z \wedge t \in z)$.

(Union) $\exists z\, (z = \bigcup s)$.

(Δ_0 Separation) $\exists z[\mathrm{Set}(z) \wedge \forall x \in z(x \in s \wedge A(x)) \wedge \forall x \in s(A(x) \rightarrow x \in z)]$
for $A(a)$ in Δ_0.

Induction Axioms.

(Δ_0-FOUND) $\mathrm{Tran}(s) \wedge \forall x \in s(\forall y \in x\, A(y) \rightarrow A(x)) \rightarrow \forall x \in s\, A(x)$
whenever $A(a)$ is Δ_0.

$(\mathrm{Ind})^*$ $\bar{0} \in s \wedge \forall xy \in \mathrm{N}[\mathrm{SUC}(y,x) \wedge y \in s \rightarrow x \in s] \rightarrow \forall x \in \mathrm{N}\, x \in s$.

As *logical rules* of **BT** we choose *Modus Ponens* and the following quantifier rules:

$$A \rightarrow F(a) \vdash A \rightarrow \forall x F(x)$$

$$F(a) \rightarrow A \vdash \exists F(x) \rightarrow A$$

$$A \rightarrow (a \in s \rightarrow F(a)) \vdash A \rightarrow \forall x \in sF(x)$$

$$(F(a) \wedge a \in s) \rightarrow A \vdash \exists x \in sF(x) \rightarrow A$$

with the proviso that a does not occur in the conclusion.

If **T** is a theory in the language $\mathcal{L}^*(\mathrm{M})$ which comprises **BT** then $\mathbf{T} \vdash A$ is meant to convey that A is deducible from the axioms of **T** via the above rules of inference.

Remark 2.11 *All non-logical axioms of* **BT** *are of the form* $G[\vec{s}]$ *where* $G[\vec{a}]$ *is a* Σ_1 *formula. Also note that none of the axioms of* **BT** *asserts that any admissible sets exist.*

Lemma 2.12 $\mathbf{ACA}_0 \subseteq \mathbf{BT}$.

Proof. The $\subseteq$ symbol is meant to convey that every theorem of $\mathbf{ACA}_0$ is a theorem of **BT** via the *-translation. This is proved by induction on the length of derivations in $\mathbf{ACA}_0$. The only interesting cases to inspect are the arithmetical comprehension axioms. The *-translation turns them into instances of Δ_0 separation. □

At this point, having introduced a great deal of the formal background for the paper, we can rejoice. Perhaps a few words about **BT** are in order. We assume that the reader is acquainted with the theory of admissible sets. The standard reference for admissible sets and an excellent presentation at that is [3]. The axioms (O6), (O8)–(O11) and Δ_0-FOUND ensure that, provably in **BT**, every set $\mathcal{A}$ which satisfies $\mathrm{Ad}(\mathcal{A})$ is a model of the theory $\mathbf{KPU}^+$ of [3] with the set of natural numbers as urelements.

Definition 2.13 The theory **KPu** comprises **BT** and has the following additional axioms.

(Δ_0-Collection) $\forall x \in s\, \exists y\, A(x,y) \to \exists z \forall x \in s \exists y \in z\, A(x,y)$ where $A(a,b) \in \Delta_0$;

(IND*) $\forall xy \in \mathrm{N}\, (\mathrm{SUC}(y,x) \wedge F(y) \to F(x)) \to \forall x \in \mathrm{N}\, F(x)$;

(FOUND) $\forall x(\forall y \in x F(y) \to F(x)) \to \forall x F(x)$,

where in the last two schemata $F(a)$ may be any formula of $\mathcal{L}^*(\mathrm{M})$.

In naming this theory **KPu** we follow the usage of [27].

Remark 2.14 One cannot prove the existence of an admissible set in **KPu**. As a result, the axioms (O6)–(O11) are immaterial as far as the proof strength of **KPu** is concerned. In more detail, letting $\mathbf{KPu}^-$ denote **KPu** restricted to the language $\mathcal{L}^*$ without the predicate symbol Ad, we have $\mathbf{KPu} \vdash A \;\Rightarrow\; \mathbf{KPu}^- \vdash A^-$ for every formula A of $\mathcal{L}^*$, where A^- results from A by replacing any occurrence $\mathrm{Ad}(s)$ in A by $\bar{0} \neq \bar{0}$ and any occurrence of M by N. This shows that **KPu** is a conservative extension of $\mathbf{KPu}^-$.

As regards the predicate Ad, it plays a role in extensions of **BT** which prove $\exists x \mathrm{Ad}(x)$. Examples of such systems are the theories **KPl** and **KPi** introduced in [27]. **KPl** axiomatizes a set universe which is a limit of admissible sets while **KPi** also demands that the universe itself be an admissible set (or class).

Definition 2.15 **KPl** is the theory **BT** + (IND*) + (FOUND) + (Lim) + (M), where (Lim) is the axiom schema $\exists y(\mathrm{Ad}(y) \wedge s \in y)$ and (M) is the axiom $\mathrm{Ad}(\mathrm{M}) \wedge \forall x \in \mathrm{M}\, \neg \mathrm{Ad}(x)$.

KPi is the theory **KPu** + **KPl**.

In what follows, theories having only restricted versions of (FOUND) or (IND*) as axioms will be of great importance. Such theories are interesting because of the following observations. In mathematics one mostly uses only limited amounts of induction. From proof theory we know that restricting the amount of induction tends to give rise to theories of much weaker proof-theoretic strength.

Definition 2.16 If **T** is a theory whose axiom schemata comprise (IND*) and (FOUND) we denote by $\mathbf{T}^w$ the theory without (FOUND) and by $\mathbf{T}^r$ the theory without (FOUND) and (IND*).

Using this convention, $\mathbf{KPl}^w$ is **BT** + (IND*) + (Lim) + (M) and $\mathbf{KPl}^r$ is **BT** + (Lim) + (M).

Remark 2.17 The combination of (Lim), (O7) and (Δ_0-FOUND) implies $\exists! y(\mathrm{Ad}(y) \wedge \forall z \in y \neg \mathrm{Ad}(z))$. Therefore $\mathbf{KPl}^r$ is a definitional extension of $\mathbf{KPl}^r$ without (M).

2.6 Some derivable consequences

We show show some basic principles that can be deduced in theories introduced so far. For future reference some will be labeled with traditional names.

Lemma 2.18 $\mathbf{KPu} \vdash F[\vec{a}] \;\Rightarrow\; \mathbf{KPl}^r \vdash \mathrm{Ad}(s) \to \forall \vec{x} \in s\, F^s[\vec{x}]$.

Proof. Proceed by induction on the length of the derivation in **KPu**. □

The main use we shall make of the foregoing lemma is that every statement that can be proved in **KPu** about the universe of sets can be transferred to $\mathbf{KPl}^r$ by relativizing it to any admissible set.

Proofs for the following four results can be found in [3], I.4.2-4.4.5.

Lemma 2.19 (Σ Persistence) *For every* Σ *formula* A *we have:*

(i) $\mathbf{BT} \vdash A^s \wedge s \subseteq t \to A^t$*;*

(ii) $\mathbf{BT} \vdash A^S \to A$*.*

Lemma 2.20 (Σ Reflection) *For* $A \in \Sigma$ *we have* $\mathbf{KPu}^r \vdash A \to \exists x\, A^x$*.*

Lemma 2.21 (Σ Collection) *For every* Σ *formula* $F(a,b)$*,*

$$\mathbf{KPu}^r \vdash \forall x \in s\, \exists y\, F(x,y) \to \exists z[\forall x \in s\, \exists y \in z\, F(x,y) \wedge \forall y \in z\, \exists x \in s\, F(x,y)].$$

Lemma 2.22 (Δ Separation) *If* $A(a)$ *is in* Σ *and* $B(a)$ *is in* Π *then*

$$\mathbf{KPu}^r \vdash \forall x\, [A(x) \leftrightarrow B(x)] \to \exists z(\mathrm{Set}(z) \wedge \forall x[x \in z \leftrightarrow A(x)]).$$

Lemma 2.23 (Σ Replacement) *For every* Σ *formula* $F(a,b)$*,*

$$\mathbf{KPu}^r \vdash \forall x \in s\, \exists! y\, F(x,y) \to \exists f[\mathrm{Fun}(f) \wedge \mathrm{dom}(f) = s \wedge \forall x \in s\, F(x, f(x))].$$

An powerful tool in set theory is definition by transfinite recursion. The most important applications are definitions of functions by Σ recursion. The axioms of **KPu** are sufficient for this task. A closer inspection of the well known proof (see [3], I.6.1)) reveals that a restricted form of foundation, dubbed (Σ-FOUND), suffices.

(Σ-FOUND) is the schema

$$\forall x[\forall y \in x\, F(y) \;\to\; F(x)] \;\to\; \forall x\, F(x) \tag{1}$$

for $F(a) \in \Sigma$.

Lemma 2.24 (Σ Recursion) $\mathbf{KPu}^r$+(Σ-FOUND) *proves all instances of* Σ *recursion,* (Σ-REC)*,*

$$\forall\alpha\forall x\exists! y\, G(\alpha, x, y) \;\to\; \forall\alpha\exists f[\mathrm{Fun}(f) \wedge \mathrm{dom}(f) = \alpha \wedge (\forall\beta < \alpha)G(\beta, f{\restriction}\beta, f(\beta))]$$

where $G(a,b,c) \in \Sigma$ *and* $f{\restriction}\beta = \{\langle\delta, z\rangle \mid \delta < \beta \wedge \langle\delta, z\rangle \in f\}$*.*

Remark 2.25 The formalization of systems of set theory with a predicate earmarked for the class of admissible sets was introduced in [27] for proof-theoretic purposes. Singling out **KPl** as a system worthy of attention owes much to the observation (see [3], V.6.12) that L_α is a model of axiom Beta (see [3, I.9.4]) if α is a limit of admissible ordinals.

3 Theories of iterated inductive definitions

In this section we introduce second theories of iterated inductive definitions formalized in the language of second order arithmetic. Till now such theories were formulated as first order theories with quantifiers just ranging over the natural numbers but with the aid of predicate symbols to represent inductively defined sets (see [10]). In this restricted language one can only talk about well orderings defined by arithmetical formulae. Moving to $\mathcal{L}_2$ enables one to reformulate these theories via set existence axioms and to talk about iterated inductive definitions where the iteration is carried out along arbitrary well orderings.

Instead of pursuing a proof-theoretic analysis of such theories directly via specific infinitary proof systems taylored to accommodate iterated inductive definitions (as e.g. in [10]), we analyze them by first embedding them into germane set theories and subsequently use the general machinery for ordinal analysis of set theories. In this way we obtain uniform and simultaneous ordinal analyses of almost all theories of inductive definitions. An example which illustrates the uniformity of the method is the theory $\mathbf{ID}_{\prec^*}$ introduced in [15]. An analysis of $\mathbf{ID}_{\prec^*}$ was carried out in [5] by means of the Ω_ν-rules. An sketch of an ordinal analysis of this theory by Pohlers' so-called method of local predicativity was adumbrated in [42]. But a full analysis using this technique didn't materialize before [71] (169 pages), and turned out to be quite a formidable task. Therefore I consider it in order to add a further proof, in particular as it is more or less a by-product of the investigation of yet stronger theories of iterated inductive definitions.

As per usual, to begin with we need to set up some terminological conventions.

Definition 3.1 Let $Q(a,b)$ be a formula of $\mathcal{L}_2$ (which may contain additional parameters, i.e. free variables). When we view Q as a binary relation we shall write sQt for $Q(s,t)$. Let $F(a)$ be an arbitrary formula of $\mathcal{L}_2$. We will use the following abbreviations:

$$\begin{array}{rcl}
\mathrm{Fld}(s) & := & \exists x(sQx \vee xQs) \quad (s \text{ is in the field of } Q);\\
\mathrm{LO}(Q) & := & \forall x \neg(xQx) \wedge \forall xy[\mathrm{Fld}(x) \wedge \mathrm{Fld}(y) \to xQy \vee x \equiv y \vee yQx]\\
 & & \wedge \forall xyz[xQy \wedge yQz \to xQz] \quad (Q \text{ is a linear order});\\
\mathrm{PROG}(Q,F) & := & \forall x[\forall y(yQx \to F(y)] \to F(x)] \quad (F \text{ is } Q \text{ progressive});\\
\mathrm{TI}(Q,F) & := & \mathrm{PROG}(Q,F) \to \forall x\, F(x) \quad (Q \text{ induction for } F);\\
\mathrm{WF}(Q) & := & \forall X\, \mathrm{TI}(Q,X) \quad (Q \text{ is well founded});\\
\mathrm{WO}(Q) & := & \mathrm{LO}(Q) \wedge \mathrm{WF}(Q) \quad (Q \text{ is a well-ordering}).
\end{array}$$

We shall use the primitive recursive pairing function $\langle m,n\rangle := \frac{1}{2}(m+n)(m+n+1)+m$. If U is a set we denote by U_s the set $\{x \mid \langle s,x\rangle \in U\}$.

We shall use the notation $F(\mathrm{P}^+)$ to convey that $F(\mathrm{P}) \in \Pi^1_0(\mathrm{P}^+)$ (see Definition 2.7). Such formulae are said to be P-*positive arithmetical formulae*. If $H(a)$ is any formula then $F(H)$ denotes the formula obtained by replacing all occurrences of the form $\mathrm{P}(t)$ and $\mathrm{P}(x)$ by $H(t)$ and $H(x)$, respectively, with the usual proviso that we may have to rename some bound variables to avoid any unintended capture of variables.

Definition 3.2 ($\mathbf{ID}_{\prec^*}$) Let $\prec$ be a linear ordering on $\mathbb{N}$, definable via an arithmetical formula $\mathrm{Q}[a,b]$ such that $\mathbf{ACA}_0 \vdash \mathrm{LO}(\prec)$. The vocabulary of $\mathbf{ID}_{\prec^*}$, $\mathcal{L}(\mathbf{ID}_{\prec^*})$, comprises that of $\mathcal{L}_1$ but in addition has a unary predicate symbol Q^* to denote the accessible part of $\prec$ and moreover for every P-positive arithmetical formula $F[\mathrm{P}^+, U, a, b]$ it has a two-place predicate symbol I^F.

The axioms of $\mathbf{ID}_{\prec^*}$ comprise those of Definition 2.3(i),(ii) and the induction schema (IND) for all formulae of $\mathcal{L}(\mathbf{ID}_{\prec^*})$. Further axioms of $\mathbf{ID}_{\prec^*}$ are the following:

$$
\begin{array}{ll}
(\mathrm{Q}^*1) & \mathrm{PROG}(\prec, \mathrm{Q}^*) \\
(\mathrm{Q}^*2) & \mathrm{PROG}(\prec, H) \rightarrow \forall x\,[\mathrm{Q}^*(x) \rightarrow H(x)] \\
(\mathrm{I}^{\mathrm{Q}^*}1) & \mathrm{Q}^*(t) \rightarrow \forall x\,(\,F[\mathrm{I}^F_t, \mathrm{I}^F_{\prec t}, t, x] \rightarrow x \in \mathrm{I}^F_t\,] \\
(\mathrm{I}^{\mathrm{Q}^*}2) & \mathrm{Q}^*(t) \,\wedge\, \forall x\,(F[H, \mathrm{I}^F_{\prec t}, t, x] \rightarrow H(x)) \rightarrow \forall x (x \in \mathrm{I}^F_t \rightarrow H(x))
\end{array}
$$

for every arithmetical formula $F[\mathrm{P}^+, U, a, b]$ and every $\mathcal{L}(\mathbf{ID}_{\prec^*})$ formula $H(a)$, where we used the notations $s \in \mathrm{I}^F_t := \mathrm{I}^F(t, s)$ and $\mathrm{I}^F_{\prec t} := \{z \mid \exists y(y \prec t \,\wedge\, z \in \mathrm{I}^F_y\}$.

All arithmetical well-orderings have an order-type less than the first recursively inaccessible ordinal ω_1^{CK}. Since the accessible part of a primitive recursive ordering can have order-type ω_1^{CK} (see [21]) it seems that $\mathbf{ID}_{\prec^*}$ may be able to axiomatize iterated inductive definitions along non-arithmetical well orderings in contrast to what we said at the beginning of this section about previous investigations of such theories in proof theory. This point will be clarified later once we have embedded $\mathbf{ID}_{\prec^*}$ into a second order system.

Definition 3.3 For formulae $F(\mathrm{P}, U, a, b)$ and $Q(a, b)$ we use the abbreviations

$$
\begin{aligned}
\mathrm{Cl}^F(V, U, s) &:= \forall x[F(V, U, s, x) \rightarrow x \in V] \\
\mathrm{IT}^F(Q, U) &:= \forall i\,[\mathrm{Cl}^F(U_i, U_{Qi}, i) \,\wedge\, \forall Y(\mathrm{Cl}^F(Y, U_{Qi}, i) \rightarrow U_i \subseteq Y)],
\end{aligned}
$$

where $U_i := \{x \mid \langle i, x\rangle \in U\}$ and $U_{Qi} := \{x \mid \exists j(jQi \,\wedge\, \langle j, x\rangle \in U\}$.

Definition 3.4 (i) Let $\mathbf{ID}^*$ be the theory $\mathbf{ACA}$ augmented by the schema

$$(\mathrm{IT}^*1) \qquad \forall x[\mathrm{WO}(\prec_x) \rightarrow \exists Z\, \mathrm{IT}^F(\prec_x, Z)]$$

for every formula $F[P^+, U, a, b]$ (having no further parameters) and every family of relations $(\prec_n)_{n\in\mathbb{N}}$, which is definable by some arithmetical formula $Q[a, b, c]$ via $s \prec_r t := Q[s, t, r]$.

(ii) $\mathbf{ID}^*_2$ arises from $\mathbf{ID}^*$ by adding the schema

$$(\mathrm{IT}^*2) \qquad \forall x \forall i \forall Z[\mathrm{WO}(\prec_x) \,\wedge\, \mathrm{IT}^F(\prec_x, Z) \,\wedge\, \mathrm{Cl}^F(H, Z_{\prec_x i}, i) \rightarrow Z_i \subseteq H]$$

with the same conditions on F as above and every formula $H(a)$.

$\mathbf{ID}^*_2$ makes greater demands than $\mathbf{ID}^*$ in that if $\mathrm{WO}(\prec_s)$ holds and U satisfies $\mathrm{IT}^F(\prec_s, U)$, then every section U_i of U will also be contained in all $\mathcal{L}_2$-definable classes closed under the operator $X \mapsto \{z \mid F[X, U_{\prec_s i}, z]\}$.

It is obvious that $\mathbf{ID}^*_2$ merely axiomatizes iterations of length $< \omega_1^{CK}$. However, the strength of $\mathbf{ID}^*_2$ is owed to the fact that the arithmetical well-orderings may depend on numerical parameters which can be quantified over. E.g. if $\prec_e$ is defined by $n \prec_e m := \exists y\, \mathrm{T}_2(e, n, m, y)$ where T_2 stands for the well-known primitive recursive predicate from Kleene's normal form theorem, then via the statement $\forall x[\mathrm{WO}(\prec_x) \to \exists Z\, \mathrm{IT}^F(\prec_x, Z_x)]$ we quantify over all iterations below ω_1^{CK}.

On the other hand, in general it is not possible to deduce $\exists Z \forall x[\mathrm{WO}(\prec_x) \to \mathrm{IT}^F(\prec_x, Z_x)]$ within $\mathbf{ID}_2^*$ which would amount to an iteration of length ω_1^{CK}.

If one lifts the restriction to arithmetical well-orderings and allows arbitrary parameters in the operator forms in the schemata (IT*1) and (IT*2) one arrives at autonomous theories of arithmetical inductive definitions which provide the natural limit of all such theories. Furthermore, we also consider the wider class of monotone inductive definitions.

Definition 3.5 (i) $\mathbf{AUT\text{-}ID}^{pos}$ is the theory $\mathbf{ACA}$ augmented by the schema

$$(\mathrm{IT}^{pos}1) \qquad \forall X[\mathrm{WO}(X) \to \exists Z\, \mathrm{IT}^F(X, Z)]$$

for every arithmetical formula $F(P^+, U, a, b)$.

(ii) $\mathbf{AUT\text{-}ID}^{mon}$ is the theory $\mathbf{ACA}$ augmented by the schema

$$(\mathrm{IT}^{mon}1) \qquad \mathrm{MON}(F) \to \forall X[\mathrm{WO}(X) \to \exists Z\, \mathrm{IT}^F(X, Z)]$$

for every arithmetical formula $F(P, U, a, b)$, where

$$\mathrm{MON}(F) := \forall i \forall x \forall X \forall Y \forall Z[X \subseteq Y \ \wedge\ F(X, Z, i, x) \to F(Y, Z, i, x)].$$

(iii) By $\mathbf{AUT\text{-}ID}_2^{pos}$ we denote the theory $\mathbf{AUT\text{-}ID}^{pos}$ plus the scheme

$$(\mathrm{IT}^{pos}2) \qquad \mathrm{WO}(R) \ \wedge\ \mathrm{IT}^F(R, U) \ \wedge\ \mathrm{Cl}^F(H, U_{Rs}, s) \to U_s \subseteq H$$

for every arithmetical formula $F(P^+, U, a, b)$ and arbitrary $\mathcal{L}_2$-formula $H(b)$.

In the same vein one defines $(\mathrm{IT}^{mon}2)$ and the theory $\mathbf{AUT\text{-}ID}_2^{mon}$.

(iv) If $\mathbf{T}$ is an $\mathcal{L}_2$ theory defined by adding axioms to $\mathbf{ACA}$, then $\mathbf{T}_0$ denotes the theory which is obtained by adding the same axioms to $\mathbf{ACA}_0$.

Remark 3.6 For P-positive formulae $F(\mathrm{P}^+, U, a, b)$, $\mathrm{MON}(F)$ is provable in pure logic. Thus axioms for monotone inductive definitions imply the corresponding axioms for positive ones.

In Definition 3.1 we defined well-foundedness of a relation in a somewhat unusual way. One can prove in $\mathbf{ACA}_0$ that our definition is equivalent to the usual one.

Lemma 3.7 $\mathbf{ACA}_0 \vdash \mathrm{WF}(R) \leftrightarrow \forall Z[Z \neq \emptyset \to \exists x \in Z\, \forall y(yRx \to y \notin Z)]$.

Proof. Exercise or see [17], 6.1.5. □

Lemma 3.8 $\mathbf{ACA}_0 \vdash \mathrm{WO}(R) \ \wedge\ \mathrm{IT}^F(R, U) \ \wedge\ \mathrm{IT}^F(R, V) \to \forall i\, (U_i = V_i)$.

Proof. Assume $\mathrm{WO}(R)$, $\mathrm{IT}^F(R,U)$, $\mathrm{IT}^F(R,V)$ but $U_i \neq V_i$ for some i. By Lemma 3.7 we can pick an R-minimal i_0 with $U_{i_0} \neq V_{i_0}$. By minimality, $U_{Ri_0} = V_{Ri_0}$, and thus $\mathrm{Cl}(U_{i_0}, V_{Ri_0}, i_0)$ as well as $\mathrm{Cl}(V_{i_0}, U_{Ri_0}, i_0)$. As a result, $V_{i_0} \subseteq U_{i_0}$ and $U_{i_0} \subseteq V_{i_0}$, yielding the contradiction $U_{i_0} = V_{i_0}$. □

Definition 3.9 We define an interpretation $^{\curlywedge} : \mathbf{ID}_{\prec^*} \longrightarrow \mathbf{ID}_2^*$, where $\prec$ is given by a formula $Q[a,b]$. Let $s \prec_r t := s \prec t \wedge (t \equiv r \vee t \prec r)$,

$$\begin{aligned} \mathrm{Acc}(\prec, U) &:= \mathrm{PROG}(\prec, U) \wedge \forall Z[\mathrm{PROG}(\prec, Z) \rightarrow U \subseteq Z], \\ \mathrm{P}^F(r,s) &:= \exists Z[\mathrm{IT}^F(\prec_r, Z) \wedge s \in Z_r] \end{aligned}$$

for every arithmetical formula $F[\mathrm{P}^+, U, a, b]$.

If A is a formula of $\mathbf{ID}_{\prec^*}$ then $A^{\curlywedge}$ arises from A by replacing all subformulas $\mathrm{Q}^*(s)$ and $\mathrm{I}^F(r,s)$ in A by $\exists X[s \in X \wedge \mathrm{Acc}(\prec, X)]$ and $\mathrm{P}^F(r,s)$, respectively.

Theorem 3.10 *If* $\mathbf{ID}_{\prec^*} \vdash A$ *then* $\mathbf{ID}_2^* \vdash A^{\curlywedge}$.

Proof. It suffices to prove the translations of axioms arising from the schemata (Q^*1), (Q^*2), ($\mathrm{I}^{\mathrm{Q}^*}1$) and ($\mathrm{I}^{\mathrm{Q}^*}2$) in $\mathbf{ID}_2^*$. We reason in the target theory $\mathbf{ID}_2^*$. Let

$$G[\mathrm{P}^+, U, a, b] := \forall y[y \prec b \rightarrow \mathrm{P}(y)].$$

There exists a set V such $\mathrm{IT}^G(\emptyset, V)$. For $S := V_{\bar{0}}$ we have $\mathrm{Acc}(\prec, S)$. As in Lemma 3.8 one shows that thereby S is uniquely determined, i.e. $\forall X[\mathrm{Acc}(\prec, X) \rightarrow X = S]$. Therefore we get $\forall x[(\mathrm{Q}^*(x))^{\curlywedge} \leftrightarrow x \in S)$ and thus $(\mathrm{PROG}(\prec, \mathrm{Q}^*))^{\curlywedge}$. As a result, provability of the translation of (Q^*2) follows with the help of (IT^*2).

To prove the translations of ($\mathrm{I}^{\mathrm{Q}^*}1$) and ($\mathrm{I}^{\mathrm{Q}^*}2$) suppose that $(\mathrm{Q}^*(r))^{\curlywedge}$ holds, i.e. $r \in S$. If $\neg\mathrm{WO}(\prec_r)$ then by Lemma 3.7 there exists a set U such that $r \in U$ and $\forall x \in U\, \exists y \in U\, (y \prec_r x)$. Letting $U^c := \{i \mid i \notin U\}$ we have $\mathrm{PROG}(\prec, U^c)$ and thus $S \subseteq U^c$ by choice of S. But this is incompatible with $r \in U$. Hence $\prec_r$ must be a well-ordering.

Now let $F[\mathrm{P}^+, U, a, b]$ be an arithmetical P-positive formula. From (IT^*1) we obtain the existence of a set V satisfying $\mathrm{IT}^F(\prec_r, V)$. By Lemma 3.8 we conclude that

$$\forall i[i \prec r \vee i \equiv r \rightarrow \forall x((\mathrm{I}^F(i,x))^{\curlywedge} \leftrightarrow x \in V_i)],$$

and hence $(\mathrm{I}^F_{\prec r})^{\curlywedge} = V_{\prec r}$ and $(\mathrm{I}^F_r)^{\curlywedge} = V_r$. From the foregoing and (IT^*2) we obtain the derivability of the $^{\curlywedge}$-translations of ($\mathrm{I}^{\mathrm{Q}^*}1$) and ($\mathrm{I}^{\mathrm{Q}^*}2$). □

Definition 3.11 *Bar induction*, abbreviated (BI), is the schema

$$\forall X\, [\mathrm{WO}(X) \rightarrow \mathrm{TI}(X, H)]$$

for every $\mathcal{L}_2$-formula $H(a)$.

Lemma 3.12 (BI) *is a consequence of* $\mathbf{AUT}\text{-}\mathbf{ID}_2^{pos}$.

Proof. Assume $\mathrm{WO}(R)$ and $\mathrm{PROG}(R,H)$. We aim at showing $\forall x\, H(x)$. Let $F(\mathrm{P}^+, U, a, b) := \forall z[zRb \to \mathrm{P}(z)]$. Owing to ($\mathrm{IT}^{pos}1$) there exists a set V such that $\mathrm{IT}^F(\emptyset, V)$. Letting $S := V_{\bar{0}}$ and employing ($\mathrm{IT}^{pos}2$) we obtain

$$\mathrm{PROG}(R,S) \quad \text{and} \quad \mathrm{PROG}(R,H) \to S \subseteq H.$$

Hence $\forall x\,(x \in S) \wedge S \subseteq H$, thus $\forall x\, H(x)$. □

Theorem 3.13 *(i)* $\mathbf{ID}_2^* \subseteq \mathbf{ID}^* + (\mathrm{BI})$.

(ii) $\mathbf{AUT\text{-}ID}_2^{pos} = \mathbf{AUT\text{-}ID}^{pos} + (\mathrm{BI})$.

(iii) $\mathbf{AUT\text{-}ID}_2^{mon} = \mathbf{AUT\text{-}ID}^{mon} + (\mathrm{BI})$.

Proof. (ii) In view of Lemma 3.12 it suffices to show that the instances of ($\mathrm{IT}^{pos}2$) can be derived in $\mathbf{AUT\text{-}ID}^{pos}$ with the help of (BI). Assume $\mathrm{WO}(R)$, $\mathrm{IT}^F(R,V)$ and $\mathrm{Cl}^F(H, V_{Rs}, s)$. Letting $G(U) := \mathrm{Cl}^F(U, V_{Rs}, s) \to V_s \subseteq U$, we have

$$\forall Z\, G(Z). \tag{2}$$

Moreover,

$$\mathbf{ACA}_0 + (\mathrm{BI}) \vdash \forall Z\, A(Z) \to A(H) \tag{3}$$

holds for every arithmetical formula $A(U)$ and arbitrary $\mathcal{L}_2$-formula $H(a)$ (see [15], Lemma 1.6.3). As $G(U)$ is arithmetical we get $V_s \subseteq H$ from (2) and (3). This shows ($\mathrm{IT}^{pos}2$).

(i) and (iii) are proved similarly, crucially using (3) and also Lemma 3.12 for the "$\supseteq$" entailments. □

Remark 3.14 Up to the year 1981, the monograph [10] gives a comprehensive account of the proof theory of iterated inductive definitions. The preface written by Feferman provides a detailed history of the subject.

4 Theories of iterated Π^1_1-comprehension

It is a classical result (see [34, 66]) that every Π^1_1-set of the structure $\mathfrak{N} = (\mathbb{N}, 0, +, \cdot)$ can be obtained as a section of a fixed point of a positive arithmetical inductive definition. As an extension of this result there is a close connection between iterated inductive definitions and iterated Π^1_1-comprehensions. The next definition provides a precise definition of the latter sort of theory.

Definition 4.1 For every Π^1_1-formula $B(U,a,b)$ let

$$\mathrm{HJ}^B(R,U) \quad := \quad \forall x \forall i[x \in U_i \leftrightarrow B(U_{Ri}, i, x)], \tag{4}$$

where $U_{Ri} = \{y \mid \exists j\,(jRi \wedge y \in U_j)\}$.

(i) (Π^1_1-TR) is the schema

$$\forall X\,[\mathrm{WO}(X) \to \exists Y\,\mathrm{HJ}^B(X,Y)]$$

where $B(U,a,b)$ is Π^1_1.

(ii) The theory of Π^1_1 *transfinite recursion*, Π^1_1-**TR**, is **ACA** augmented by the schema (Π^1_1-TR).

Lemma 4.2 ($\mathbf{ACA}_0$) *If* $\mathrm{WO}(R)$, $\mathrm{HJ}^B(R,U)$ *and* $\mathrm{HJ}^B(R,V)$, *then* $U_i = V_i$ *holds for all* i.

Proof. Analogous to Lemma 3.8. □

Lemma 4.3 $\mathbf{AUT\text{-}ID}_0^{mon} \subseteq \Pi^1_1\text{-}\mathbf{TR}_0$.

Proof. We have to show that ($\mathrm{IT}^{mon}1$) is deducible in Π^1_1-$\mathbf{TR}_0$. Let $F(\mathrm{P},U,a,b)$ be arithmetical and set $B(U,a,b) := \forall Z[\mathrm{Cl}^F(Z,U,a) \to b \in Z]$. Now assume $\mathrm{WO}(R)$ and $\mathrm{MON}(F)$. Invoking (Π^1_1-TR), there exists a set S such that $\mathrm{HJ}^B(R,S)$. For all i we then have

$$S_i \quad = \quad \{x \mid \forall Z[\mathrm{Cl}^F(Z,S_{Ri},i) \to x \in Z]\} \tag{5}$$

and hence

$$\forall Z[\mathrm{Cl}^F(Z,S_{Ri},i) \to S_i \subseteq Z]. \tag{6}$$

Thus if $F(S_i,S_{Ri},i,s)$ and $\mathrm{Cl}^F(U,S_{Ri},i)$ hold, then from (6) we obtain $S_i \subseteq U$ and, since $\mathrm{MON}(F)$, we have $F(U,S_{Ri},i,s)$ and consequently $s \in U$. So in view of (5) the preceding argument shows that $F(S_i,S_{Ri},i,s) \to s \in S_i$ holds. Thence

$$\mathrm{Cl}(S_i,S_{Ri},i). \tag{7}$$

(6) and (7) yield $\mathrm{IT}^F(R,S)$. □

Corollary 4.4 *For every* Π^1_1*-formula* $F(\mathrm{P},U,a,b)$,

$$\Pi^1_1\text{-}\mathbf{TR}_0 \vdash \mathrm{MON}(F) \to \forall X[\mathrm{WO}(X) \to \exists Z\,\mathrm{IT}^F(X,Z)].$$

Proof. Since (Σ^1_1-AC) is deducible in Π^1_1-$\mathbf{CA}_0$ (cf. 5.11(i)), the formula $\mathrm{Cl}^F(V,U,a,b)$ is provably equivalent to a Σ^1_1-formula in Π^1_1-$\mathbf{TR}_0$, thus $B(U,a,b)$ is equivalent to a Π^1_1-formula. □

To prove the inclusion Π^1_1-$\mathbf{TR}_0 \subseteq \mathbf{AUT\text{-}ID}_0^{pos}$, we shall need the inductive characterization of Π^1_1 classes mentioned at the beginning of this section. Unfortunately, the usual proofs in the literature (cf. [3], VI.1.11 and [25], III.3.2) cannot be directly carried out in $\mathbf{AUT\text{-}ID}_0^{pos}$ since they utilize ordinals or use Π^1_1-comprehension. Albeit Π^1_1-comprehension is a consequence of $\mathbf{AUT\text{-}ID}_0^{pos}$, we cannot use it at this point since this is part of what we want to prove. $\mathbf{ACA}_0$ suffices for the desired characterization of Π^1_1 classes.

Lemma 4.5 *For every* Π^1_1*-formula* $B(U,a,b)$ *one can construct an arithmetical formula* $F(\mathrm{P}^+,U,a,b)$ *such that*

$$\mathbf{ACA}_0 \vdash \forall X \forall i \forall z\, [B(Y,i,z) \leftrightarrow \forall Z\, [\mathrm{Cl}^F(Z,Y,i) \to \langle z, \bar{1}\rangle \in Z]].$$

Proof. By the Π^1_1 normal form theorem (cf. [25, IV.1.]) one finds an arithmetical formula $Q(U,a,b,c,d)$ such that with $c \prec^b d := Q(U,a,b,c,d)$ one has

$$\begin{aligned} \mathbf{ACA}_0 &\vdash \forall z\, [B(U,a,z) \leftrightarrow \mathrm{WO}(\prec^z)]; && (8)\\ \mathbf{ACA}_0 &\vdash \forall z \forall x\, [\mathrm{Fld}(x,\prec^z) \to (x \prec^z \bar{1} \vee x \equiv \bar{1})]. && (9) \end{aligned}$$

((9) follows from the fact that $\prec^s$ is a relation on codes of sequences of natural numbers and $\bar{1}$ encodes the empty sequence.) Immediately from (9) we have

$$\forall x\, [x \prec^s \bar{1} \to x \in V) \to \mathrm{TI}(\prec^s, V). \tag{10}$$

Define

$$\begin{aligned} F(Z^+,U,a,\langle b,c\rangle) &:= \forall x\, (x \prec^b c \to y \in V),\\ A(V,b,c) &:= \mathrm{TI}(\prec^b, V) \vee \forall y(y \prec^b c \to y \in V),\\ C(b,c) &:= \forall Z\, [\mathrm{Cl}^F(Z,U,a) \to \langle b,c\rangle \in Z). \end{aligned}$$

We aim at showing

$$\forall Y A(Y,s,t) \leftrightarrow C(s,t). \tag{11}$$

"⇒": $\mathrm{Cl}^F(R,U,a)$ implies $\mathrm{PROG}(\prec^s, R_s)$, thus $\forall y(y \in R_s) \vee (t \in R_s)$ since $A(R_s,s,t)$ holds. Thus $\langle s,t\rangle \in R$.

"⇐": Given a set V, let $V^* := \{\langle z,u\rangle \mid A(V,z,u)\}$. Suppose that $F(V^*,U,a,\langle b,c\rangle)$. Then $\forall x[x \prec^b c \to A(V,b,x)]$, thus $\mathrm{TI}(\prec^b,V) \vee \forall xy\, [x \prec^b c \wedge y \prec^b x \to y \in V]$. Consequently, we have $\mathrm{TI}(\prec^b,V) \vee \neg\mathrm{PROG}(\prec^b,V) \vee \forall x\, [x \prec^b c \to x \in V]$, hence $A(V,b,c)$, thus $\langle b,c\rangle \in V^*$. As a result we have shown $\mathrm{Cl}^F(V^*,U,a)$. Thence the assumption $C(s,t)$ yields $\langle s,t\rangle \in V^*$, so that $A(V,s,t)$ holds.

We have thus shown (11). From (10) we can conclude

$$A(V,s,\bar{1}) \leftrightarrow \mathrm{TI}(\prec^s, V). \tag{12}$$

Combining (8), (11), and (12) we arrive at $\forall z\, [B(U,a,z) \leftrightarrow C(z,\bar{1})]$, as desired. □

Theorem 4.6 *(i)* Π^1_1-$\mathbf{TR}_0 = \mathbf{AUT\text{-}ID}_0^{pos} = \mathbf{AUT\text{-}ID}_0^{mon}$.

(ii) Π^1_1-$\mathbf{TR} = \mathbf{AUT\text{-}ID}^{pos} = \mathbf{AUT\text{-}ID}^{mon}$.

(iii) Π^1_1-$\mathbf{TR} + (\mathrm{BI}) = \mathbf{AUT\text{-}ID}_2^{pos} = \mathbf{AUT\text{-}ID}_2^{mon}$.

Proof. (i) implies (ii) and by Theorem 3.13 also (iii). For (i), in view of Lemma 4.3, it suffices to show Π^1_1-$\mathbf{TR}_0 \subseteq \mathbf{AUT\text{-}ID}_0^{pos}$. Let $B(U, a, b)$ be Π^1_1 and choose $F(\mathrm{P}^+, U, a, b)$ as in Lemma 4.5. Let

$$G(\mathrm{P}^+, U, a, b) := F(\mathrm{P}, \{x \mid \langle x, \bar{1}\rangle \in U\}, a, b).$$

On account of ($\mathrm{IT}^{pos}1$), for every well ordering R there exists a set V such that

$$\mathrm{IT}^G(R, V). \tag{13}$$

(13) implies

$$\forall Z\,[\mathrm{Cl}^F(Z, \{x \mid \langle x, \bar{1}\rangle \in V_{Ra}\}, a) \to V_a \subseteq Z]$$

and $\mathrm{Cl}^F(V_a, \{x \mid \langle x, \bar{1}\rangle \in V_{Ra}\}, a)$, which by choice of $F(\mathrm{P}, U, a, b)$ entails

$$\forall z\,[B(\{x \mid \langle x, \bar{1}\rangle \in V_{Ra}\}, a, z) \to \langle z, \bar{1}\rangle \in V_a]. \tag{14}$$

With $V^* := \{\langle i, z\rangle \mid \langle i, \langle z, \bar{1}\rangle\rangle \in V\}$, (14) implies $\forall iz\,[B(V^*_{Ri}, i, z) \leftrightarrow z \in V^*_i]$, and hence $\mathrm{HJ}^B(R, V^*)$. □

Corollary 4.7 *Owing to Corollary 4.4, the identities of Theorem 4.6 also hold for iterated* Π^1_1 *inductive definitions instead of iterated arithmetical inductive definitions.*

Remark 4.8 (i) The main purpose of this section was preparatory work for the interpretation of Π^1_1-$\mathbf{TR}_0$ into systems of set theory. The equivalences of Theorem 4.6 lend itself to rather transparent interpretations into theories of iterated admissibility.

(ii) Historically, reductions of subsystems of second order arithmetic played an important role in proof theory (cf. [10, 15, 19]). Theorem 4.6 can be viewed as a tribute to those times.

(iii) The idea of proof of Lemma 4.3 using the characterization of a fixed point I_Γ of an operator Γ by means of $I_\Gamma = \bigcap\{X \mid \Gamma(X) \subseteq X\}$ is of course the standard one.

5 Set theories of iterated admissibility

Theories of iterated inductive definitions have a canonical interpretation in set theories of iterated admissibility. The structure theory of Σ_+-inductive definitions on admissible sets (cf. [3, VI.2]) will be useful here.

Definition 5.1 (i) Let $\mathfrak{D}[\alpha, f]$ denote the conjunction of the formulae $\mathrm{Ord}(\alpha)$, $\mathrm{Fun}(f)$, $\mathrm{dom}(f) = \alpha$, and

$$(\forall\beta < \alpha)[\mathrm{Ad}(f(\beta)) \wedge (\forall\eta < \beta)\,(f(\eta) \in f(\beta)) \wedge (\forall x \in f(\beta))[\mathrm{Ad}(x) \to (\exists\eta < \beta)\,(x = f(\beta))]].$$

(ii) (AUT-Ad) $:= \forall\alpha\exists f\,\mathfrak{D}[\alpha, f]$.

(iii) $\mathbf{AUT\text{-}KPl} := \mathbf{KPl} + $ (AUT-Ad).

(iv) $(\mathrm{Ad}^*) := (\forall \alpha \in \mathrm{M})\, \exists f\, \mathfrak{D}[\alpha, f]$.

(v) $\mathbf{KPl}^* := \mathbf{KPl} + (\mathrm{Ad}^*)$.

AUT-KPl axiomatizes a set universe that has as many admissible sets as ordinals and in which the admissible sets are linearly ordered, whereas $\mathbf{KPl}^*$ only asserts that there are at least as many admissible sets as there are ordinals below ω_1^{CK}, i.e. ordinals in the least admissible set above the urelement structure of the natural numbers.

Lemma 5.2 *(i)* $\mathbf{KPl}^r \vdash \mathfrak{D}[\alpha, f] \wedge \mathfrak{D}[\alpha, g] \to f = g$.

(ii) $\mathbf{KPl}^* \vdash (\forall \alpha \in \mathrm{M})\, \exists! f\, \mathfrak{D}[\alpha, f]$.

Proof. (i): Suppose that $\mathfrak{D}[\alpha, f]$ and $\mathfrak{D}[\alpha, g]$. We show $\forall \beta < \alpha\, (f(\beta) = g(\beta)$ by induction on β, a principle justified by (Δ_0-FOUND). Let $\beta < \alpha$ and assume by induction hypothesis that $f \restriction \beta = g \restriction \beta$ (where $h \restriction \beta$ is the restriction of the function h to the domain β). Since $\mathrm{Ad}(f(\beta))$ and $\mathrm{Ad}(g(\beta))$ hold it follows from axiom (O7) that $f(\beta) \in g(\beta) \vee f(\beta) = g(\beta) \vee g(\beta) \in f(\beta)$. As $\mathrm{Ad}(f(\beta))$ and $\mathfrak{D}[\alpha, g]$ hold, $f(\beta) \in g(\beta)$ would entail the existence of an ordinal $\eta < \beta$ such that $f(\beta) = g(\eta) = f(\eta)$, contradicting $\mathfrak{D}[\alpha, f]$. Likewise one can rule out that $g(\beta) \in f(\beta)$. Hence $f(\beta) = g(\beta)$.

Finally, $f \restriction \alpha = g \restriction \alpha$ yields $f = g$.

(ii) is an immediate consequence of (i). □

Lemma 5.3 *Let* $\mathfrak{D}_0[a, \alpha, f]$ *be the conjunction of the following formulas:* $\mathrm{Ord}(\alpha)$, $\mathrm{Fun}(f)$, $\mathrm{dom}(f) = \alpha \cup \{0\}$, $\mathrm{Ad}(a)$, $f(0) = a$, *and*

$$(\forall \beta < \alpha)[\mathrm{Ad}(f(\beta)) \wedge (\forall \eta < \beta)\, (f(\eta) \in f(\beta)) \wedge (\forall x \in f(\beta))[\mathrm{Ad}(x) \wedge a \in x \to (\exists \eta < \beta)\, x = f(\beta)]].$$

We then have $\mathbf{AUT}\text{-}\mathbf{KPl}^r \vdash \mathrm{Ad}(a) \to \forall \alpha\, \exists! f\, \mathfrak{D}_0[a, \alpha, f]$.

Proof. Uniqueness of f can be proved as in Lemma 5.2. To prove existence suppose $\mathrm{Ad}(a)$. Invoking the axiom (Lim), there are admissible sets b and c such that $a, \alpha \in b$ and $b \in c$. Δ_0 separation relativized to c ensures the existence of $\rho := \{\eta \in b \mid \mathrm{Ord}(\eta)\}$ with $\rho \in c$. We also have $\rho \notin b$. Moreover, by (AUT-Ad) there exists a function g such that $\mathfrak{D}[\rho + \rho, g]$. The existence of the ordinal $\rho + \rho \in c$ can be established in the usual way since c is admissible. Using (Δ_0-FOUND) one easily shows that $(\forall \eta < \rho + \rho)\, \eta \in g(\eta + 1)$. Since $\rho \in g(\rho + 1)$ and $\rho \notin a$, the axiom (O7) ensures that $a \in g(\rho + 1)$. Thus there exists $\delta < \rho + 1$ such that $a = g(\delta)$. Also $\alpha < \rho$. The desired function f can be defined by $f(\eta) := g(\delta + \eta)$ for $\eta < \alpha$. One easily verifies that $\mathfrak{D}_0[a, \alpha, f]$. □

Definition 5.4

$$\begin{aligned}
\mathrm{Fld}(s, r) &:= \exists x\, [\langle x, s\rangle \in r \vee \langle s, x\rangle \in r]. \\
\mathrm{Lo}(a, r) &:= r \subseteq a \times a \wedge \text{“}r \text{ is a linear ordering” (cf. 3.1)}. \\
\mathrm{Wf}(a, r) &:= r \subseteq a \times a \wedge \forall x[x \neq \emptyset \wedge x \subseteq a \to (\exists y \in x)\, (\forall z \in x)\, \langle z, y\rangle \notin r]. \\
\mathrm{Wo}(a, r) &:= \mathrm{Lo}(a, r) \wedge \mathrm{Wf}(a, r).
\end{aligned}$$

Definition 5.5 (Axiom Beta) If r is a well-founded relation on a, i.e. $\mathrm{Wf}(a,r)$, then f is said to be a *collapsing for* r if $\mathrm{Collab}(a,r,f)$ holds, where

$$\mathrm{Collab}(a,r,f) \ := \ \mathrm{Fun}(f) \wedge \mathrm{dom}(f) = a \wedge (\forall x \in a)\,(f(x) = \{f(y) \mid \langle y,x\rangle \in r\}).$$

Axiom Beta (cf. [3, I.9.4]) is the assertion $\forall u \forall v\, \exists f\, [\mathrm{Wf}(u,v) \rightarrow \mathrm{Collab}(u,v,f)]$.

Theorem 5.6 $\mathbf{KPl}^r$ *proves axiom Beta. Inspection of the usual proof actually shows that* $\mathbf{KPl}^r$ *proves something stronger, namely that if* $\mathrm{Wf}(a,r)$, $\langle a,r\rangle \in b$, *and* $\mathrm{Ad}(b)$, *then the function which is collapsing for* r *is also an element of* b.

Proof. The proof is just a slight variation of the standard proof from $\mathbf{KP}$ + (Σ_1-separation) in [3, Theorem 9.6]: Just do the definition of the function F inside an admissible set $\mathbb{A}$ which contains the well-founded relation r. Then Σ_1 separation can be replaced by Δ_0 separation involving $\mathbb{A}$ as a parameter. For more details see [32, Theorem 4.6]. □

Theorem 5.7 *Every instance of* (BI) *is a theorem of* $\mathbf{KPl}$ *(via the translation* * *of Definition 2.5).*

Proof. By means of axiom Beta every well-ordering $\prec$ on $\mathbb{N}$ is order-isomorphic to an ordinal α. As a result, the schema of transfinite on $\prec$ is implied by (FOUND). For more details see [32, Lemma 7.1]. □

Lemma 5.8 (Iterated inductive definitions in AUT-KPl^r and KPl*.) *For* $A[\mathrm{P}^+,b,c,d,\vec{t}]$ *in* $\Delta_0(\mathrm{P}^+)$ *let*

$$\mathrm{Cl}^A_{\mathrm{N}}(a,b,c,\vec{t}) \ := \ \forall j \in \mathrm{N}(A[a,b,c,j,\vec{t}] \rightarrow j \in a)$$

and $\mathrm{IT}^A_{\mathrm{N}}(r,a,\vec{t})$ *be the formula*

$$a \subseteq \mathrm{N} \times \mathrm{N} \wedge (\forall i \in \mathrm{N})[\mathrm{Cl}^A_{\mathrm{N}}((a)_i,(a)_{ri},i,\vec{t}) \wedge \forall z \subseteq \mathrm{N}(\mathrm{Cl}^A_{\mathrm{N}}(z,(a)_{ri},i,\vec{t}) \rightarrow (a)_i \subseteq z)],$$

where $(a)_i := \{k \in \mathrm{N} \mid \langle i,k\rangle \in a\}$ *and* $(a)_{ri} := \{k \in \mathrm{N} \mid (\exists m \in \mathrm{N})(\langle m,i\rangle \in r \wedge \langle m,k\rangle \in a)\}$.

(i) $\mathbf{AUT\text{-}KPl}^r \vdash \mathrm{Wo}(\mathrm{N},r) \rightarrow \exists y\, \mathrm{IT}^A_{\mathrm{N}}(r,y,\vec{t})$.

(ii) $\mathbf{KPl}^* \vdash \mathrm{Wo}(\mathrm{N},r) \wedge r,t \in \mathrm{M} \rightarrow \exists y\, \mathrm{IT}^A_{\mathrm{N}}(r,y,\vec{t})$.

Proof. (ii): Suppose $\mathrm{Wo}(\mathrm{N},r)$ and $r,\vec{t} \in \mathrm{M}$. Let $S := \{k \in \mathrm{N} \mid \mathrm{Fld}(k,r)\}$. Then $S \in \mathrm{M}$ and $\mathrm{Wo}(S,r)$. By Theorem 5.6 there exists a function $h \in \mathrm{M}$ such that $\mathrm{Collab}(S,r,h)$. Then $\mathrm{rng}(h)$ is an ordinal $\alpha \in \mathrm{M}$, $h : S \rightarrow \alpha$ is bijective and, moreover, $(\forall ij \in S)(\langle i,j\rangle \in r \rightarrow h(i) < h(j))$. By ($\mathrm{Ad}^*$) there exists a function f such that $\mathfrak{D}[\alpha,f]$. In particular, $f(0) \in \mathrm{M}$. Using axiom (Lim) there exists an admissible set K such that $\alpha, f, \mathrm{M} \in K$. Let $K_\beta := f(\beta)$ for $\beta < \alpha$. Within the admissible K we simultaneously define a function g with $\mathrm{dom}(g) = \alpha$ and a sequence of functions $(f_\beta)_{\beta<\alpha}$ by Σ recursion as follows:

$$f_\beta(\xi) \ := \ \{k \in \mathrm{N} \mid A[\bigcup\{f_\beta(\gamma) \mid \gamma < \xi\}, \bigcup\{g(\delta) \mid \delta < \beta\}, h^{-1}(\beta), k, \vec{t}]\},$$
$$g(\beta) \ := \ \bigcup \mathrm{rng}(f_\beta).$$

For $i \in \mathrm{N} \setminus S$ let f_i be a function with $\mathrm{dom}(f_i) = \{\xi \mid \xi \in \mathrm{M}\}$ defined by Σ recursion in K via

$$f_i(\xi) \;:=\; \{k \in \mathrm{N} \mid A(\bigcup\{f_i(\gamma) \mid \gamma < \xi\}, \emptyset, i, k, \vec{t}]\}.$$

Also let

$$a \;:=\; \{\langle i,k\rangle \mid i \in S \,\wedge\, k \in g(h(i))\} \;\cup\; \{\langle i,k\rangle \mid i \in \mathrm{N} \setminus S \,\wedge\, k \in \mathrm{rng}(f_i)\}.$$

By construction, $f, g, (f_\beta)_{\beta<\alpha}$, and a are elements of K. To begin with we show that for $i \in \mathrm{N} \setminus S$,

$$\mathrm{Cl}^A_{\mathrm{N}}((a)_i, (a)_{ri}, i, \vec{t}), \tag{15}$$

$$\forall z \subseteq \mathrm{N}\, [\mathrm{Cl}^A_{\mathrm{N}}(z, (a)_{ri}, i, \vec{t}) \to (a)_i \subseteq z]. \tag{16}$$

Proof of (15): We have $(a)_{ri} = \emptyset$ since $i \in \mathrm{N} \setminus S$. As M is admissible, f_i is Σ definable in M. If $A[(a)_i, \emptyset, i, k, \vec{t}]$ holds for some $k \in \mathrm{N}$ then $j \in (a)_i \leftrightarrow \exists \xi \in \mathrm{M}\,(j \in f_i(\xi))$, and therefore, since $(a)_i$ occurs positively, utilizing Σ reflection in M we arrive at

$$\exists \delta \in \mathrm{M}\, A[\bigcup\{f_i(\xi) \mid \xi < \delta\}, \emptyset, i, k, \vec{t}],$$

thus $\exists \delta \in \mathrm{M}\,(k \in f_i(\delta))$, so $k \in (a)_i$. This verifies (15).

Proof of (16): let $z \subseteq \mathrm{N}$ and suppose $\mathrm{Cl}^A_{\mathrm{N}}(z, \emptyset, i, \vec{t})$. By transfinite induction on $\xi \in \mathrm{M}$ we show that $\forall \xi \in \mathrm{M}\,(f_i(\xi) \subseteq z)$, yielding (16). So suppose inductively that $\bigcup\{f_i(\gamma) \mid \gamma < \xi\} \subseteq z$. Then

$$f_i(\xi) \;=\; \{k \in \mathrm{N} \mid A[\bigcup\{f_i(\gamma) \mid \gamma < \xi\}, \emptyset, i, k, \vec{t}]\} \;\subseteq\; \{k \in \mathrm{N} \mid A[z, \emptyset, i, k, \vec{t}]\}.$$

As $\mathrm{Cl}^A_{\mathrm{N}}(z, \emptyset, i, \vec{t})$ holds, the latter implies $f_i(\xi) \subseteq z$.

Next we address the case when $i \in S$ and to this end show, by induction on $\beta < \alpha$, that

$$g \restriction \beta \in K_\beta. \tag{17}$$

Suppose that $g \restriction \delta \in K_\delta$ for all $\delta < \beta$. Then also $\forall \delta < \beta(g \restriction \delta \in K_\beta)$, and the sequence $(g \restriction \delta)_{\delta<\beta}$ is thus Σ definable in the admissible K_β. Note that since $\alpha \in \mathrm{M}$ and $\beta < \alpha$, we have $\beta \in K_\beta$. Using Σ replacement (cf. Theorem 2.23) inside K_β, we then have $(g \restriction \delta)_{\delta<\beta} \in K_\beta$. If β is a limit ordinal we have $g \restriction \beta = \bigcup\{g \restriction \delta \mid \delta < \beta\} \in K_\beta$. Suppose β is a successor $\rho + 1$. Then $g \restriction \rho$ and $\mathrm{dom}(f_\rho)$ are elements of K_β. Thus, by Σ recursion in K_β, f_ρ belongs to K_β, too. Therefore, $g \restriction \beta = g \restriction \rho \cup \{\langle \rho, \bigcup \mathrm{rng}(f_\rho)\rangle\} \in K_\beta$. As a result, transfinite induction on β establishes (17).

Now assume $i \in S$ and $i = h^{-1}(\beta)$. We have to show (15) and (16) for i. From (17) and the definition of a we get $(a)_{ri} \in K_\beta$. Also $(a)_i = \mathrm{rng}(f_\beta)$ and, for $\xi \in K_\beta$,

$$f_\beta(\xi) \;=\; \{k \in \mathrm{N} \mid A[\bigcup\{f_\beta(\gamma) \mid \gamma < \xi\}, (a)_{ri}, i, k, \vec{t}]\}.$$

So f_β is definable by Σ recursion in K_β. From $A[(a)_i, (a)_{ri}, i, k, \vec{t}]$ it follows, by Σ reflection in K_β, that

$$\exists \xi \in K_\beta\, A[\bigcup\{f_\beta(\gamma) \mid \gamma < \xi\}, (a)_{ri}, i, k, \vec{t}],$$

and hence $k \in (a)_i$, thereby showing (15) for $i \in S$. (16) can be shown for $i \in S$ in the same way as for $i \in \mathrm{N} \setminus S$.

(i) can be shown in the same way as (ii), except for a small change which consists in choosing an admissible set b such that $r, \vec{t} \in b$ and invoking Lemma 5.3 to ensure the existence of a function f with $\mathfrak{D}_0[b, \alpha, f]$. □

Theorem 5.9 *Via the translation* * *of definition 2.5 we have*

(i) Π^1_1-$\mathbf{TR}_0 \subseteq \mathbf{AUT}$-$\mathbf{KPl}^r$.

(ii) Π^1_1-$\mathbf{TR} \subseteq \mathbf{AUT}$-$\mathbf{KPl}^w$.

(iii) Π^1_1-$\mathbf{TR} + (\mathrm{BI}) \subseteq \mathbf{AUT}$-$\mathbf{KPl}$.

(iv) $\mathbf{ID}_{\prec^*} \stackrel{\hat{}}{\longrightarrow} \mathbf{ID}^* + (\mathrm{BI}) \subseteq \mathbf{KPl}^*$

where in (iv) the first the translation $\hat{}$ *stems from Definition 3.9.*

Proof. (i) follows from Theorem 4.6(i) and Lemma 5.8(i). (ii) is an immediate consequence of (i) as does (iii) if viewed in conjunction with Theorem 5.7. It reamins to show (iv). For $Q[a, b, c]$ arithmetical, we have $r_i := \{\langle j, k\rangle \mid j, k \in \mathrm{N} \wedge Q[i, j, k]^*\} \in \mathrm{M}$ for $i \in \mathrm{N}$. Therefore Lemma 5.8(ii) and Theorem 5.7 imply that $\mathbf{ID}^* + (\mathrm{BI}) \subseteq \mathbf{KPl}^*$. The first entailment via $\hat{}$ is a consequence of Theorem 3.10 and Theorem 3.13(i). □

Finally we would like to find a set-theoretic pendant to Π^1_1-$\mathbf{TR} + \Sigma^1_2$-$\mathbf{AC}$. We take this as an opportunity to introduce a few more traditional axiom schemata considered in second order arithmetic (cf. [17]).

Definition 5.10 Let $\mathcal{F}$ be a collection of formulae in $\mathcal{L}_2$.

$$\begin{array}{rcl}
(\mathcal{F}\text{-CA}) & := & \{\exists Z \forall x\, [x \in Z \leftrightarrow F(x)] \mid F(a) \in \mathcal{F}\}. \\
(\mathcal{F}\text{-AC}) & := & \{\forall x \exists Y\, H(x, Y) \rightarrow \exists Z \forall x\, H(x, Z_x) \mid H(a, U) \in \mathcal{F}\}. \\
(\mathcal{F}\text{-DC}) & := & \{\forall X \exists Y\, G(X, Y) \rightarrow \forall W \exists Z\, [W = Z_{\bar{0}} \wedge \forall x\, G(Z_x, Z_{x+1})] \mid G(U, V) \in \mathcal{F}\}. \\
(\Delta^1_2\text{-CA}) & := & \{\forall x\, [A(x) \leftrightarrow B(x)] \rightarrow \exists Z \forall x\, [x \in Z \leftrightarrow A(x)] \mid A(a) \in \Pi^1_2, B(a) \in \Sigma^1_2\}.
\end{array}$$

If (S) denotes any of the above schemata, then $\mathbf{S}$ stands for the theory $\mathbf{ACA} + (S)$.

The following well-known relationships can be found in [18, Theorem 2.3.1].

Theorem 5.11 *(i)* Σ^1_1-$\mathbf{AC}_0 \subseteq \Pi^1_1$-$\mathbf{CA}_0$.

(ii) Δ^1_2-$\mathbf{CA} = \Sigma^1_2$-$\mathbf{AC} = \Sigma^1_2$-$\mathbf{AC}$.

Theorem 5.12 Π^1_1-$\mathbf{TR} + \Sigma^1_2$-$\mathbf{AC} \subseteq \mathbf{KPi}^w + \mathbf{AUT}$-$\mathbf{KPl}^w$.

Proof. In view of Theorem 5.9(ii) it suffices to show Σ^1_2-$\mathbf{AC} \subseteq \mathbf{KPi}^w$. But this inclusion is a consequence of Theorem 5.11 and Theorem 7.2, a result we shall show later. □

Remark 5.13 (i) Theorem 5.9 crucially uses Lemma 5.8 which is essentially a generalization of Gandy's Theorem (cf. [3, VI.2.6]) to the iterated scenario.

(ii) Theories of iterated admissibility were also considered by Jäger in [30]. However, in the theories in [30] iterated admissibility is couched in terms of inference rules and they come also equipped with an extended Bar rule. As a result, they are different from the theories considered here. There are several conjectures about the proof-theoretic strength of such theories stated in [30]. These conjectures turn out to be true as they are corollaries of results in this paper. Details will be spelled out at the appropriate places.

6 Theories of iterated choices and Δ^1_2 comprehension

Let $\mathbf{N}$ be the standard structure of the natural numbers with language $\mathcal{L}_1$. Every level $\mathrm{L}(\alpha)_{\mathbf{N}}$ of the constructible hierarchy above $\mathbf{N}$ (for a precise definition see [3, II]) can be viewed as a structure of the language $\mathcal{L}^*$ wherein the predicate symbol Ad is interpreted by the class $\{\mathrm{L}(\beta)_{\mathbf{N}} \mid \beta < \alpha \text{ and } \mathrm{L}(\beta)_{\mathbf{N}} \text{ is admissible}\}$.

If $\mathbf{T}$ is a theory with language $\mathcal{L}^*$, then the structures $\mathrm{L}(\alpha)_{\mathbf{N}}$ satisfying and $\mathrm{L}(\alpha)_{\mathbf{N}} \models \mathbf{T}$ are said to be the *standard models* of $\mathbf{T}$.

The smallest standard model of the theories $\mathbf{AUT\text{-}KPl}^r$, $\mathbf{AUT\text{-}KPl}^w$ and $\mathbf{AUT\text{-}KPl}$ is $\mathrm{L}(\mathrm{g}_1(0))_{\mathbf{N}}$ where the mapping $\xi \mapsto \mathrm{g}_0(\xi)$ enumerates the admissible ordinals $\geq \omega_1^{CK}$ and their limits, and (recursively) for $\alpha > 0$, $\xi \mapsto \mathrm{g}_\alpha(\xi)$ enumerates the common fixed points of all the functions g_β with $\beta < \alpha$.

All further $\mathcal{L}_2$-theories to be introduced in this section and in section 8 will comprise $\Delta^1_2\text{-}\mathbf{CA}_0$ and will turn out to be subtheories of $\Delta^1_2\text{-}\mathbf{CA}+(\mathrm{BI})$. On the set-theoretic side they correspond to theories in strength between $\mathbf{KPi}^r$ and $\mathbf{KPi}$. The difference in proof-theoretic strength between the latter two theories is enormous, albeit both theories have the same minimal standard model $\mathrm{L}(\iota_0)_{\mathbf{N}}$ with ι_0 being the least recursively inaccessible ordinal. As a result, the minimal standard model is hardly indicative of the proof-theoretic strength of these theories. A better measure is provided by the minimal Π_2-model.

Definition 6.1 $\mathrm{L}(\alpha)_{\mathbf{N}}$ is a Π_2-*model* of a set theory $\mathbf{T}$, whenever

$$\mathbf{T} \vdash F \;\;\Rightarrow\;\; \mathrm{L}(\alpha)_{\mathbf{N}} \models F \tag{18}$$

holds for all set-theoretic Π_2 sentences.

(The notion of a Π_2-model appears to have been introduced in [32].)

As far as the theories $\mathbf{AUT\text{-}KPl}^r$, $\mathbf{AUT\text{-}KPl}^w$ and $\mathbf{AUT\text{-}KPl}$ are concerned, $\mathrm{L}(\mathrm{g}_1(0))_{\mathbf{N}}$ is also their minimal Π_2-model. The theories $\mathbf{T}$ with $\mathbf{KPi}^r \subseteq \mathbf{T} \subseteq \mathbf{KPi}$ we are going to study next, though, will have their minimal Π_2-model $\mathrm{L}(\alpha)_{\mathbf{N}}$ at an ordinal $\alpha \leq \Gamma^{\mathrm{g}}_0 := \min\{\rho \mid \mathrm{g}_\rho(0) = \rho\}$. The main cause for the widely diverging Π_2-models of such theories is to be found in the amount of induction principles they incorporate. In conjunction with stronger induction principles, the pivotal principle of Σ collection gives rise to recursion principles which allow one to prove the existence of ever larger admissible sets.

To analyze the gap between $\Pi^1_1\text{-}\mathbf{TR}$ and $\Delta^1_2\text{-}\mathbf{CA} + (\mathrm{BI})$ we consider iterations of principles stronger than Π^1_1-comprehension.

Definition 6.2 (i) Δ^1_2-**TR** is the theory **ACA** augmented by the schema of transfinite Δ^1_2 recursion, (Δ^1_2-TR),

$$\forall R[\mathrm{WO}(R) \wedge \forall X \forall iy\, [B(X,i,y) \leftrightarrow A(X,i,y)] \rightarrow \exists Z \forall iy\, [y \in Z_i \leftrightarrow B(Z_{Ri}, i, y)]]$$

with $B(U,a,b) \in \Pi^1_2$ and $A(U,a,b) \in \Sigma^1_2$.

(ii) Σ^1_2-**TRDC** (Π^1_1-**TRDC**, respectively) is the theory **ACA** augmented by the schema of transfinitely iterated Σ^1_2 (Π^1_1) dependent choices, (Σ^1_2-TRDC) (Π^1_1-TRDC, respectively),

$$\forall R[\mathrm{WO}(R) \wedge \forall i \forall X \exists Y\, C(X,Y,i) \rightarrow \exists Z \forall i\, C(Z_{Ri}, Z_i, i)]$$

where $C(U,V,a) \in \Sigma^1_2$ ($C(U,V,a) \in \Pi^1_1$, respectively).

As it turns out, Π^1_1 dependent choices are as strong as Σ^1_2 dependent choices.

Lemma 6.3 Π^1_1-$\mathbf{TRDC}_0$ = Σ^1_2-$\mathbf{TRDC}_0$.

Proof. we have to show "$\supseteq$". Let $C(U,V,a)$ be a formula $\exists W\, A(U,V,W,a)$ with $A(U,V,S,a)$ Π^1_1. Suppose that $\mathrm{WO}(R)$ and $\forall i \forall X \exists Y\, C(X,Y,i)$. Then also

$$\forall i \forall X \exists Y\, A(\{z \mid \langle z, \bar{0}\rangle \in X\}, \{z \mid \langle z, \bar{0}\rangle \in Y\}, \{z \mid \langle z, \bar{1}\rangle \in Y\}, i)$$

and by (Π^1_1-TRDC) there exists V such that

$$\forall i\, A(\{z \mid \langle z, \bar{0}\rangle \in V_{Ri}\}, \{z \mid \langle z, \bar{0}\rangle \in V_i\}, \{z \mid \langle z, \bar{1}\rangle \in V_i\}, i), \tag{19}$$

and hence

$$\forall i\, C(\{z \mid \langle z, \bar{0}\rangle \in V_{Ri}\}, \{z \mid \langle z, \bar{0}\rangle \in V_i\}, i). \tag{20}$$

Letting $V^* : \{\langle i, z\rangle \mid \langle i, \langle z, \bar{0}\rangle\rangle \in V\}$, (20) implies $\forall i\, C(V^*_{Ri}, V^*_i, i)$. □

Lemma 6.4 Δ^1_2-$\mathbf{CA}_0 \subseteq \Sigma^1_2$-$\mathbf{TRDC}_0$.

Let $F(a), \neg G(a) \in \Sigma^1_2$. Suppose that $\forall x\, [G(x) \leftrightarrow F(x)]$. Then

$$\forall x \forall X \exists Y\, [(F(x) \wedge Y = \{\bar{0}\}) \vee (\neg G(x) \wedge Y = \{\bar{1}\})]. \tag{21}$$

Applying (Σ^1_2-TRDC) to (21) and the well ordering $\emptyset$, there exists a set V such that

$$\forall x\, [(F(x) \wedge V_x = \{\bar{0}\}) \vee (\neg G(x) \wedge V_x = \{\bar{1}\})].$$

With $V' := \{x \mid V_x = \{\bar{0}\}\}$ we obtain the desired $\forall x\, [x \in V' \leftrightarrow F(x)]$. □

Lemma 6.5 Δ^1_2-$\mathbf{TR}_0 \subseteq \Sigma^1_2$-$\mathbf{TRDC}_0$.

Proof. Let $B(U,a,b), \neg A(U,a,b) \in \Sigma^1_2$. Moreover, suppose that $\mathrm{WO}(R)$ and

$$\forall X \forall iy\, [B(X,i,y) \leftrightarrow A(X,i,y)]. \tag{22}$$

By Lemma 6.4, (22) imples

$$\forall i \forall X \exists Y \forall y\, [B(X,i,y) \leftrightarrow y \in Y]. \tag{23}$$

The formula $\forall y\, [B(X,i,y) \leftrightarrow y \in Y]$, in view of (22) and the fact that $(\Sigma^1_2\text{-AC}) \subseteq \Sigma^1_2\text{-}\mathbf{TRDC}_0$, is equivalent to a Σ^1_2 formula. Hence, with $(\Sigma^1_2\text{-TRDC})$, from (23) we obtain

$$\exists Z \forall iy\, [B(Z_{Ri}, i, y) \leftrightarrow y \in Z_i].$$

□

To facilitate the interpretation of $\Sigma^1_2\text{-}\mathbf{TRDC}_0$ into set theory without choice, we first reduce this theory to a theory $\Pi^1_1\text{-}\mathbf{TRK}_0$.

Definition 6.6 Π^1_1-**TRK** is the theory $\Delta^1_2\text{-}\mathbf{CA} + (\Pi^1_1\text{-TRK})$ where

$$(\Pi^1_1\text{-TRK}) \qquad \forall R\, [\mathrm{WO}(R) \wedge \forall i \forall X \exists! Y\, D(X,Y,i) \rightarrow \exists Z \forall i\, D(Z_{Ri}, Z_i, i)]$$

with $D(U,V,a) \in \Pi^1_1$.

Lemma 6.7 $\Pi^1_1\text{-}\mathbf{TRK}_0 = \Pi^1_1\text{-}\mathbf{TRDC}_0$.

For the proof of Lemma 6.7 we need to show that a certain result from descriptive set theory is provable in $\Delta^1_2\text{-}\mathbf{CA}_0$.

Lemma 6.8 (Π^1_1 uniformization) *For every Π^1_1 formula $A[\vec{S}, V, \vec{a}\,]$ there exists a Π^1_1 formula $H[\vec{S}, V, \vec{a}\,]$ such that provably in $\Pi^1_1\text{-}\mathbf{CA}_0$ we have*

(i) $\forall Y\, (H[\vec{S}, V, \vec{a}\,] \rightarrow A[\vec{S}, V, \vec{a}\,])$;

(ii) $\exists Y\, A[\vec{S}, Y, \vec{a}\,] \rightarrow \exists! Y\, H[\vec{S}, Y, \vec{a}\,]$.

Proof. [65, Lemma VI.2.1] □

Proof of Lemma 6.7: "$\subseteq$" follows from Lemma 6.3 and 6.4. For "$\supseteq$" let $A[\vec{S}, U, V, b, \vec{a}\,] \in \Pi^1_1$ and assume

$$\mathrm{WO}(R) \wedge \forall i \forall X \exists Y\, A[\vec{S}, X, Y, i, \vec{a}\,]. \tag{24}$$

By Lemma 6.8 there is a Π^1_1 formula $H[\vec{S}, U, V, b, \vec{a}\,]$ such that

$$\forall i \forall X \forall Y\, (H[\vec{S}, X, Y, i, \vec{a}\,] \rightarrow A[\vec{S}, X, Y, i, \vec{a}\,]) \tag{25}$$

$$\forall i \forall X \exists! Y\, H[\vec{S}, X, Y, i, \vec{a}\,]. \tag{26}$$

With the aid of $(\Pi^1_1\text{-TRK})$, (26) yields

$$\exists Z \forall i\, H[\vec{S}, Z_{Ri}, Z_i, i, \vec{a}\,]. \tag{27}$$

(25) and (27) imply $\exists Z \forall i\, A[\vec{S}, Z_{Ri}, Z_i, i, \vec{a}\,]$. □

Lemma 6.9 Π^1_1-$\mathbf{TRK}_0 \subseteq \Delta^1_2$-$\mathbf{TR}_0$.

Proof. Assume $\mathrm{WO}(R) \wedge \forall i \forall X \exists ! Y\, C(X, Y, i)$ for some Π^1_1 formula $C(U, V, a)$. Let $B(U, a, b) := \exists Y\, [C(U, Y, a) \wedge b \in Y]$ and $A(U, a, b) := \forall Y\, [C(U, Y, a) \to b \in Y]$. By assumption,

$$\forall X \forall i \forall y\, [B(X, i, y) \leftrightarrow A(X, i, y)]. \tag{28}$$

Using (Δ^1_2-TR), (28) yields the existence of a set S such that for all i,

$$S_i = \{y \mid \exists Y\, [C(S_{Ri}, Y, i) \wedge y \in Y]\}.$$

As $\forall i \exists ! Y\, C(S_{Ri}, Y, i)$ it follows that $\forall i\, C(S_{Ri}, S_i, i)$. □

By Lemmata 6.3, 6.5, 6.7, and 6.9 we have the following:

Theorem 6.10 Δ^1_2-$\mathbf{TR}_0 = \Sigma^1_2$-$\mathbf{TRDC}_0 = \Pi^1_1$-$\mathbf{TRDC}_0 = \Pi^1_1$-$\mathbf{TRK}_0$.

Lemma 6.8 also yields the following.

Theorem 6.11 Σ^1_2-$\mathbf{AC}_0 = \Delta^1_2$-$\mathbf{CA}_0$.

Another natural route to approach Δ^1_2-$\mathbf{CA}$ + (BI) from below is to consider restrictions of the bar induction schema (BI).

Definition 6.12 If $\mathcal{F}$ is a collection of $\mathcal{L}_2$-formulas, we let

$$(\mathcal{F}\text{-BI}) \quad := \quad \{\forall X\, [\mathrm{WO}(X) \to \mathrm{TI}(X, F)] \mid F(a) \in \mathcal{F}\}.$$

It worthwhile noting that (Π^1_2-BI) is already deducible in Δ^1_2-$\mathbf{CA}$.

Theorem 6.13 (Π^1_2-BI) $\subseteq \Sigma^1_2$-$\mathbf{DC}_0 \subseteq \Delta^1_2$-$\mathbf{CA}$.

Proof. The second inclusion follows from Theorem 5.11(ii). To show the first inclusion we argue in Σ^1_2-$\mathbf{DC}_0$. Suppose we have a counter-example to (Π^1_2-BI). Then there is a formula $H(a) = \forall X\, A(X, a)$ with $A(X, a) \in \Sigma^1_1$, and there exists a well-ordering $\prec$ and a number k such that

$$\mathrm{PROG}(\prec, H) \wedge \neg H(k). \tag{29}$$

Let variables $f, g, h, \ldots$ range over functions from N^{N}, where we identify f with the set $\{\langle n, f(n)\rangle \mid n \in \mathrm{N}\}$.

Since $\neg H(k)$ holds, there exists a set S such that $\neg A(S, k)$. Let f' be defined by $f'(0) = k$ and

$$f'(x+1) = \begin{cases} 0 \text{ if } x \in S \\ 1 \text{ if } x \notin S\,. \end{cases}$$

Letting $N(h) := \{x \mid h(x+1) = 0\}$ for $h \in \mathrm{N}^{\mathrm{N}}$, we have

$$f'(0) = k \quad \wedge \quad \neg A(N(f'), k). \tag{30}$$

Since PROG$(\prec, h)$ we get $\forall i\,[\exists X\,\neg A(X,i) \to \exists Y \exists j(j \prec i \,\wedge\, \neg A(Y,j))]$, whence

$$\forall f \exists g\,[\neg A(N(f), f(0)) \to g(0) \prec f(0) \,\wedge\, \neg A(N(g), g(0))]. \tag{31}$$

Applying (Σ^1_2-DC) to (31), we obtain a function h such that

$$h_0 = f' \,\wedge\, \forall i\,[\neg A(N(h_i), h_i(0)) \to h_{i+1}(0) \prec h_i(0) \,\wedge\, \neg A(N(h_{i+1}), h_{i+1}(0))], \tag{32}$$

where h_i denotes the function $x \mapsto h(\langle i, x\rangle)$.

Using induction (for a Π^1_1 formula), (30) and (32) imply

$$\forall i\,[\neg A(N(h_i), h_i(0)) \,\wedge\, h_{i+1}(0) \prec h_i(i)],$$

violating the assumption that $\prec$ is a well-ordering. □

The dual formula class, though, provides a strengthening.

Theorem 6.14 Σ^1_2-$\mathbf{TRDC}_0 \subseteq \Delta^1_2$-$\mathbf{CA}_0 + (\Sigma^1_2$-BI$)$.

Proof. According to Theorem 6.10 it suffices to show (Π^1_1-TRK) $\subseteq \Delta^1_2$-$\mathbf{CA}_0 + (\Sigma^1_2$-BI$)$. So suppose we have a Π^1_1 formula $C(U, V, a)$ such that

$$\mathrm{WO}(R) \,\wedge\, \forall i \forall X \exists! Y\, C(X, Y, i). \tag{33}$$

Let $F(U, a) := \forall j\,[(jRa \,\vee\, j \equiv a) \to C(U_{Rj}, U_j, j)]$. As WO$(R)$ we get

$$F(U, a) \,\wedge\, F(V, a) \to \forall j\,[(jRa \,\vee\, j \equiv a) \to U_j = V_j]. \tag{34}$$

We show

$$\forall i \exists Z\, F(Z, i) \tag{35}$$

by induction on R. From $\forall x\,[xRi \to \exists Z\, F(Z, x)]$ it follows by (34) and with the help of (Δ^1_2-CA) that there exists a set S such that

$$S \;=\; \{\langle x, y\rangle \mid xRi \,\wedge\, \exists Z\,(F(Z, x) \,\wedge\, y \in Z_x)\} = \{\langle x, y\rangle \mid xRi \,\wedge\, \forall Z\,[F(Z, x) \to y \in Z_x)\}.$$

Moreover, owing to (33), there exists a set V such that $C(S_{Ri}, V, i)$.
Letting $S^* := S \cup \{\langle i, y\rangle \mid y \in V\}$ we have $F(S^*, i)$. Thus (35) follows by (Σ^1_2-BI).

In view of (34) and (35), we can apply (Δ^1_2-CA) to show that

$$U := \{\langle i, y\rangle \mid \exists Z\,[F(Z, i) \,\wedge\, y \in Z_i)\}$$

is set. Moreover, by (34) and (35), we also have $\forall i\, C(U_{Ri}, U_i, i)$, showing ($\Pi^1_1$-TRK). □

Corollary 6.15 (Π^1_2-BI) $\subseteq \Delta^1_2$-$\mathbf{CA}_0 + (\Sigma^1_2$-BI$)$.

Proof. This follows from Theorems 6.13 and 6.14. □

Remark 6.16 We shall later see that Σ^1_2-$\mathbf{TRDC}_0$ and Δ^1_2-$\mathbf{CA}_0 + (\Sigma^1_2$-BI$)$ have the same prooftheoretic ordinal. Using standard arguments this imples that both theories prove the same Π^1_1 sentences. Indeed, this result can be improved. Both theories prove the same Π^1_3 sentences, but this stronger conservativity result cannot be simply gleaned from the proof-theoretic ordinal. One has to scrutinize the whole series of reductions to arrive at it.

7 Set theories with recursion schemata

As in the case of $\mathcal{L}_2$-theories of iterated Π^1_1-comprehension, one can also single out a set-theoretic counterpart to (Σ^1_2-TRDC). Ignoring the latter's choice aspects, Σ-recursion lends itself as a pendant to (Σ^1_2-TRDC). In order to interpret Σ^1_2-$\mathbf{TRDC}_0$ in $\mathbf{KPi}^r$ + (Σ-REC), we need a "quantifier theorem" which reduces Σ^1_2 formulae of $\mathcal{L}_2$ to set-theoretic Σ_1 formulas, thereby reducing the number of unbounded set quantifiers by one. In the case of **ZF** this is a standard result. (cf. [12, CH.5,7.14]).

Theorem 7.1 *To any Σ^1_2 $\mathcal{L}_2$-formula $B[\vec{a},\vec{U}\,]$ one can assign a Σ_1 formula $B_\sigma[\vec{a},\vec{b}\,]$ of $\mathcal{L}^*$ such that*

$$\mathbf{KPl}^r \vdash \vec{a} \in \mathrm{N} \wedge b \subseteq \mathrm{N} \rightarrow (B[\vec{a},\vec{b}\,]^* \leftrightarrow B_\sigma[\vec{a},\vec{b}\,]).$$

Proof. The crucial step in the well known proof (usually carried out in **ZF**) consists in realizing that via the Π^1_1 normal form (the equivalence (8) in the proof of Lemma 4.5), every Π^1_1 formula is equivalent to a Σ_1 formula exploiting axiom Beta, and consequently every Π^1_1 formula is Δ_1. Since axiom Beta is provable in $\mathbf{KPl}^r$ by Theorem 5.6, the desired result follows. For more details see [32, Theorem 7.1]. □

Theorem 7.2 Δ^1_2-$\mathbf{CA}_0 \subseteq \mathbf{KPi}^r$.

Immediate by the latter Theorem, using Δ separation (Theorem 2.22) in $\mathbf{KPi}^r$. □

Lemma 7.3 (Embedding Lemma for Π^1_1-$\mathbf{TRK}_0$) Π^1_1-$\mathbf{TRK}_0 \subseteq \mathbf{KPi}^r$ + (Σ-REC).

Proof. By Theorem 7.1 it suffices to show (Π^1_1-TRK) $\subseteq \mathbf{KPi}^r$ + (Σ-REC). Let $A[a, U, V, \vec{d}, \vec{S}\,] \in \Sigma^1_2$. Let $j_1, \ldots, j_k \in \mathrm{N}$, $s_1, \ldots, s_n \subseteq \mathrm{N}$ and, letting $B(a, b, c) := (A[a, b, c, \vec{j}, \vec{s}\,])^*$, assume that

$$\mathrm{Wo}(r, \mathrm{N}) \quad \wedge \quad \forall i \in \mathrm{N}\, \forall x \subseteq \mathrm{N}\, \exists! y\, [y \subseteq \mathrm{N} \wedge B(i, x, y)]. \tag{36}$$

We have to show that

$$\exists z \subseteq \mathrm{N} \times \mathrm{N}\, (\forall i \in \mathrm{N})\, B(i, (z)_{ri}, (z)_i) \tag{37}$$

(with $(z)_{ri}$ and $(z)_i$ being defined as in Lemma 5.8). By Theorem 7.1 there exists a Σ_1 formula $B_\sigma(a, b, c)$ such that

$$\forall i \in \mathrm{N}\, \forall x \subseteq \mathrm{N}\, \forall y \subseteq \mathrm{N}\, [B(i, x, y) \leftrightarrow B_\sigma(i, x, y)]. \tag{38}$$

Letting $S := \{i \in \mathrm{N} \mid \mathrm{Fld}(i, r)\}$, by Theorem 5.6 there exists a function h such that h is collapsing for r, i.e. $\mathrm{Collab}(S, r, h)$. Whence h is a bijection from S onto $\alpha_h := \mathrm{rng}(h)$ satisfying $\forall ij\, [\langle i, j\rangle \in r \rightarrow h(i) < h(j)]$. Let

$$\begin{aligned} F(\beta, a, b) &:= F_0(\beta, a, b) \vee F_1(\beta, a, b), \\ F_0(\beta, a, b) &:= (\alpha_h \leq \beta \vee \bigcup \mathrm{rng}(a) \not\subseteq \mathrm{N}) \wedge b = \emptyset, \\ F_1(\beta, a, b) &:= \beta < \alpha_h \wedge \bigcup \mathrm{rng}(a) \subseteq \mathrm{N} \wedge b \subseteq \mathrm{N} \wedge B_\sigma(h^{-1}(\beta), \bigcup \mathrm{rng}(a), b). \end{aligned}$$

In $\mathbf{KPl}^r$, $F(\beta, a, b)$ is equivalent to a Σ formula. In view of (36) and (38) we have $\forall\beta\forall x\exists! y\, F(\beta, x, y)$, whence, using ($\Sigma$-REC), there exists a function f, such that

$$\mathrm{dom}(f) = \alpha_h \quad \wedge \quad (\forall\beta < \alpha_h)\, F(\beta, f \upharpoonright \beta, f(\beta))). \tag{39}$$

From (36) we get $(\forall\beta < \alpha_h)\, [\bigcup \mathrm{rng}(f \upharpoonright \beta) \subseteq \mathrm{N} \wedge f(\beta) \subseteq \mathrm{N}$ by (Δ_0-FOUND). Whence from (39) we can conclude that for all $i \in S$,

$$\bigcup \mathrm{rng}(f \upharpoonright h(i)) \subseteq \mathrm{N} \wedge f(h(i)) \subseteq \mathrm{N} \wedge B_\sigma(i, \bigcup \mathrm{rng}(f \upharpoonright h(i)), f(h(i))). \tag{40}$$

With $X := \{\langle i, j\rangle \mid i \in S \wedge j \in f(h(i))\}$, (38) and (40) yield

$$(\forall i \in S)\, B(i, (X)_{ri}, (X)_i) \wedge X \subseteq \mathrm{N} \times \mathrm{N}. \tag{41}$$

Moreover, (36) implies $(\forall i \in \mathrm{N}\backslash S)\, \exists! y\, [y \subseteq \mathrm{N} \wedge B(i, \emptyset, y)]$, so that with the help of Σ replacement there exists a function g satisfying

$$\mathrm{dom}(g) = \mathrm{N} \setminus S \wedge (\forall i \in \mathrm{N} \setminus S)\, [g(i) \subseteq \mathrm{N} \wedge B(i, \emptyset, g(i))]. \tag{42}$$

Letting $Y := X \cup \{\langle i, j\rangle \mid i \in \mathrm{N} \setminus S \wedge j \in g(i)\}$, (41) and (42) entail that

$$Y \subseteq \mathrm{N} \times \mathrm{N} \wedge (\forall i \in \mathrm{N})\, B(i, (Y)_{ri}, (Y)_i),$$

confirming (37). □

From Theorem 6.10 and Lemma 7.3 we get the following.

Theorem 7.4 *Σ^1_2-$\mathbf{TRDC}_0$ = Δ^1_2-$\mathbf{TR}_0 \subseteq \mathbf{KPi}^r$ + (Σ-REC).*

The theories Σ^1_2-$\mathbf{TRDC}_0$ and Σ^1_2-$\mathbf{TRDC}$ will later be interpreted in a semi-formal system of ramified set theory. This, however, will only provide a partial interpretation for Σ_1 formulae with free variables. To bring about this interpretation it is technically advisable to reduce (Σ-REC) to a simpler schema of Ad-valued recursion on ordinals.

Definition 7.5 For $F(a, b, c)$ a Δ_0 formula, we denote by $\mathcal{C}^F(\alpha, f)$ the formula

$$\mathrm{Ord}(\alpha) \wedge \mathrm{Fun}(f) \wedge \mathrm{dom}(f) = \alpha \wedge$$
$$(\forall\beta < \alpha)\, [\mathrm{Ad}(f(\beta)) \wedge F(\beta, f \upharpoonright \beta, f(\beta)) \wedge (\forall x \in f(\beta))(\mathrm{Ad}(x) \rightarrow \neg F(\beta, f \upharpoonright \beta, x))].$$

Note that $\mathcal{C}^F(\alpha, f)$ is also Δ_0. By (Ad-REC) we denote the schema

$$\forall\beta\forall x\exists y[\mathrm{Ad}(y) \wedge F(\beta, x, y)] \rightarrow \forall\alpha\exists f\, \mathcal{C}^F(\alpha, f)$$

where $F(a, b, c)$ is Δ_0.

As in Lemma 5.2 one proves

Lemma 7.6 *$\mathbf{KPl}^r \vdash \mathcal{C}^F(\alpha, g) \wedge \mathcal{C}^F(\alpha, g) \rightarrow f = g$.*

The "trick" of replacing the premiss $\forall\beta\forall x\exists! y\, F(\beta, x, y)$ by $\forall\beta\forall x\exists y[\mathrm{Ad}(y) \wedge F(\beta, x, y)]$ allows one to relinquish one's hold on the uniqueness requirement for y since admissible sets are well-ordered on the basis of $\mathbf{KPl}^r$.

Theorem 7.7 *$\mathbf{KPi}^r$ + (Ad-REC) = $\mathbf{KPi}^r$ + (Σ-REC).*

Proof. For a proof see [43, Satz 5.7]. A proof will also be supplied in the sequel to this paper. □

8 Systems with Bar rules and other induction principles

An alternative to restricting the schema (BI) to specific syntactic complexity classes of formulae (as in ($\mathcal{F}$-BI)) consists in directing the attention to the well-ordering over which transfinite induction is allowed in that one requires them to be provably well-ordered.

Definition 8.1 (i) The *Bar rule*, BR, is the rule of inference

$$\frac{\mathrm{WO}(\prec)}{\mathrm{TI}(\prec, F)}$$

with $\prec$ being a primitive recursive relation and $F(a)$ any formula of $\mathcal{L}_2$.

(ii) BR(impl-Σ^1_2) is the rule

$$\frac{\exists! X\,(\mathrm{WO}(X)\,\wedge\, G[X])}{\forall X\,(\mathrm{WO}(X)\,\wedge\, G[X] \rightarrow \mathrm{TI}(X, H))}$$

where $G[U]$ is a Σ^1_2 formula (without additional parameters) and $H(a)$ is an arbitrary $\mathcal{L}_2$ formula.

(iii) BI(impl-Σ^1_2) denotes the schema

$$\exists! X\,(\mathrm{WO}(X)\,\wedge\, G[X]) \rightarrow \forall X\,(\mathrm{WO}(X)\,\wedge\, G[X] \rightarrow \mathrm{TI}(X, H))$$

where $G[U]$ is a Σ^1_2 formula (without additional parameters) and $H(a)$ is an arbitrary $\mathcal{L}_2$ formula.

The Quantifier Theorem 7.1 and Axiom Beta suggest set-theoretic equivalences to the foregoing induction principles.

Definition 8.2 (i) FOUNDR(impl-Σ(M)) is the rule of inference

$$\frac{\exists! x\,(x \in \mathrm{M}\,\wedge\, F[x]^{\mathrm{M}})}{\forall x[x \in \mathrm{M}\,\wedge\, F[x]^{\mathrm{M}}\,\wedge\, \forall y(\forall z \in y\, H(z) \rightarrow H(y)) \rightarrow (\forall y \in x)\, H(y)]}$$

with $F[a]$ a Σ formula and $H(a)$ any formula of $\mathcal{L}^*$.

(ii) FOUNDR(impl-Σ) is the rule of inference

$$\frac{\exists! x\, F[x]}{\forall x[F[x]\,\wedge\, \forall y(\forall z \in y\, H(z) \rightarrow H(y)) \rightarrow (\forall y \in x)\, H(y)]}$$

with $F[a]$ a Σ formula and $H(a)$ any formula of $\mathcal{L}^*$.

(iii) FOUND(impl-Σ) denotes the schema

$$\exists! x\, F[x] \rightarrow \forall x[F[x]\,\wedge\, \forall y(\forall z \in y\, H(z) \rightarrow H(y)) \rightarrow (\forall y \in x)\, H(y)]$$

where $F[a]$ is a Σ formula and $H(a)$ is any formula of $\mathcal{L}^*$.

Remark 8.3 The rule BR(impl-Σ^1_2) is, on the basis of Δ^1_2-**CA**, much stronger than the rule BR whereas BR(impl-Σ^1_2) is still much weaker than (BI). The difference in strength between (BI) and BR(impl-Σ^1_2) is of course owed to the fact that the first is a rule while the second is a schema. But one can say something more illuminative about it. As it turns out, BR(impl-Σ^1_2) and BI(impl-Σ^1_2) are of the same strength (on the basis of Δ^1_2-**CA**), in actuality the theories Δ^1_2-**CA**+BR(impl-Σ^1_2) and Δ^1_2-**CA**+BI(impl-Σ^1_2) prove the same Π^1_1 statements. Thus the main difference between BR(impl-Σ^1_2) and (BI) is to be found in the premiss of BI(impl-Σ^1_2) requiring the well-ordering to be describable via a Σ^1_2 formula without parameters. Analogous remarks apply to the corresponding set-theoretic principles. The theme is explored in more detail in [46].

The next lemma relates (in a weak sense) the $\mathcal{L}_2$ versions of Definition 8.1 to their set-theoretic counterparts.

Lemma 8.4 *(i)* Δ^1_2-**CA** + BR(impl-Σ^1_2) $\subseteq$ $\mathbf{KPl}^w$ + FOUNDR(impl-Σ) = $\mathbf{KPi}^r$ + FOUNDR(impl-Σ).

(ii) Σ^1_2-**TRDC** + BR $\subseteq$ $\mathbf{KPi}^w$ + (Σ-REC) + FOUNDR(impl-Σ(M)).

(iii) Σ^1_2-**TRDC** + BR(impl-Σ^1_2) $\subseteq$ $\mathbf{KPi}^w$ + (Σ-REC) + FOUNDR(impl-Σ).

Proof. (i) The first identity is obvious since FOUNDR(impl-Σ) implies all instances of (IND)*. Let $T := \Delta^1_2$-**CA** + BR(impl-Σ^1_2) and $T' := \mathbf{KPi}^w$ + FOUNDR(impl-Σ). We want to show

$$T \vdash A \;\Rightarrow\; T' \vdash A^*$$

by induction on the length of the derivation in T. Owing to Theorem 7.2 it suffices to assume that A is the consequence of an inference BR(impl-Σ^1_2). Then A is of the form

$$\forall X[\mathrm{WO}(X) \wedge F[X] \to \mathrm{TI}(X, H)]$$

with $F[U] \in \Sigma^1_2$. Moreover, inductively we have

$$T' \vdash (\exists! X(\mathrm{WO}(X) \wedge F[X]))^*.$$

We now argue in T'. By Theorem 7.1 there exists a Σ formula $F'[a]$ such that

$$\forall x \subseteq \mathrm{N}\,(F'[x] \leftrightarrow \mathrm{Wo}(\mathrm{N}, x) \wedge F[x]^*).$$

Let r be the unique well-ordering on N which satisfies $F'[r]$. Via Axiom Beta there exist a unique ordinal α and order isomorphism between r and α. As a result, α has an implicit Σ definition, so that with the help of FOUNDR(impl-Σ) we have transfinite induction on α for arbitrary formulae. Via the order isomorphism f we then obtain A^*.

The proof of (iii) is analogous to (i), using Lemma 7.3.

(ii) is also proved similarly. The only extra consideration one has to employ is the following. For a primitive recursive well-ordering $\prec$ we have $r := \{\langle i, j\rangle \mid i \prec j\} \in \mathrm{M}$ and therefore the function f which is collapsing for r is an element of M, thus r is order isomorphic to an ordinal in M, which possesses an implicit $\Sigma(\mathrm{M})$ definition. □

Below we shall list some results whose proofs are too long to be incorporated in the first part of this paper. They will be supplied in the second part.

Theorem 8.5 *(i)* $\mathbf{AUT\text{-}KPl}^r$, $\mathbf{KPi}^w + \mathrm{FOUND}(\text{impl-}\Sigma)$, *and* $\mathbf{KPi}^w + \mathrm{FOUNDR}(\text{impl-}\Sigma)$ *prove the same* Σ_1 *sentences.*

(ii) $\mathbf{KPi}^w + (\Sigma\text{-REC}) + \mathrm{FOUND}(\text{impl-}\Sigma)$ *and* $\mathbf{KPi}^w + (\Sigma\text{-REC}) + \mathrm{FOUNDR}(\text{impl-}\Sigma)$ *prove the same* Σ_1 *sentences.*

Proof. See [43], Satz 6.5. The proof (which is long) will be in incorporated in the second part of this paper. □

The following two results show that the strength of (Σ-FOUND) is already encapsulated in (Σ-REC).

Theorem 8.6 $\mathbf{KPi}^r + (\Sigma\text{-FOUND})$ *and* $\mathbf{KPi}^r + (\Sigma\text{-REC})$ *prove the same* Π_2 *sentences.*

Proof. See [43], Satz 7.1. The proof (which is long) will be in incorporated in the second part of this paper. □

Theorem 8.7 $\mathbf{KPi}^w + (\Sigma\text{-FOUND})$ *and* $\mathbf{KPi}^w + (\Sigma\text{-REC})$ *prove the same* Π_2 *sentences.*

Proof. See [43], Satz 7.20. The proof will be in the second part of this paper. □

The next result shows

Theorem 8.8 $\mathbf{AUT\text{-}KPl}^r + \mathbf{KPi}^r$ *and* $\mathbf{AUT\text{-}KPl}^r$ *prove the same* Π_2 *sentences.*

Proof. See [43], Satz 7.20. The proof will be in the second part of this paper. □

Theorem 8.9 *For every* Π_2 *sentence* F,

$$\mathbf{KPi}^w + \mathrm{FOUND}(\text{impl-}\Sigma) \vdash F \quad \Rightarrow \quad \mathbf{AUT\text{-}KPl}^r \vdash F.$$

Proof. See [43], Satz 7.22. The proof will be in the second part of this paper. □

Theorem 8.10 *For every* Σ *sentence* G,

$$\mathbf{AUT\text{-}KPl}^r \vdash G \quad \Rightarrow \quad \mathbf{KPi}^w + \mathrm{FOUND}(\text{impl-}\Sigma) \vdash G.$$

Proof. See [43], Satz 7.23. The proof will be in the second part of this paper. □

II. WELL-ORDERING PROOFS

An ordinal α is said to be *provable* in a theory T (whose language encompasses $\mathcal{L}_2$) if there exists a recursive well-ordering $\prec$ whose order-type is α such that $\mathrm{T} \vdash \mathrm{WO}(\prec)$. In this chapter we try to give lower bounds for the provable ordinals of the various theories introduced in chapter I. That the results are indeed optimal will be shown in chapter III which will form the main chunk of the sequel to the present paper.

9 The functions φ_α and Φ_α

$\alpha, \beta, \gamma, \delta, \xi, \zeta, \rho$ will always denote ordinals. λ will be reserved for limit ordinals. Let $\alpha \mapsto \omega^\alpha$ be the ordinal function which enumerates the additive principal ordinals, i.e. the ordinals $\alpha > 0$ satisfying $(\forall \eta < \alpha)\, \eta + \alpha = \alpha$. This function is also a normal function since it is strictly increasing $\alpha < \beta \Rightarrow \omega^\alpha < \omega^\beta$ and satisfies $\omega^\lambda = \sup\{\omega^\eta \mid \eta < \lambda\}$.

Definition 9.1 Inductive definition of the classes $\mathrm{Cr}(\alpha)$:

1. Cr(0 is the class of additive principal ordinals.
2. φ_α is the function that enumerates $\mathrm{Cr}(\alpha)$, i.e. $\varphi_\alpha(\xi)$ is the ξth member of $\mathrm{Cr}(\alpha)$.
3. $\mathrm{Cr}(\alpha + 1) = \{\rho \mid \varphi_\alpha(\rho) = \rho\}$.
4. $\mathrm{Cr}(\lambda) = \bigcap\{\mathrm{Cr}(\xi) \mid \xi < \alpha\}$.

Definition 9.2 Inductive definition of the classes $\mathrm{Kr}(\alpha)$:

1. Kr(0) is the class of uncountable cardinals.
2. Φ_α is the function that enumerates $\mathrm{Kr}(\alpha)$.
3. $\mathrm{Kr}(\alpha + 1) = \{\rho \mid \Phi_\alpha(\rho) = \rho\}$.
4. $\mathrm{Kr}(\lambda) = \bigcap\{\mathrm{Kr}(\xi) \mid \xi < \alpha\}$.

On account of their definitions, the classes $\mathrm{Cr}(\alpha)$ and $\mathrm{Kr}(\alpha)$ are unbounded and closed in the ON (:=the class of ordinals) and thus every function φ_α and Φ_α is a normal function f, i.e. strictly increasing and continuous ($f(\lambda) = \sup\{f(\xi) \mid \xi < \lambda\}$ for limits λ).

In what follows we write $\varphi\alpha\beta$ for $\varphi_\alpha(\beta)$ and $\Phi\alpha\beta$ for $\Phi_\alpha(\beta)$.

The following three lemmas are proved for ϕ in [61, section 13], but the same proof works for Φ as well.

Lemma 9.3 *Let f be one of the functions φ or Φ. Suppose that $\alpha = f\gamma\delta$ and $\beta = f\xi\eta$.*

(i) $\alpha = \beta$ holds if and only if one of the following three statements holds:

1. *$\gamma < \xi$ and $\delta = f\xi\eta$.*
2. *$\gamma = \xi$ and $\delta = \eta$.*
3. *$\xi < \gamma$ and $f\gamma\delta = \eta$.*

(ii) $\alpha < \beta$ holds if and only if one of the following three statements holds:

1. *$\gamma < \xi$ and $\delta < f\xi\eta$.*
2. *$\gamma = \xi$ and $\delta < \eta$.*
3. *$\xi < \gamma$ and $f\gamma\delta < \eta$.*

Lemma 9.4 *(i)* $\varphi\alpha 0 < \varphi\beta 0 \Leftrightarrow \Phi\alpha 0 < \Phi\beta 0 \Leftrightarrow \alpha < \beta$.

(ii) $\alpha, \beta \leq \varphi\alpha\beta$ *and* $\alpha, \beta \leq \Phi\alpha\beta$.

Lemma 9.5 *For every* $\rho \in \mathrm{Cr}(0)$ $(\rho \in \mathrm{Kr}(0))$ *there exist unique ordinals* β, γ *such that* $\gamma < \rho$ *and* $\rho = \varphi\beta\gamma$ $(\rho = \Phi\beta\gamma)$.

Definition 9.6 (i) $\alpha =_{nf} \varphi\beta\gamma \quad :\Leftrightarrow \quad \alpha = \varphi\beta\gamma$ and $\beta, \gamma < \alpha$.

(ii) $\alpha =_{nf} \Phi\beta\gamma \quad :\Leftrightarrow \quad \alpha = \Phi\beta\gamma$ and $\beta, \gamma < \alpha$.

(iii) $\alpha =_{nf} \alpha_1 + \ldots + \alpha_n \quad :\Leftrightarrow \quad \alpha = \alpha_1 + \ldots + \alpha_n,\ \alpha_1, \ldots, \alpha_n \in \mathrm{Cr}(0)$ and $\alpha > \alpha_1 \geq \ldots \geq \alpha_n$.

The normal forms of Definition 9.6 are unique representations of ordinals owing to Lemma 9.3.

Definition 9.7 (i) $\mathrm{SC} := \{\alpha \mid \varphi\alpha 0 = \alpha\}$.

(ii) $\Gamma_0^\Phi := \min\{\alpha \mid \Phi\alpha 0 = \alpha\}$.

Lemma 9.8 $\Gamma_0^\Phi = \sup\{\rho_n \mid n < \omega\}$ *where* $\rho_0 = \Phi 00$ *and* $\rho_{n+1} = \Phi\rho_n 0$.

Proof. As in [61, Theorem 14.16]. □

By $\mathfrak{R}$ we shall denote the class of uncountable regular cardinals. $\alpha \mapsto \Omega_\alpha$ is the mapping which enumerates the class $\mathfrak{R}_0 := \mathrm{Kr}(0) \cup \{0\}$. In more traditional notation we have $\Omega_\alpha = \aleph_\alpha$ for all $\alpha > 0$. The regular uncountable cardinals $< \Gamma_0^\Phi$ can be characterized as follows:

Theorem 9.9 *If* $\kappa \in \mathfrak{R}$ *and* $\kappa < \Gamma_0^\Phi$ *then there exists a unique* ξ *such that* $\kappa = \Omega_{\xi+1}$.

Proof. Let $\kappa \in \mathfrak{R}$ and $\kappa < \Gamma_0^\Phi$. By Lemma 9.4, $\kappa \leq \Phi\kappa 0 < \Phi(\kappa+1)0$. Hence there is largest ordinal β such that $\kappa \in \mathrm{Kr}(\beta)$. Thus $\kappa = \Phi\beta\delta$ for some $\delta < \kappa$. If δ were a limit we would have $\kappa = \sup\{\Phi\beta\xi \mid \xi < \delta\}$ and κ would be singular. As a result, $\kappa = \Phi\beta(\eta+1)$ for some η or $\kappa = \Phi\beta 0$.

If $\beta = \kappa = \Phi\beta 0$ one could show, by induction on n, utilizing Lemma 9.3(ii), that $\rho_n < \kappa$, contradicting $\kappa < \Gamma_0^\Phi$. Hence $\beta < \kappa$. Now one could show the cofinality of κ to be the same as that of β if $\kappa = \Phi\beta 0$ and β were a limit, making κ singular. Likewise, if $\beta = \zeta + 1$ and $\kappa = \Phi\beta 0$ one could show that the cofinality of κ is ω, and similarly if $\kappa = \Phi\beta(\eta+1)$ and $\beta = \zeta + 1$ the cofinality of κ would be ω, too. As a result, since κ is regular $> \omega$ we must have $\beta = 0$. Therefore $\kappa = \Omega_1$ or $\kappa = \Omega_{\xi+1}$, where $\xi = \eta + 1 + 1$ if $\eta < \omega$ and $\xi = \eta$ otherwise. □

In what follows, the properties of the functions φ and Φ exhibited in this section will be used frequently and mostly tacitly.

10 The set of ordinals, $\mathrm{OT}(\Phi)$

This section introduces an ordinal representation system sufficient unto the task of expressing the proof-theoretic ordinals of all the theories considered so far. There will be no proofs in this section since they would be similar (with minor modifications) to those in [9] or [53]. **ZFC** will suffice as a background theory for showing the existence of the various functions.

We use the following conventions: (α, β), $(\alpha, \beta]$, $[\alpha, \beta)$, and $[\alpha, \beta]$ denote the intervals of ordinals between α and β in the obvious sense. For a set of ordinals A we use the abbreviations $A < \alpha \ := \ (\forall \eta \in A)\, \eta < \alpha$ and $A \leq \alpha \ := \ (\forall \eta \in A)\, \eta \leq \alpha$. Variables ν, μ, τ are understood to range over elements from $\mathfrak{R}_0$.

Definition 10.1 By recursion on α we define the sets of ordinals ordinals $\mathrm{C}_\nu(\alpha)$ and the ordinals $\psi\nu\alpha$. The sets $\mathrm{C}_\nu(\alpha)$ themselves are defined inductively by the following clauses:

($\mathrm{C}_\nu 1$) $[0, \nu] \subseteq \mathrm{C}_\nu(\alpha)$.

($\mathrm{C}_\nu 2$) $\xi, \eta \in \mathrm{C}_\nu(\alpha) \Rightarrow \xi + \eta \in \mathrm{C}_\nu(\alpha)$.

($\mathrm{C}_\nu 3$) $\xi, \eta \in \mathrm{C}_\nu(\alpha) \Rightarrow \varphi\xi\eta \in \mathrm{C}_\nu(\alpha)$.

($\mathrm{C}_\nu 4$) $\xi, \eta \in \mathrm{C}_\nu(\alpha) \Rightarrow \Phi\xi\eta \in \mathrm{C}_\nu(\alpha)$.

($\mathrm{C}_\nu 5$) $\xi < \alpha$ and $\xi, \mu \in \mathrm{C}_\nu(\alpha) \Rightarrow \psi\mu\xi \in \mathrm{C}_\nu(\alpha)$.

($\mathrm{C}_\nu 6$) $\psi\nu\alpha = \min\{\eta \mid \eta \notin \mathrm{C}_\nu(\alpha)\}$.

Definition 10.2 (i) $\alpha^+ := \min\{\kappa \in \mathfrak{R} \mid \alpha < \kappa\}$.

(ii) $\mathrm{S}(\alpha) := \min\{\mu \in \mathfrak{R}_0 \mid \alpha < \mu^+\}$.

Proposition 10.3 *(i)* $\alpha \leq \beta \Rightarrow \mathrm{C}_\nu(\alpha) \subseteq \mathrm{C}_\nu(\beta)$.

(ii) $\psi\nu\alpha \in (\nu, \nu^+)$.

(iii) $\nu < \Gamma_0^\Phi \Rightarrow \mathrm{C}_\nu(\alpha) \subseteq \Gamma_0^\Phi$.

(iv) $\psi\nu\alpha \in \mathrm{SC}$.

(v) $\psi\nu\alpha \notin \mathfrak{R}_0$.

(vi) $\psi\nu\alpha = \mathrm{C}_\nu(\alpha) \cap \nu^+$.

Proposition 10.4 *Let* $\alpha \in \mathrm{C}_\nu(\alpha)$ *and* $\beta \in \mathrm{C}_\nu(\beta)$.

(i) $\psi\nu\alpha = \psi\mu\beta$ *if and only if* $\nu = \mu$ *and* $\alpha = \beta$.

(i) $\psi\nu\alpha < \psi\mu\beta$ *if and only if* $\nu < \mu$ *or* $\nu = \mu \wedge \alpha < \beta$.

Definition 10.5 $\alpha =_{nf} \psi\nu\beta \ :\Leftrightarrow \ (\alpha = \psi\nu\beta \wedge \beta \in \mathrm{C}_\nu(\beta)$.

Definition 10.6 The set of ordinals $\mathrm{OT}(\Phi)$ and the complexity $\mathrm{G}\alpha < \omega$ for $\alpha \in \mathrm{OT}(\Phi)$ are defined inductively by the following clauses:

($\mathfrak{T}1$) $0 \in \mathrm{OT}(\Phi)$ and $\mathrm{G}(0) = 0$.

($\mathfrak{T}2$) $\alpha =_{nf} \alpha_1 + \ldots + \alpha_n \wedge \alpha_1, \ldots, \alpha_n \in \mathrm{OT}(\Phi) \Rightarrow$
$$\alpha \in \mathrm{OT}(\Phi) \wedge \mathrm{G}\alpha = \max\{\mathrm{G}\alpha_1, \ldots, \mathrm{G}\alpha_n\} + 1.$$

($\mathfrak{T}3$) $\alpha =_{nf} \varphi\beta\gamma \wedge \beta, \gamma \in \mathrm{OT}(\Phi) \Rightarrow \alpha \in \mathrm{OT}(\Phi) \wedge \mathrm{G}\alpha = \max\{\mathrm{G}\beta, \mathrm{G}\gamma\} + 1$.

($\mathfrak{T}4$) $\alpha =_{nf} \Phi\beta\gamma \wedge \beta, \gamma \in \mathrm{OT}(\Phi) \Rightarrow \alpha \in \mathrm{OT}(\Phi) \wedge \mathrm{G}\alpha = \max\{\mathrm{G}\beta, \mathrm{G}\gamma\} + 1$.

($\mathfrak{T}5$) $\alpha =_{nf} \psi\nu\gamma \wedge \nu, \gamma \in \mathrm{OT}(\Phi) \Rightarrow \alpha \in \mathrm{OT}(\Phi) \wedge \mathrm{G}\alpha = \max\{\mathrm{G}\nu, \mathrm{G}\gamma\} + 1$.

It follows from Lemma 9.5 and Proposition 10.3(iv),(v) that every ordinal $\alpha \in \mathrm{OT}(\Phi)$ enters $\mathrm{OT}(\Phi)$ owing to exactly one of the rules ($\mathfrak{T}1$)-($\mathfrak{T}5$). As a result the inductive definition of $\mathrm{OT}(\Phi)$ is deterministic, thus $\mathrm{G}\alpha$ is well-defined.

Theorem 10.7 $\mathrm{OT}(\Phi) = \mathrm{C}_0(\Gamma_0^{\Phi})$.

Every element of $\mathrm{OT}(\Phi)$ can be uniquely named via a term built up from the "symbols" $0, +, \varphi, \Phi, \psi$. At this point we have not yet established that thereby $\mathrm{OT}(\Phi)$ with its ordering gives rise to a decidable well ordering. This can be achieved by showing that questions such as whether $\gamma < \Phi\beta\gamma$ in ($\mathfrak{T}4$) and whether $\beta \in \mathrm{C}_\nu(\beta)$ in ($\mathfrak{T}5$) can be decided. To this end we exhibit several lemmata which will entail the decidabilty of $(\mathrm{OT}(\Phi), <)$.

Definition 10.8 The set of ordinals $\mathrm{K}_\nu\alpha$ for $\alpha \in \mathrm{OT}(\Phi)$ and $\nu \in \mathfrak{R}_0$ are defined inductively by the following clauses:

($\mathrm{K}_\nu 1$) $\mathrm{K}_\nu 0 = \emptyset$.

($\mathrm{K}_\nu 2$) $\mathrm{K}_\nu\alpha = \bigcup\{\mathrm{K}_\nu\alpha_j \mid j = 1, \ldots, n\}$ if $\alpha =_{nf} \alpha_1 + \ldots + \alpha_n$.

($\mathrm{K}_\nu 3$) $\mathrm{K}_\nu\alpha = \mathrm{K}_\nu\beta \cup \mathrm{K}_\nu\gamma$ if $\alpha =_{nf} \varphi\beta\gamma$ or $\alpha =_{nf} \Phi\beta\gamma$.

($\mathrm{K}_\nu 4$) Let $\alpha =_{nf} \psi\mu\beta$.
$$\mathrm{K}_\nu\alpha = \begin{cases} \emptyset & \text{if } \mu < \nu \\ \{\beta\} \cup \mathrm{K}_\nu\beta \cup \mathrm{K}_\nu\mu & \text{if } \nu \leq \mu. \end{cases}$$

Lemma 10.9 *For $\alpha \in \mathrm{OT}(\Phi)$ we have $\alpha \in \mathrm{C}_\nu(\beta) \Leftrightarrow \mathrm{K}_\nu\alpha < \beta$.*

Definition 10.10 Sets $\mathrm{e}(\alpha)$ and $\mathrm{E}(\alpha)$ are defined inductively as follows:

1. $\mathrm{e}(0) = \mathrm{E}(0) = \emptyset$.
2. $\mathrm{e}(0) = \mathrm{E}(0) = \emptyset$ if $\alpha =_{nf} \alpha_1 + \ldots + \alpha_n$.
3. $\mathrm{e}(\alpha) = \{\beta\}$ and $\mathrm{E}(\alpha) = \emptyset$ if $\alpha =_{nf} \varphi\beta\gamma$.
4. $\mathrm{e}(\alpha) = \{\alpha\}$ and $\mathrm{E}(\alpha) = \{\beta\}$ if $\alpha =_{nf} \Phi\beta\gamma$.

5. $\mathrm{e}(\alpha) = \{\alpha\}$ and $\mathrm{E}(\alpha) = \emptyset$ if $\alpha =_{nf} \psi\nu\beta$.

Lemma 10.11 *Let $\alpha, \beta, \gamma \in \mathrm{OT}(\Phi)$.*

(i) If $\alpha = \varphi\beta\gamma$ then $\alpha =_{nf} \varphi\beta\gamma \Leftrightarrow [\mathrm{e}(\gamma) \leq \beta \wedge (\beta \notin \mathrm{SC} \vee \gamma > 0)]$.

(ii) If $\alpha = \Phi\beta\gamma$ then $\alpha =_{nf} \Phi\beta\gamma \Leftrightarrow \mathrm{E}(\gamma) \leq \beta$.

Proof. We only remark that it is essential for (ii) to hold that $\beta < \Phi\beta 0$ holds for all $\beta \in \mathrm{OT}(\Phi)$ by Theorem 10.7. □

Definition 10.12 A coding function

$$\ulcorner\ \urcorner : \mathrm{OT}(\Phi) \longrightarrow \mathbb{N}$$

is defined as follows: 1. $\ulcorner 0 \urcorner = (0)$. 2. $\ulcorner\alpha\urcorner = (1, \ulcorner\alpha_1\urcorner, \ldots, \ulcorner\alpha_n\urcorner)$ if $\alpha =_{nf} \alpha_1 + \ldots + \alpha_n$. 3. $\ulcorner\alpha\urcorner = (2, \ulcorner\beta\urcorner, \ulcorner\gamma\urcorner)$ if $\alpha =_{nf} \varphi\beta\gamma$. 4. $\ulcorner\alpha\urcorner = (3, \ulcorner\beta\urcorner, \ulcorner\gamma\urcorner)$ if $\alpha =_{nf} \Phi\beta\gamma$. 5. $\ulcorner\alpha\urcorner = (4, \ulcorner\nu\urcorner, \ulcorner\gamma\urcorner)$ if $\alpha =_{nf} \psi\nu\gamma$. Here $(\ldots)$ stands for some fixed primitive recursive coding of tuples of natural numbers.

Let

$$\ulcorner \mathrm{OT}(\Phi) \urcorner := \{\ulcorner\alpha\urcorner \mid \alpha \in \mathrm{OT}(\Phi)\}$$

and define an ordering $\prec$ on $\mathbb{N}$ via

$$n \prec m \ :\Leftrightarrow\ \exists\alpha, \beta \in \mathrm{OT}(\Phi)\, (\alpha < \beta \wedge n = \ulcorner\alpha\urcorner \wedge m = \ulcorner\beta\urcorner).$$

If one now combines Lemma 9.3, Proposition 10.4, Lemma 10.9 and Lemma 10.11 one sees that $\ulcorner\mathrm{OT}(\Phi)\urcorner$ is a primitive recursive set equipped with a primitive recursive ordering $\prec$ such that $(\mathrm{OT}(\Phi), <)$ and $\ulcorner\mathrm{OT}(\Phi)\urcorner, \prec)$ are isomorphic.

In what follows we shall no longer distinguish between $(\mathrm{OT}(\Phi), <)$ and its arithmetization $\ulcorner\mathrm{OT}(\Phi)\urcorner, \prec)$. Via this identification, SC beomes a primitive recursive predicate and the functions $\mathrm{S}, \mathrm{K}, \mathrm{G}, \mathrm{e}, \mathrm{E}, \xi \mapsto \omega^\xi, \alpha \mapsto \Omega_\alpha, \varphi, \Phi, \psi$ can be viewed as primitive recursive functions acting on $\ulcorner\mathrm{OT}(\Phi)\urcorner$. In particular, all these relations and functions are definable in the language of arithmetic, $\mathcal{L}_1$.

Conventions 10.13 Lower case Greek letters $\alpha, \beta, \gamma, \delta, \xi, \eta, \sigma, \zeta, \vartheta$ will range over arbitrary elements of $\mathrm{OT}(\Phi)$ for the remainder of this paper while ν, μ, τ will be reserved for elements of $\mathrm{OT}(\Phi) \cap \mathfrak{R}_0$. Quantifiers $\forall\alpha, \exists\alpha, \ldots$ will exclusively range over elements of $\mathrm{OT}(\Phi)$, too.

11 Distinguished sets

By a well-ordering proof in a given theory T we mean a proof formalizable in T which shows that a certain ordinal representation system (or a subset of it) is well-ordered. The notion of a *distinguished set* (of ordinals) (in German: ausgezeichnete Menge) will be central to carrying out well ordering proofs in the various subtheories of second order arithmetic introduced in earlier sections. A theory of distinguished sets developed for this purpose emerged in the works of Buchholz and Pohlers [4, 6, 7].

As a base theory in which all the results of this section can be proved one can take Π^1_1-$\mathbf{CA}_0$. It also worthwhile to point out that all the proofs work when the underlying logic is changed to intuitionistic logic. The principle of excluded third gets applied only to decidable properties (actually primitive recursive predicates). Thus all the proofs can be formalized in Π^1_1-$\mathbf{CA}^i_0$, the intuitionistic version of Π^1_1-$\mathbf{CA}_0$.

We introduce another operation on $\mathrm{OT}(\Phi)$ which will play an important role in the remainder of this paper.

Definition 11.1 The *strongly critical subterms of level* μ of α are defined inductively as follows:

1. $\mathrm{SC}_\mu(0) = \emptyset$.
2. $\mathrm{SC}_\mu(\alpha) = \{\alpha\}$ if $\alpha \in \mathrm{SC} \cap \mu^+$.
3. $\mathrm{SC}_\mu(\alpha) = \bigcup\{\mathrm{SC}_\mu(\alpha_i) \mid i = 1, \ldots, n\}$ if if $\alpha =_{nf} \alpha_1 + \ldots + \alpha_n$.
4. $\mathrm{SC}_\mu(\alpha) = \mathrm{SC}_\mu(\beta) \cup \mathrm{SC}_\mu(\gamma)$ if $\alpha =_{nf} \varphi\beta\gamma$.
5. $\mathrm{SC}_\mu(\alpha) = \mathrm{SC}_\mu(\beta) \cup \mathrm{SC}_\mu(\gamma)$ if $\alpha =_{nf} \Phi\beta\gamma$ and $\mu^+ \leq \alpha$.
6. $\mathrm{SC}_\mu(\alpha) = \mathrm{SC}_\mu(\beta) \cup \mathrm{SC}_\mu(\gamma)$ if $\alpha =_{nf} \psi\nu\gamma$ and $\mu^+ \leq \alpha$.

Definition 11.2 Let $U \subseteq \mathrm{OT}(\Phi)$ and $F(a)$ be an $\mathcal{L}_2$-formula.

(i) $U \cap \alpha := \{\eta \in U \mid \eta < \alpha\}$.

(ii) $U \cap \alpha \subseteq F \;:\Leftrightarrow\; (\forall \eta \in U \cap \alpha)\, F(\eta)$.

(iii) $\mathrm{Prg}(U, F) \;:\Leftrightarrow\; \forall \eta \in U\, [U \cap \eta \subseteq F \to F(\eta)]$.

(iv) $\mathrm{W}[U] := \{\eta \in U \mid \forall Y[\mathrm{Prg}(U, Y) \to U \cap \eta \subseteq Y]\}$.

(v) $\mathrm{M}^U_\mu := \{\eta < \mu^+ \mid (\forall \eta \in U \cap \mu)\mathrm{SC}_\nu(\eta) \subseteq U\}$.

(vi) $\mathrm{W}^U_\mu := \mathrm{W}[\mathrm{M}^U_\mu]$.

Remark 11.3 (i) If $<_U$ denotes the restriction of $<$ to U and $F_U(a)$ is the formula $a \in U \to F(a)$ then $\mathrm{Prg}(U, F) \leftrightarrow \mathrm{PROG}(<_U, F_U)$ holds with PROG defined in Definition 3.1.

(ii) M^U_μ is always a set by arithmetical comprehension. To show that $\mathrm{W}[U]$ and W^U_μ are sets one can use Π^1_1 comprehension. $\mathrm{W}[U]$ and W^U_μ can also be shown to be sets in any theory which proves that the accessible part of an ordering R on $\mathbb{N}$ (where R is assumed to be a set) is a set. A case in point is constructive Zermelo-Fraenkel set theory with the regular extension axiom, $\mathbf{CZF} + \mathrm{REA}$ (see [1, 2]). Actually the fragment $\mathbf{CZF}^r + \mathrm{REA}$ of $\mathbf{CZF} + \mathrm{REA}$ suffices. Here $\mathbf{CZF}^r$ denotes $\mathbf{CZF}$ with $\in$-induction restricted to bounded formulae. To place this theory into perspective, $\mathbf{CZF}^r + \mathrm{REA}$ and Π^1_1-$\mathbf{CA}_0$ are of the same strength.

The next Lemma lists basic properties of $\mathrm{W}[U]$, M^U_μ and W^U_μ.

Lemma 11.4 *(i)* $\mathrm{Prg}(U,S) \to \mathrm{W}[U] \subseteq S$.

(ii) $\mathrm{Prg}(U,\mathrm{W}[U])$.

(iii) $U \subseteq V \wedge \mathrm{Prg}(U,S) \to \mathrm{Prg}(V,\{\eta \mid \eta \in U \to \eta \in S\})$.

(iv) $\mathrm{Prg}(\mathrm{W}[U],S) \to \mathrm{W}[U] \subseteq S$.

(v) $\mathrm{W}[\mathrm{W}[U]] = \mathrm{W}[U]$.

(vi) $\mathrm{W}[U \cap \alpha] \subseteq \mathrm{W}[U]$.

(vii) $\mathrm{W}[U \cap \alpha] \subseteq \mathrm{W}[U]$.

(viii) $\alpha \in \mathrm{W}^U_\mu \leftrightarrow \alpha \in \mathrm{M}^U_\mu \wedge \mathrm{M}^U_\mu \cap \alpha \subseteq \mathrm{W}^U_\mu$.

Proof. (i) and (vii) are immediate by going back to the definitions.

(ii) Let $\alpha \in U$ and $U \cap \alpha \subseteq \mathrm{W}[U]$. By (i) we have $U \cap \alpha \subseteq S$ for every S satisfying $\mathrm{Prg}(U,S)$. Thence $\alpha \in \mathrm{W}[U]$.

(iii) Assume that $U \subseteq V$ and $\mathrm{Prg}(U,S)$ hold and also that $\alpha \in V$ and

$$V \cap \alpha \subseteq \{\eta \mid \eta \in U \to \eta \in S\}.$$

Then $U \cap \alpha = U \cap V \cap \alpha \subseteq S$, thus $\alpha \in U \to \alpha \in S$, i.e. $\alpha \in \{\eta \mid \eta \in U \to \eta \in S\}$.

(iv) Suppose $\mathrm{Prg}(W[U],S)$. (iii) implies $\mathrm{Prg}(U,\{\eta \mid \eta \in \mathrm{W}[U] \to \eta \in S\})$. Therefore, by (i), we also have $\mathrm{W}[U] \subseteq \{\eta \mid \eta \in \mathrm{W}[U] \to \eta \in S\}$, and hence $\mathrm{W}[U] \subseteq S$.

(v) $\mathrm{W}[\mathrm{W}[U]] \subseteq \mathrm{W}[U]$ holds by definition. Using (ii) we have $\mathrm{Prg}(\mathrm{W}[U],\mathrm{W}[\mathrm{W}[U]])$, hence, by (iv), $\mathrm{W}[U] \subseteq \mathrm{W}[\mathrm{W}[U]]$.

(vi) From

$$\eta \in U \cap \alpha \wedge \forall Y(\mathrm{Prg}(U \cap \alpha, Y) \to U \cap \alpha \cap \eta \subseteq Y)$$

we deduce that $\forall Y(\mathrm{Prg}(U,Y) \to U \cap \eta \subseteq Y)$, thence $\eta \in \mathrm{W}[U]$.

(viii) By (ii) we have $\mathrm{Prg}(\mathrm{M}^U_\mu,\mathrm{W}^U_\mu)$. W^U_μ is also a set. Thus (viii) follows. □

Definition 11.5 (i) A set $U \subseteq \mathrm{OT}(\Phi)$ is said to be *distinguished* if (D1) and (D2) are satisfied:

(D1) $(\forall \alpha \in U)\, \mathrm{S}\alpha \in U$.

(D2) $(\forall \mu \in U)\, U \cap \mu^+ = \mathrm{W}^U_\mu$.

(ii) We shall use the abbreviation $\mathrm{Ds}(U)$ to convey that U is a distinguished set. Variables P and Q will always refer to distinguished sets.

(iii) $\mathfrak{W} := \{\eta \mid \exists X\,[\mathrm{Ds}(X) \wedge \eta \in X]\}$.

Note that $\mathfrak{W}$ cannot be shown to be a set in our background theory Π^1_1-$\mathbf{CA}_0$ (nor actually in any of the other theories we investigate in this paper).

Lemma 11.6 *Recall that the letters Q and P are reserved for distinguished sets.*

(i) $Q \subseteq \mathrm{W}[Q]$ *and hence* $Q = \mathrm{W}[Q]$

(ii) $\mathrm{Prg}(Q, V) \to Q \subseteq V$.

Proof. (i) Let $\alpha \in Q$. Then $\mathrm{S}\alpha \in Q$ by (D1) and hence $Q \cap \alpha^+ = \mathrm{W}^Q_{\mathrm{S}\alpha}$. So by Lemma 11.4(v),(vi) we arrive at $\alpha \in Q \cap \alpha^+ = \mathrm{W}^Q_{\mathrm{S}\alpha} = \mathrm{W}[\mathrm{W}^Q_{\mathrm{S}\alpha}] = \mathrm{W}[Q \cap \alpha^+] \subseteq \mathrm{W}[Q]$.
(ii) is an immediate consequence of (i) and Lemma 11.4(i). □

Owing to Lemma 11.6(ii) we have transfinite induction over $<_Q := < \cap (Q \times Q)$ for arbitrary sets. Thus if we want to show that $Q \subseteq V$ holds for a set V it suffices to prove that

$$\forall \beta (\beta \in Q \wedge Q \cap \beta \subseteq V \to \beta \in V).$$

Specifically we have $\mathrm{WO}(<_Q)$.

Lemma 11.7 *(i)* $\nu \leq \mu \wedge \beta \in \mathrm{SC}_\mu(\alpha) \to \mathrm{SC}_\nu(\beta) \subseteq \mathrm{SC}_\nu(\alpha)$.

(ii) $\alpha \in Q \wedge \mu \in Q \to \mathrm{SC}_\mu(\alpha) \subseteq Q$.

(iii) $\mu \in \mathrm{M}^Q_\mu \to (\forall \nu \in Q)\mathrm{SC}_\mu(\alpha) \subseteq Q$.

(iv) $\mu \in \mathrm{M}^Q_\mu \wedge \mu \leq Q \to \mu \in Q$.

Proof. (i) follows by induction on $\mathrm{G}\alpha$
(ii) 1. Suppose $\mu < \mathrm{S}\alpha$. Then (D1) and (D2) imply that $\alpha \in Q \cap \alpha^+ = \mathrm{W}^Q_{\mathrm{S}\alpha} \subseteq \mathrm{M}^Q_{\mathrm{S}\alpha}$. As $\mu \in Q \cap \mathrm{S}\alpha$ we see that $\mathrm{SC}_\mu(\alpha) \subseteq Q$ by definition of $\mathrm{M}^Q_{\mathrm{S}\alpha}$.
2. Suppose $\mu \geq \mathrm{S}\alpha$. From (D2) it then follows that $\alpha \in \mathrm{W}^Q_\mu \subseteq \mathrm{M}^Q_\mu$. For $\nu \in Q \cap \mu$ we thus have $\mathrm{SC}_\nu(\alpha) \subseteq Q$, and from (i) we conclude that $(\forall \beta \in \mathrm{SC}_\mu(\alpha))\, \mathrm{SC}_\nu(\beta) \subseteq Q$. Therefore $\mathrm{SC}\mu(\alpha) \subseteq \{\alpha\} \cup \mathrm{M}^Q_\mu \cap \alpha$. By Lemma 11.4(viii) we get $\mathrm{SC}_\mu(\alpha) \subseteq \mathrm{W}^Q_\mu \subseteq Q$ as $\alpha \in \mathrm{W}^Q_\mu$.
(iii) will be proved by transfinite induction on Q (i.e. $<_Q$).
1. If $\nu \in Q \cap \mu^+$ then the desired assertion follows in the case $\nu < \mu$ from the definition of M^Q_μ and in the case $\nu = \mu$ from (ii).
2. If $\nu \in Q$ and $\mu < \nu$ then by induction hypothesis we have $(\forall \tau \in Q \cap \nu)\mathrm{SC}_\tau(\mu) \subseteq Q$. and consequently $\mu \in \mathrm{M}^Q_\nu$. From $\nu \in Q \cap \nu^+ = \mathrm{W}^Q_\nu$ we obtain by Lemma 11.4(viii) that $\mathrm{M}^Q_\nu \cap \nu \subseteq \mathrm{W}^Q_\nu$, whence $\mu \in \mathrm{W}^Q_\nu$. Since $\mathrm{SC}_\nu(\mu) \subseteq \{\mu\}$ we arrive at the desired assertion.
(iv) follows directly from (iii). □

Lemma 11.8 $Q \cap \mu^+ \subseteq \mathrm{W}^Q_\mu$.

Proof. Let $\alpha \in Q \cap \mu^+$. Then $\alpha \in \mathrm{W}^Q_{\mathrm{S}\alpha}$ and so by Lemma 11.4(viii), $\mathrm{M}^Q_{\mathrm{S}\alpha} \cap \alpha \subseteq \mathrm{W}^Q_{\mathrm{S}\alpha}$. In view of Lemma 11.4(vi) it suffices to show that $\alpha \in \mathrm{W}[\mathrm{M}^Q_\mu \cap \alpha^+]$. Lemma 11.7(ii) yields $\alpha \in \mathrm{M}^Q_\mu \cap \alpha^+$. Using Lemma 11.4(iii), $\mathrm{Prg}(\mathrm{M}^Q_\mu \cap \alpha^+, U)$ implies

$$\mathrm{Prg}(\mathrm{M}^Q_{\mathrm{S}\alpha}, \{\eta \mid \eta \in \mathrm{M}^Q_\mu \cap \alpha^+ \to \eta \in U\}),$$

and further, by Lemma 11.4(i),

$$\mathrm{M}_\mu^Q \cap \alpha \subseteq \mathrm{M}_{\mathrm{S}\alpha}^Q \cap \alpha \subseteq \mathrm{W}_{\mathrm{S}\alpha}^Q \subseteq \{\eta \mid \eta \in \mathrm{M}_\mu^Q \cap \alpha^+ \to \eta \in U\},$$

thence $\mathrm{M}_\mu^Q \cap \alpha^+ \cap \alpha \subseteq U$. This shows $\alpha \in \mathrm{W}[\mathrm{M}_\mu^Q \cap \alpha^+]$. □

Proposition 11.9 $\mu \in \mathrm{M}_\mu^Q \wedge \mathrm{M}_\mu^Q \cap \mu \subseteq Q \to \mu \in \mathrm{W}_\mu^Q \wedge \mathrm{Ds}(\mathrm{W}_\mu^Q)$.

Proof. By Lemma 11.8, $\mathrm{M}_\mu^Q \cap \mu \subseteq Q$ implies $\mathrm{M}_\mu^Q \cap \mu = \mathrm{W}_\mu^Q \cap \mu$. Thus, by Lemma 11.4(viii), $\mu \in \mathrm{M}_\mu^Q$ implies $\mu \in \mathrm{W}_\mu^Q$.

Next we show that W_μ^Q is a distinguished set.

Ad (D1): If $\alpha \in \mathrm{W}_\mu^Q \cap \mu$ then $\mathrm{S}\alpha \in Q \cap \mu \subseteq \mathrm{W}_\mu^Q \cap \mu$. From $\alpha \in \mathrm{W}_\mu^Q$ and $\mu \leq \alpha$ we obtain $\mathrm{S}\alpha = \mu \in \mathrm{W}_\mu^Q$.

Ad (D2): For $\tau \leq \mu$ we have $(*)$ $\mathrm{W}_\mu^Q \cap \tau = Q \cap \tau$ since $\mathrm{M}_\mu^Q \cap \mu \subseteq Q$ yields $\mathrm{W}_\mu^Q \cap \tau \subseteq Q$, and so, by Lemma 11.8, $Q \cap \tau \subseteq \mathrm{W}_\mu^Q$ holds. Now let $P := \mathrm{W}_\mu^Q$ and suppose $\nu \in P$. By $(*)$, we then have $P \cap \nu = Q \cap \nu$, and thus, by Lemma 11.4(viii), $(**)$ $\mathrm{W}_\nu^P = \mathrm{W}_\nu^Q$. For $\nu < \mu$, $(*)$ entails $\nu \in Q$ and therefore $\mathrm{W}_\nu^P = \mathrm{W}_\nu^Q = Q \cap \nu^+ \overset{(*)}{=} \mathrm{W}_\mu^Q \cap \nu^+ = P \cap \nu^+$. If $\nu = \mu$, then $(**)$ yields $\mathrm{W}_\mu^Q = P = P \cap \mu^+$. □

Vacuously $\emptyset$ is a distinguished set. Proposition 11.9 yields the existence of non-trivial distinguished sets. For example, $\mathrm{W}_0^\emptyset$ is a distinguished set.

Lemma 11.10 $\mathrm{Prg}(P \cup Q, U) \to P \cup Q \subseteq U$.

Proof. Suppose $\mathrm{Prg}(P \cup Q, U)$. Then we have

$$P \cap \alpha \subseteq U \to \mathrm{Prg}(Q, \{\eta \mid \eta < \alpha \to \eta \in U\}), \text{ and}$$
$$P \cap \alpha \subseteq U \wedge Q \cap \alpha \subseteq U \wedge \alpha \in P \to \alpha \in U.$$

Therefore, by Lemma 11.6(ii), we have

$$P \cap \alpha \subseteq U \wedge \alpha \in P \to \alpha \in U,$$

i.e. $\mathrm{Prg}(P, U)$ holds, and consequently $P \subseteq U$ by Lemma 11.6(ii). Similarly one shows that $Q \subseteq U$. □

Lemma 11.11 $\mu \in P \cup Q \wedge \mu \leq P \wedge \mu \leq Q \to P \cap \mu^+ = Q \cap \mu^+$.

Proof. We use induction on $P \cup Q$, i.e. Lemma 11.10. Let $\mu \in P$ and suppose $\mu \leq Q$. The induction hypothesis yields $P \cap \mu = Q \cap \mu$ and, by Lemma 11.4(vii), we conclude that $\mu \in P \cap \mu^+ = \mathrm{W}_\mu^P = \mathrm{W}_\mu^Q \subseteq \mathrm{M}_\mu^Q$, and hence $\mu \in Q$ by Lemma 11.7(iv). As a result, $P \cap \mu^+ = \mathrm{W}_\mu^P = \mathrm{W}_\mu^Q = Q \cap \mu^+$. The same arguments can be used if $\mu \in Q$ and $\mu \leq P$. □

Proposition 11.12 $\alpha \in Q \to Q \cap \alpha^+ = \mathfrak{W} \cap \alpha^+$.

Proof. Let $\alpha \in Q$. $Q \cap \alpha^+ \subseteq \mathfrak{W} \cap \alpha^+$ is obvious by definition of $\mathfrak{W}$. Let $\eta \in \mathfrak{W} \cap \alpha^+$. Then there exists a distinguished set P such that $\eta \in P \cap \alpha^+$. Thus $\mathrm{S}\eta \in P \cup Q$, $\mathrm{S}\eta \leq \eta \in P$ and $\mathrm{S}\eta \leq \alpha \in Q$. Therefore $\eta \in P \cap \eta^+ = Q \cap \eta^+ \subseteq Q \cap \alpha^+$ using Lemma 11.11. □

Next we study closure properties shared by all distinguished sets.

Proposition 11.13 *(i)* $\alpha, \beta \in Q \to \alpha + \beta \in Q$.

(ii) $\alpha, \beta \in \mathfrak{W} \to \alpha + \beta \in \mathfrak{W}$.

Proof. (ii) is an immediate consequence of (i) in view of Proposition 11.12. In the proof of (i) let $X := \mathrm{M}^Q_{\mathrm{S}\alpha}$, $Y := \mathrm{W}^Q_{\mathrm{S}\alpha}$ and $U := \{\xi \mid \alpha + \xi \in Y\}$. Suppose $\alpha, \beta \in Q$. If $\mathrm{S}\alpha < \mathrm{S}\beta$ then $\alpha + \beta = \beta \in Q$. Now assume $\mathrm{S}\beta \leq \mathrm{S}\alpha$. Then we have $Q \cap \alpha^+ = Y$ and $\alpha, \beta \in Y$. Moreover we have

$$\eta \in X \wedge X \cap \eta \subseteq U \to \alpha + \eta \in X \wedge X \cap (\alpha + \eta) \subseteq Y,$$

so that with Lemma 11.4(viii) we get $\eta \in X \wedge X \cap \eta \subseteq U \to \alpha + \eta \in Y$. As a result, $\mathrm{Prg}(X, U)$ holds, and thus $Y \subseteq U$ by Lemma 11.4(i), hence $\alpha + \beta \in Y \subseteq Q$. □

Lemma 11.14 *Letting* $\mathfrak{F}(\alpha, \beta)$ *be the formula*

$$\alpha, \beta \in Q \wedge (\forall \xi \in Q \cap \alpha)(\forall \eta \in Q)(\varphi\xi\eta \in Q) \wedge (\forall \eta \in Q \cap \beta)(\varphi\alpha\eta \in Q),$$

the following are true:

(i) $\mathfrak{F}(\alpha, \beta) \wedge \mu = \max\{\mathrm{S}\alpha, \mathrm{S}\beta\} \wedge \gamma \in \mathrm{M}^Q_\mu \cap \varphi\alpha\beta \to \gamma \in Q$.

(ii) $\mathfrak{F}(\alpha, \beta) \to \varphi\alpha\beta \in Q$.

Proof. We show (i) by induction on $\mathrm{G}\gamma$. $\mathfrak{F}(\alpha, \beta)$ implies $\alpha, \beta \in Q \cap \mu^+ = \mathrm{W}^Q_\mu$. We distinguish cases according to the shape of γ. The assertion is trivially true if $\gamma = 0$. Let $\gamma =_{nf} \gamma_1 + \ldots + \gamma_n$. Then $\gamma_1, \ldots, \gamma_n \in \mathrm{M}^Q_\mu \cap \varphi\alpha\beta$, and thus by the induction hypothesis, $\gamma_1, \ldots, \gamma_n \in Q$, so $\gamma \in Q$ by Proposition 11.13. If $\gamma \in \mathrm{SC}$ then $\gamma \leq \alpha \vee \gamma \leq \beta$, and therefore, as $\alpha, \beta \in \mathrm{W}^Q_\mu$ and $\gamma \in \mathrm{M}^Q_\mu$, it follows from Lemma 11.4(viii) that $\gamma \in \mathrm{W}^Q_\mu \subseteq Q$.

The last case to consider is when $\gamma =_{nf} \varphi\xi\eta$ for some ξ, η. Then $\xi, \eta \in \mathrm{M}^Q_\mu \cap \varphi\alpha\beta$ and the induction hypothesis yields $\xi, \eta \in Q$. If $\xi \leq \alpha$ then $\gamma \in Q$ follows from $\mathfrak{F}(\alpha, \beta)$. If $\alpha < \xi$ then $\gamma < \beta$ must hold, and with the aid of Lemma 11.4(viii) we conclude that $\gamma \in Q$.

(ii) By (i) we have

$$\mathfrak{F}(\alpha, \beta) \wedge \mu = \{\mathrm{S}\alpha, \mathrm{S}\beta\} \to \mathrm{M}^Q_\mu \cap \varphi\alpha\beta \subseteq \mathrm{W}^Q_\mu.$$

By Lemma 11.7(ii) we also have

$$\mathfrak{F}(\alpha, \beta) \wedge \mu = \max\{\mathrm{S}\alpha, \mathrm{S}\beta\} \to \varphi\alpha\beta \in \mathrm{M}^Q_\mu.$$

Thus, by Lemma 11.4(viii),

$$\mathfrak{F}(\alpha, \beta) \wedge \mu = \max\{\mathrm{S}\alpha, \mathrm{S}\beta\} \to \varphi\alpha\beta \in \mathrm{W}^Q_\mu,$$

and hence $\mathfrak{F}(\alpha, \beta) \to \varphi\alpha\beta \in Q$. □

Proposition 11.15 *(i)* $\alpha, \beta \in Q \to \varphi\alpha\beta \in Q$.

(ii) $\alpha, \beta \in \mathfrak{W} \to \varphi\alpha\beta \in \mathfrak{W}$.

Proof. Again, by Proposition 11.12, (ii) is an immediate consequence of (i). Let $\alpha \in Q$, $U := \{\xi \mid (\forall \eta \in Q)(\varphi\xi\eta \in Q)\}$ and $V := \{\eta \mid \varphi\alpha\eta \in Q\}$. Lemma 11.14(ii) yields

$$(\forall \xi \in Q \cap \alpha)(\forall \eta \in Q)(\varphi\xi\eta \in Q) \to \mathrm{Prg}(Q, V)$$

and hence, using Lemma 11.6(ii),

$$(\forall \xi \in Q \cap \alpha)(\forall \eta \in Q)(\varphi\xi\eta \in Q) \to Q \subseteq V.$$

The latter implies $\mathrm{Prg}(Q, U)$, whence $Q \subseteq U$. □

Corollary 11.16 *(i)* $\mathrm{S}\alpha \leq \mu \wedge \mu \in Q \wedge \mathrm{SC}_\mu(\alpha) \subseteq Q \to \alpha \in Q$.

(ii) $\mathrm{S}\alpha \leq \mu \wedge \mu \in \mathfrak{W} \wedge \mathrm{SC}_\mu(\alpha) \subseteq Q \to \alpha \in \mathfrak{W}$.

Proof. This follows from Propositions 11.13 and 11.15. □

Lemma 11.17 *(i)* $\beta \in Q \wedge \alpha \in \mathrm{M}^Q_{\mathrm{S}\beta} \cap \beta \to \alpha \in Q$.

(ii) $\beta \in \mathfrak{W} \wedge \alpha \in \mathrm{M}^Q_{\mathrm{S}\beta} \cap \beta \to \alpha \in \mathfrak{W}$.

(i) $\beta \in Q$ implies $\beta \in Q \cap \beta^+ = \mathrm{W}^Q_{\mathrm{S}\beta}$. Therefore, by Lemma 11.4(viii), $\alpha \in \mathrm{W}^Q_{\mathrm{S}\beta} \subseteq Q$. (ii) is an immediate consequence of (i). □

Definition 11.18 $\mathfrak{B}^Q_\mu := \{\alpha \mid (\forall \tau \in Q \cap \mu)[\mathrm{K}_\tau\alpha < \alpha \to \psi\tau\alpha \in Q\}$.

Lemma 11.19 *Assume* $\alpha \in \mathrm{M}^Q_\mu$, $\mathrm{M}^Q_\mu \cap \alpha \subseteq \mathfrak{B}^Q_\mu$, $\nu \in Q \cap \mu$, $\mathrm{K}_\nu\alpha < \alpha$ *and* $\gamma \in \mathrm{M}^Q_\nu \cap \psi\nu\alpha$. *Then* $\gamma \in Q$.

Proof. We proceed by induction on $\mathrm{G}\gamma$.
If $\gamma \leq \nu$ then $\gamma \in Q$ by Lemma 11.17(i). Now let $\nu < \gamma$.
1. $\gamma =_{nf} \gamma_1 + \ldots + \gamma_n$ By the induction hypothesis we get $\gamma_1, \ldots, \gamma_n \in Q$ and hence $\gamma \in Q$ by Lemma 11.13.
2. $\gamma =_{nf} \varphi\xi\eta$. By the induction hypothesis we get $\xi, \eta \in Q$ and hence $\gamma \in Q$ by Lemma 11.15.
3. $\gamma =_{nf} \Phi\xi\eta$. Then we would have $\gamma \leq \nu$ since $\gamma < \nu^+$, but this we ruled out. So this case cannot occur.
4. $\gamma =_{nf} \psi\nu\eta$. Then $\eta < \alpha$. By Lemma 11.7(i), $\gamma \in \mathrm{M}^Q_\nu$ entails that

$$(\forall \tau \in Q \cap \nu)(\forall \beta \in \mathrm{SC}_\nu(\eta))\ \mathrm{SC}_\tau(\beta) \subseteq \mathrm{SC}_\tau(\eta) \subseteq Q.$$

Since $\mathrm{SC}_\nu(\eta) < \psi\nu\eta < \psi\nu\alpha$, the latter entails that $\mathrm{SC}\nu(\eta) \subseteq \mathrm{M}^Q_\nu \cap \psi\nu\alpha$, and therefore, by the induction hypothesis, $\mathrm{SC}_\nu(\eta) \subseteq Q$. As a result we have shown that

$$(\forall \tau \leq \nu)[\tau \in Q \cap \mu \to \mathrm{SC}_\tau(\eta) \subseteq Q]. \tag{43}$$

Via a subsidiary induction on Q we shall show that

$$(\forall \tau \in Q \cap \mu)\, \mathrm{SC}_\tau(\eta) \subseteq Q. \tag{44}$$

Let $\tau \in Q \cap \mu$. In view of (43) we may assume that $\nu < \tau$. The subsidiary induction hypothesis yields $(\forall \tau' \in Q \cap \tau)\, \mathrm{SC}_{\tau'}(\eta) \subseteq Q$, which implies $\mathrm{SC}_\tau(\eta) \subseteq \mathrm{M}^Q_\tau$. Since $\nu < \tau$, $\mathrm{K}_\nu \eta < \eta$ and $\mathrm{K}_\nu \alpha < \alpha$ hold, we conclude that $\mathrm{K}_\tau \eta < \eta$, $\mathrm{K}_\tau \alpha < \alpha$ and $\mathrm{SC}_\tau(\eta) < \psi\tau\eta < \psi\tau\alpha$. Therefore we have $\mathrm{SC}_\tau(\eta) \subseteq \mathrm{M}^Q_\tau \cap \psi\tau\alpha$ and consequently, by applying the main induction hypothesis, $\mathrm{SC}_\tau(\eta) \subseteq Q$. This completes the proof of (44).

Finally, from (44) we conclude that $\eta \in \mathrm{M}^Q_\mu \cap \alpha \subseteq \mathfrak{B}^Q_\mu$, yielding $\gamma = \psi\nu\eta \in Q$. □

Lemma 11.20 $\mathrm{Prg}(\mathrm{M}^Q_\mu, \mathfrak{B}^Q_\mu)$.

Proof. Let $\alpha \in \mathrm{M}^Q_\mu$ and $\mathrm{M}^Q_\mu \cap \alpha \subseteq \mathfrak{B}^Q_\mu$. We have to show $\alpha \in \mathfrak{B}^Q_\mu$. So suppose $\nu \in Q \cap \mu$ and $\mathrm{K}_\nu \alpha < \alpha$. By Lemma 11.19 we have $\mathrm{M}^Q_\nu \cap \psi\nu\alpha \subseteq \mathrm{W}^Q_\nu$. For $\tau \in Q \cap \nu$ it holds $\mathrm{SC}_\tau(\psi\nu\alpha) = \mathrm{SC}_\tau(\nu) \cup \mathrm{SC}_\tau(\alpha)$ and therefore, using Lemma 11.7(ii), $\mathrm{SC}_\tau(\psi\nu\alpha) \subseteq Q$ since $\nu \in Q$ and $\alpha \in \mathrm{M}^Q_\mu$. Thus $\psi\nu\alpha \in \mathrm{M}^Q_\nu$, so that by Lemma 11.4(viii) we have $\psi\nu\alpha \in \mathrm{W}^Q_\nu \subseteq Q$. This shows $\alpha \in \mathfrak{B}^Q_\mu$. □

Lemma 11.21 *(i)* $\alpha, \nu \in Q \wedge \mathrm{K}_\nu \alpha < \alpha \to \psi\nu\alpha \in Q$.

(ii) $\alpha, \nu \in \mathfrak{W} \wedge \mathrm{K}_\nu \alpha < \alpha \to \psi\nu\alpha \in \mathfrak{W}$.

Proof. (ii) is a consequence of (i). For (i), let $\tau := \max\{\mathrm{S}\alpha, \mathrm{S}\nu\}$ and $\mu := \tau^+$. By Lemmata 11.20 and 11.4(i), we have $\mathrm{W}^Q_\mu \subseteq \mathfrak{B}^Q_\mu$. Therefore, since $\tau \in Q$, we have $Q \cap \mu \subseteq \mathfrak{B}^Q_\mu$, and hence $\psi\nu\alpha \in Q$. □

Lemma 11.22 $(\forall j \in U)\mathrm{Ds}(Q_j) \to \mathrm{Ds}(\bigcup\{Q_j \mid j \in U\})$.

Proof. Suppose $\mathrm{Ds}(Q_j)$ holds for all $j \in U$. Using arithmetical comprehension,

$$Z := \bigcup\{Q_j \mid j \in U\}$$

is a set. If $\alpha \in Z$ there exists $j \in U$ such that $\alpha \in Q_j$, thus $\mathrm{S}\alpha \in Q_j \subseteq Z$, showing that Z satisfies (D1). To verify (D2), suppose $\mu \in Z$. Then $\mu \in Q_i$ for some $i \in U$. Owing to Proposition 11.12 it follows that

$$\mathfrak{W} \cap \mu^+ = Q_i \cap \mu^+ \subseteq Z \cap \mu^+ \subseteq \mathfrak{W} \cap \mu^+,$$

and thus $Q_i \cap \mu^+ = Z \cap \mu^+$. By applying Lemma 11.4(vii), we see that $\mathrm{W}^Z_\mu = \mathrm{W}^{Q_i}_\mu = Q_i \cap \mu^+ = Z \cap \mu^+$. □

12 Well-ordering proofs in Π^1_1-$\mathbf{TR}_0$, Π^1_1-$\mathbf{TR} + \Delta^1_2$-$\mathbf{CA}$ and Δ^1_2-$\mathbf{CA} + \mathrm{BR}(\text{impl-}\Sigma^1_2)$.

Lemma 12.1 $\nu < \mathrm{S}\alpha \to \mathrm{SC}_\nu(\mathrm{S}\alpha) \subseteq \mathrm{SC}_\nu(\alpha)$.

Proof. We use induction on $\mathrm{G}\alpha$.
1. If $\alpha =_{nf} \alpha_1 + \ldots + \alpha_n$ or $\alpha =_{nf} \varphi\xi\beta$ the assertion follows immediately from the induction hypothesis.
2. $\alpha =_{nf} \psi\mu\beta$. Then $\mathrm{S}\alpha = \mu$ and $\mathrm{SC}_\nu(\mu) \subseteq \mathrm{SC}_\nu(\alpha)$.
3. $\alpha =_{nf} \Phi\xi\beta$. Then $\mathrm{S}\alpha = \alpha$. □

Proposition 12.2 Π^1_1-$\mathbf{TR}_0 \vdash \forall\alpha(\alpha \in \mathfrak{W} \to \Omega_\alpha \in \mathfrak{W})$.

Proof. We argue informally in Π^1_1-$\mathbf{TR}_0$. Let $\alpha \in \mathfrak{W}$. Then there exists a distinguished set Q such that $\alpha \in Q$. By Lemma 11.6(ii), $<\restriction Q$ is a well-ordering, thus, using (Π^1_1-TR), there exists a set X such that for all $\beta \in Q$,

$$X_\beta = \mathrm{W}^{X_{Q\beta}}_{\Omega_\beta} \cup Q \text{ where } X_{Q\beta} := \bigcup\{X_\eta \mid \eta \in Q \cap \beta\} \text{ and } X_\eta := \{z \mid \langle \eta, z\rangle \in X\}.$$

We now show by induction on Q that for all $\beta \in Q$,

$$\Omega_\beta \in X_\beta \wedge \mathrm{Ds}(X_\beta) \wedge X_{Q\beta} \subseteq X_\beta. \tag{45}$$

Let $\beta \in Q$. The induction hypothesis, in conjunction with Lemma 11.22, yields

$$\mathrm{Ds}(X_{Q\beta}) \wedge (\forall\xi \in Q \cap \beta)\,(\Omega_\xi \in X_{Q\beta}).$$

As $0 \in \mathrm{W}^\emptyset_0 \subseteq Q$, we have $\Omega_0 = 0 \in X_0$ and hence (45) holds when $\beta = 0$. Now let $0 < \beta$. If $\nu \in X_{Q\beta} \cap \Omega_\beta$ we can use Lemma 11.7(ii) to conclude that $\mathrm{SC}_\nu(\Omega_\beta) = \mathrm{SC}_\nu(\beta) \subseteq X_{Q\beta}$ since $\beta \in Q \subseteq X_{Q\beta}$. This shows

$$\Omega_\beta \in \mathrm{M}^{X_{Q\beta}}_{\Omega_\beta}. \tag{46}$$

Now let $\delta \in \mathrm{M}^{X_{Q\beta}}_{\Omega_\beta} \cap \Omega_\beta$ and $\mathrm{S}\delta = \Omega_\sigma$. We want to show $\delta \in X_{Q\beta}$. We may assume that $\beta < \Omega_\beta$ since otherwise we have $\beta = \Omega_\beta$ and thus $\mathrm{M}^{X_{Q\beta}}_{\Omega_\beta} \cap \Omega_\beta = \mathrm{M}^Q_\beta \cap \beta \subseteq Q \subseteq X_{Q\beta}$ using Lemmata 11.11, 11.4(vii) and 11.17(i).

Case 1: $\mathrm{S}\delta \le \mathrm{S}\beta$ or there exists $\xi \in Q \cap \beta$ such that $\mathrm{S}\delta \le \Omega_\xi$. Then, by Corollary 11.16, we obtain $\delta \in X_{Q\beta}$.

Case 2: $(\forall\xi \in Q \cap \beta)(\Omega_\xi < \Omega_\sigma)$ and $\mathrm{S}\beta < \mathrm{S}\delta = \Omega_\sigma$. In this case we have $\mathrm{S}\beta \in X_{Q\beta} \cap \Omega_\beta$, thus, using Lemma 12.1, we arrive at

$$\mathrm{SC}_{\mathrm{S}\beta}(\sigma) = \mathrm{SC}_{\mathrm{S}\beta}(\Omega_\sigma) \subseteq \mathrm{SC}_{\mathrm{S}\beta}(\delta) \subseteq X_{Q\beta},$$

and hence $\mathrm{SC}_{\mathrm{S}\beta}(\sigma) \subseteq X_{Q\beta} \cap (\mathrm{S}\beta)^+$. An application of Lemma 11.11 yields $\mathrm{SC}_{\mathrm{S}\beta}(\sigma) \subseteq Q$, and since $\sigma < \beta$ and $\mathrm{S}\sigma \le \mathrm{S}\beta$ we conclude that $\sigma \in Q \cap \beta$ by employing Lemma 11.16. However, this is an impossibility since we assumed that $(\forall\xi \in Q \cap \beta)(\Omega_\xi < \Omega_\sigma)$. Thus Case 2 is ruled out.

In sum, we have shown that

$$\mathrm{M}_{\Omega_\beta}^{X_{Q\beta}} \subseteq X_{Q\beta}. \tag{47}$$

In view of the Lemmata 11.9 and 11.22 we can deduce $\Omega_\beta \in X_\beta \wedge \mathrm{Ds}(X_\beta)$ from (46 and (47). Moreover, by Lemma 11.22, we have $X_{Q\beta} \cap \Omega_\beta \subseteq \mathrm{W}_{\Omega_\beta}^{X_{Q\beta}}$, and hence

$$X_{Q\beta} = (X_{Q\beta} \cap \Omega_\beta) \cup Q \subseteq X_\beta.$$

This completes the proof of (46). Letting $Z := \bigcup\{X_\beta \mid \beta \in Q\}$, we can use Lemma 11.22 and (46) to conclude that $\mathrm{Ds}(Z)$ and $(\forall \beta \in Q)\,(\Omega_\beta \in Z)$, hence $\Omega_\alpha \in \mathfrak{W}$. □

Corollary 12.3 *Let $\mathfrak{C}[U,\beta,\gamma,Q]$ be the Π^1_1 formula $\gamma \in Q \vee \gamma \in \mathrm{W}^U_{\Omega_\beta}$. Put $\Xi_0 := 1$ and $\Xi_{n+1} := \Omega_{\Xi_n}$. Let $\mathbf{T}$ be the theory Π^1_1-$\mathbf{CA}_0$ plus the additional rule*

$$\frac{\exists! Q\,(F[Q] \wedge \mathrm{Ds}(Q))}{\forall P\,(F[P] \wedge \mathrm{Ds}(P) \rightarrow \exists X (\forall \beta \in P) \forall \gamma (\gamma \in X_\beta \leftrightarrow \mathfrak{C}[X_{P\beta},\gamma,P]))} \tag{48}$$

with the proviso that $F[Q]$ is an arithmetical formula.

For all n we then have

$$\mathbf{T} \vdash \Xi_n \in \mathfrak{W}.$$

Proof. We proceed by metainduction on n. For $n = 0$ this obvious. Let $n = m + 1$. By the the induction hypothesis, we have $\mathbf{T} \vdash \Xi_m \in \mathfrak{W}$. Let $\mu := \Xi_m$. Arguing in $\mathbf{T}$, there exists a distinguished set Q such that $\mu \in Q$ and $Q = Q \cap \mu^+$. Owing to Lemma 11.11, Q is uniquely determined via this description. Thus $\exists! P\, F[P]$, where $F[P] := (\mu \in P \wedge P \cap \mu^+ = P)$. Since μ can be described via an arithmetical formula, too, we can use the above rule to infer that there exists a set X such that $(\forall \beta \in Q)\forall \gamma(\gamma \in Z_\beta \leftrightarrow \mathfrak{C}[X_{Q\beta},\gamma,Q])$. Inspection of the proof of Proposition 12.2 shows that the existence of X is what is needed to conclude that $\Omega_\mu \in \mathfrak{W}$, i.e. $\Xi_n \in \mathfrak{W}$. □

Corollary 12.4 *For all n, Δ^1_2-$\mathbf{CA} + \mathrm{BR}(\text{impl-}\Sigma^1_2) \vdash \Xi_n \in \mathfrak{W}$.*

Proof. As a corollary of the proof of Theorem 6.14 one has that the theorems of Δ^1_2-$\mathbf{CA}$ + $\mathrm{BR}(\text{impl-}\Sigma^1_2)$ are closed under the inference rule (48). Thus, by Corollary 12.3, the claim is true. □

Lemma 12.5 *Let $\mathbf{T}^*$ be the theory $\mathbf{KPl}^r$ augmented by the rule*

$$\frac{\exists! \alpha\, A[\alpha]}{\forall \beta \forall x\,(A[\beta] \rightarrow \exists f\, \mathfrak{D}_0[x,\beta,f])} \tag{49}$$

for every Σ formula $A[\beta]$ and $\mathfrak{D}_0[x,\beta,f]$ be defined as in Lemma 5.3.

With $\mathbf{T}$ being the theory of Corollary 12.3 we then have

$$\mathbf{T} \subseteq \mathbf{T}^*.$$

(To avoid possible confusion I hasten to remark that quantifiers $\forall\beta, \exists\beta, \ldots$ in theories with language $\mathcal{L}_2$ are still supposed to range over $\mathrm{OT}(\Phi)$ while the same quantifiers in the context of $\mathcal{L}^*$-theories are supposed to range over set-theoretic ordinals.)

Proof. It is easy to show that Π^1_1-$\mathbf{CA}_0 \subseteq \mathbf{KPl}^r$: By Lemma 2.5, more precisely (8), Π^1_1 formulae are equivalent to formulae saying that certain arithmetical relations (which may contain set parameters) are well-founded, and thus, by Theorem 5.6, they are Δ_1 on any admissible set which houses the parameters of this formula. Therefore in $\mathbf{KPl}^r$ one has comprehension for Π^1_1 formulas. (see Theorem 5.6). So it suffices to establish the closure of the $\mathbf{T}^*$-provable formulae under the rule (48) (modulo the *-translation). Suppose

$$\mathbf{T}^* \vdash (\exists! Q(F[Q] \wedge \mathrm{Ds}(Q)))^*.$$

Since Π^1_1 formulae are provably Δ_1 in $\mathbf{KPl}^r$ and the formula $\mathrm{Ds}(Q)$ is arithmetical in Π^1_1, $\mathrm{Ds}(Q)$ is provably Δ_1 in $\mathbf{KPl}^r$. Moreover, by Theorem 5.6, Q is order-isomorphic to an ordinal α which will then have a provable Σ_1 definition in $\mathbf{T}^*$. By rule (49) there exist a function f with $\mathfrak{D}_0[Q, \alpha, f]$. Picking an admissible set K with $Q, \alpha, f \in K$, we can now proceed as in the proof of Lemma 5.8 to arrive at the conclusion of the rule (49). □

Adding Δ^1_2-$\mathbf{CA}$ to Π^1_1-$\mathbf{TR}$ enables to show that much bigger ordinals belong to $\mathfrak{W}$.

Lemma 12.6 Π^1_1-$\mathbf{TR} + \Delta^1_2$-$\mathbf{CA} \vdash (\forall \delta < \psi 00)\, [\Phi 1 \delta \in \mathfrak{W} \rightarrow \Phi 1(\delta + 1) \in \mathfrak{W}]$.

Proof. Let $\delta < \psi 00$ and suppose that $\Phi 1 \delta \in \mathfrak{W}$. By employing arithmetical comprehension there exists a function $f : \mathbb{N} \longrightarrow \mathrm{OT}(\Phi)$ such that $f(0) = \Phi 1 \delta$ and $f(k+1) = \Omega_{f(k)}$. Using Proposition 12.2 and (IND) we obtain

$$(\forall k \in \mathbb{N}) \exists X\, [\mathrm{Ds}(X) \wedge f(k) \in X \wedge f(k) < \Phi 1(\delta + 1)]. \tag{50}$$

Since by Lemma 6.11 (Σ^1_2-AC) is available in our background theory, we may infer from (50) the existence of a set Y such that

$$(\forall k \in \mathbb{N})[\mathrm{Ds}(Y_k) \wedge f(k) \in Y_k].$$

Letting $Z := \bigcup \{Y_k \mid k \in \mathbb{N}\}$ (which is a set by arithmetical comprehension), we conclude with the help of Lemma 11.22 that Z is a distinguished set. Using induction on $\mathrm{G}\alpha$ one easily establishes that

$$(\forall \alpha < \Phi 1(\delta + 1))(\exists k \in \mathbb{N})\, \alpha < f(k). \tag{51}$$

Using (Π^1_1-CA), $U := \mathrm{W}^Z_{\Phi 1(\delta+1)}$ is a set.

If $\nu \in Z \cap \Phi 1(\delta + 1)$ then $\mathrm{SC}_\nu(\Phi 1(\delta + 1)) = \mathrm{SC}_\nu(1) \cup \mathrm{SC}_\nu(\delta + 1) = \emptyset$, and therefore $\Phi 1(\delta + 1) \in \mathrm{M}^Z_{\Phi 1(\delta+1)}$. If $\beta \in \mathrm{M}^Z_{\Phi 1(\delta+1)} \cap \Phi 1(\delta + 1)$ then, by (51), there exists $\nu \in Z \cap \Phi 1(\delta + 1)$ with $\mathrm{S}\beta \leq \nu$, whence, by Corollary 11.16(i), $\beta \in Z$. Thus, in the light of Proposition 11.9, the foregoing observations show that $\Phi 1(\delta + 1) \in U$ and $\mathrm{Ds}(U)$, whence $\Phi 1(\delta + 1) \in \mathfrak{W}$. □

Lemma 12.7 *Let* $\omega_0 := \varphi 00$, $\omega_{n+1} := \varphi 0 \omega_n$ *and* $varepsilon_0 := \varphi 10$. *Then, for all* $n < \omega$,

$$\Pi^1_1\text{-}\mathbf{TR} + \Delta^1_2\text{-}\mathbf{CA} \vdash (\forall \alpha < \omega_n)\, \Phi 1 \alpha \in \mathfrak{W}.$$

Proof. For every (meta) n,

$$\mathbf{ACA} \vdash (\forall\alpha < \omega_n)[(\forall\delta < \alpha)F(\delta) \to F(\alpha)] \to (\forall\alpha < \omega_n)\, F(\alpha)$$

for every $\mathcal{L}_2$ formula $F(\alpha)$.

Therefore it suffices to infer $\Phi 1\alpha \in \mathfrak{W}$ from the assumptions $\alpha < \omega_n$ and $(\forall\delta < \alpha)\, \Phi 1\delta \in \mathfrak{W}$.

If $\alpha = \gamma+1$ for some γ then $\Phi 1\alpha \in \mathfrak{W}$ is a consequence of 12.6. For $\alpha = 0$ note that $\Phi 10 \in \mathfrak{W}$ holds by employing a modification of the proof of 12.6 whereby one defines $f : \mathbb{N} \longrightarrow \mathrm{OT}(\Phi)$ by $f(0) = \Omega_1$ and $f(k+1) = \Omega_{f(k)}$.

Now assume that α is a limit. By assumption we have $(\forall\delta < \alpha)\exists X\, (\Phi 1\delta \in X \wedge \mathrm{Ds}(X))$. Applying ($\Sigma^1_2$-AC) we find a set Y such that

$$(\forall\delta < \alpha)[\Phi 1\delta \in Y_\delta \wedge \mathrm{Ds}(Y_\delta)].$$

Letting $Z := \bigcup\{Y_\delta \mid \delta < \alpha\}$ and $U := \mathrm{W}^Z_{\Phi 1\alpha}$, 11.22 tells us that Z is a distinguished set. For $\nu \in Z \cap \Phi 1\alpha$ we have $\mathrm{SC}_\nu(\Phi 1\alpha) = \emptyset$ as $\alpha < \psi 00$; and hence $\Phi 1\alpha \in \mathrm{M}^Z_{\Phi 1\alpha}$. For every $\beta \in \mathrm{M}^Z_{\Phi 1\alpha} \cap \Phi 1\alpha$ there exists $\gamma < \alpha$ with $\mathrm{S}\beta \leq \Phi 1\gamma$, and thus, using 11.16(i), it follows that $\beta \in Z$. Thus, applying 11.9, the foregoing yields that $\Phi 1\alpha \in U \wedge \mathrm{Ds}(U)$, thereby verifying $\Phi 1\alpha \in \mathfrak{W}$. □

Lemma 12.8 *For $\alpha \in \mathrm{OT}(\Phi)$ let $<_\alpha$ be the restriction of $<$ to ordinals $< \alpha$, i.e. $\beta <_\alpha \gamma \Leftrightarrow \beta < \gamma < \alpha$. We shall write $\mathrm{WO}(\alpha)$ rather than $\mathrm{WO}(<_\alpha)$. Then:*

$$\Pi^1_1\text{-}\mathbf{CA}_0 \vdash \alpha \in \mathfrak{W} \wedge \alpha < \Omega_1 \to \mathrm{WO}(\alpha).$$

Proof. Let $\alpha \in \mathfrak{W} \cap \Omega_1$. Then there exists a distinguished set Q such that $\alpha \in Q \cap \Omega_1$. Since $\mathrm{S}\alpha = 0 \in Q$, it follows that $\alpha \in Q \cap 0^+ = \mathrm{W}[\{\eta \mid \eta < \Omega_1\}]$, and hence $\mathrm{WO}(\alpha)$. □

Lemma 12.9 *With Ξ_n being defined as in 12.3, the following hold:*

(i) $\Xi_n < \Xi_{n+1}$ *and* $\mathrm{K}_0\Xi_n = \emptyset$, *hence* $\psi 0\Xi_n \in \mathrm{OT}(\Phi)$.

(ii) For every $\alpha < \Phi 10$ *there exists* n *such that* $\Xi_n > \alpha$.

(iii) For every $\beta < \psi 0(\Phi 10)$ *there exists* n *such that* $\beta < \psi 0\Xi_n$.

Proof. (i) can be easily shown by induction on n. (ii) follows by induction on $\mathrm{G}\alpha$, while (iii) follows from (ii) using induction on $\mathrm{G}\beta$. □

Definition 12.10 Let $\mathbf{T}$ be a theory whose language is $\mathcal{L}_2$ or $\mathcal{L}^*$. We say that an ordinal α is *provable in* $\mathbf{T}$ if there exists a primitive recursive well-ordering whose order-type is α such that $\mathbf{T} \vdash \mathrm{WO}(\prec)$.

The *proof-theoretic ordinal* of $\mathbf{T}$ is the least ordinal not provable in $\mathbf{T}$, or, equivalently, it is the supremum of the provable ordinals of $\mathbf{T}$. We denote this ordinal by $|\mathbf{T}|$.

Theorem 12.11 *(i)* $\psi 0(\Phi 10) \leq |\Pi^1_1\text{-}\mathbf{TR}_0|$.

(ii) $\psi 0(\Phi 10) \leq |\Delta^1_2\text{-}\mathbf{CA} + \mathrm{BR}(\text{impl-}\Sigma^1_2)|$.

(iii) Letting $\mathbf{T}$ *be any of the theories of 12.3 or 12.5 it holds that* $\psi 0(\Phi 10) \leq |T|$.

(iv) $\psi 0(\Phi 1\varepsilon_0) \leq |\Pi^1_1\text{-}\mathbf{TR} + \Delta^1_2\text{-}\mathbf{CA}|$.

Proof. (i) follows from 12.2, 12.9 and 12.8. (ii) is a consequence of 12.4, 12.9 and 12.8. (iii) follows from 12.3 and 12.5 using 12.9 and 12.8. (iv) is a consequence of 12.7, 12.9 and 12.8. □

13 Well-ordering proofs in Π^1_1-TR and Π^1_1-TR + (BI)

Building on 12.2, we will prove lower bounds for the theories mentioned in this section's title. We will also use techniques which were developed in [6] and [7], paragraph 13.

Using (BI) we can strengthen 11.6 as follows.

Lemma 13.1 *For every* $\mathcal{L}_2$ *formula* $F(a)$,

$$\Pi^1_1\text{-}\mathbf{CA} + (\mathrm{BI}) \vdash \mathrm{Prg}(\mathfrak{W}, F) \to \mathfrak{W} \subseteq F.$$

Proof. By 11.6 we have $\forall X(\mathrm{Prg}((Q, X) \to Q \subseteq X)$, which in the presence of (BI) yields $\mathrm{Prg}(Q, F) \to Q \subseteq F$ for every $\mathcal{L}_2$ formula $F(a)$ (cf. [15, Lemma 1.6.3]). Assuming $\mathrm{Prg}(\mathfrak{W}, F)$ and $\alpha \in \mathfrak{W}$, we use 11.12 to infer the existence of a distinguished set P with $\alpha \in P$ and $\mathfrak{W} \cap \alpha^+ = P \cap \alpha^+$. Therefore we have $\mathrm{Prg}(P, F)$, so $P \subseteq F$, and thence $F(\alpha)$. □

With the help of 13.1 we can strengthen some of the results of section 11. Using (BI), the proof of 11.19 carries over to $\mathfrak{W}$, yielding the following strengthening of 11.20.

Lemma 13.2 $\Pi^1_1\text{-}\mathbf{CA} + (\mathrm{BI}) \vdash \mathrm{Prg}(\mathrm{M}^{\mathfrak{W}}_\mu, \mathfrak{B}^{\mathfrak{W}}_\mu)$.

For the next Lemma the employment of (IND) is crucial.

Lemma 13.3 $\Pi^1_1\text{-}\mathbf{TR} \vdash \mathrm{M}^{\mathfrak{W}}_{\Phi 10} \cap \Phi 10 = \mathfrak{W} \cap \Phi 10$.

Proof. Let f be the primitive recursive function $f : \omega \longrightarrow \mathrm{OT}(\Phi)$ defined by $f(0) = 1$ and $f(k+1) = \Omega_{f(k)}$. With help of (IND), 12.2 and 12.9(ii) yield

$$(\forall k < \omega)\, f(k) \in \mathfrak{W} \wedge (\forall \alpha < \Phi 10)(\exists k < \omega)\, \alpha < f(k). \tag{52}$$

Let $\xi \in \mathrm{M}^{\mathfrak{W}}_{\Phi 10} \cap \Phi 10$. Then, according to (52), there exists $k < \omega$ with $\mathrm{S}\xi \leq f(k)$. By 11.16 we then get $\xi \in \mathfrak{W} \cap \Phi 10$. Conversely, if $\xi, \mu \in \mathfrak{W} \cap \Phi 10$ we have $\mathrm{SC}_\mu(\xi) \subseteq \mathfrak{W}$ by 11.7(ii), whence $\xi \in \mathrm{M}^{\mathfrak{W}}_{\Phi 10} \cap \Phi 10$. □

Definition 13.4 By $\mathfrak{I}(U, \alpha)$ we shall refer to the schema

$$\mathrm{Prg}(U, F) \to \alpha \in U \wedge U \cap \alpha \subseteq F$$

where $F(a)$ is an arbitrary formula of $\mathcal{L}_2$.

Lemma 13.5 Π^1_1-**TR** + (BI) $\vdash \mathfrak{I}(\mathrm{M}^{\mathfrak{W}}_{\Phi 10}, (\Phi 10) + 1)$.

Proof. Let $X := \mathrm{M}^{\mathfrak{W}}_{\Phi 10}$ and $\tau := \Phi 10$. According to 13.3 we have $X \cap \tau = \mathfrak{W} \cap \tau$ which implies

$$\mathrm{Prg}(X, F) \to \mathrm{Prg}(\mathrm{W}, \{\xi \mid \xi < \tau \to F(\xi)\}),$$

and which, with the help of 13.1, implies $\mathrm{Prg}(X, F) \to \mathfrak{W} \cap \tau \subseteq F$. The latter yields

$$\mathrm{Prg}(X, F) \to X \cap \tau \subseteq F.$$

Since also $\tau, \tau + 1 \in X$, the desired assertion follows. □

Definition 13.6 For every formula $F(a)$ we define the "Gentzen jump"

$$F^j(\gamma) \;:=\; \forall\delta\, [\mathrm{M}^{\mathfrak{W}}_{\Phi 10} \cap \delta \subseteq F \to \mathrm{M}^{\mathfrak{W}}_{\Phi 10} \cap (\delta + \omega^\gamma) \subseteq F].$$

Lemma 13.7 *The following are deducible in* Π^1_1-**TR**:

(i) $F^j(\gamma) \to \mathrm{M}^{\mathfrak{W}}_{\Phi 10} \cap \omega^\gamma \subseteq F$.

(ii) $\mathrm{Prg}(\mathrm{M}^{\mathfrak{W}}_{\Phi 10}, F) \to \mathrm{Prg}(\mathrm{M}^{\mathfrak{W}}_{\Phi 10}, F^j)$.

Proof. (i) is obvious. (ii) Let $M := \mathrm{M}^{\mathfrak{W}}_{\Phi 10}$. Then $M \cap (\delta + \omega^\gamma) \subseteq F$ is to proved under the assumptions (a) $\mathrm{Prg}(M, F)$, (b) $\gamma \in M \wedge M \cap \gamma \subseteq F^j$ and (c) $M \cap \delta \subseteq F$. So let $\eta \in M \cap (\delta + \omega^\gamma)$.

1. $\eta < \delta$: Then $F(\eta)$ is a consequence of (c).
2. $\eta = \delta$: Then $F(\eta)$ follows from (c) and (a).
3. $\delta < \eta < \delta + \omega^\gamma$: Then there exist $\gamma_1, \ldots, \gamma_k < \gamma$ such that $\eta = \delta + \omega^{\gamma_1} + \ldots + \omega^{\gamma_k}$ and $\gamma_1 \geq \ldots \geq \gamma_k$. $\eta \in M$ implies $\gamma_1, \ldots, \gamma_k \in M \cap \gamma$. Through applying (b) and (c) we obtain $M \cap (\delta + \omega^{\gamma_1}) \subseteq F$. By iterating this procedure we eventually arrive at $F(\delta + \omega^{\gamma_1} + \ldots + \omega^{\gamma_k})$, so $F(\eta)$ holds.

□

Lemma 13.8 *Let* $\delta_0 := (\Phi 10) + 1$, $\delta_{n+1} := \omega^{\delta_n}$ *and* $M := \mathrm{M}^{\mathfrak{W}}_{\Phi 10}$. *Then:*

$$\Pi^1_1\text{-}\mathbf{TR} + (\mathrm{BI}) \vdash \mathfrak{I}(M, \delta_n).$$

Proof. Proof by meta-induction on n. For $n = 0$ this follows from 13.5. Now let $n = m + 1$. Inductively we have $\mathrm{Prg}(M, F^j) \to F^j(\delta_m)$ for every formula $F(a)$. An application of 13.7 yields $\mathrm{Prg}(M, F) \to M \cap \delta_n \subseteq F$. Since trivially $\delta_n \in M$, we have shown $\mathfrak{I}(M, \delta_n)$. □

Theorem 13.9 $\psi 0 \varepsilon_{(\Phi 10)+1} \leq |\Pi^1_1\text{-}\mathbf{TR} + (\mathrm{BI})|$.

Proof. 13.2 and 13.8 yield $\delta_n \in \mathfrak{B}^{\mathfrak{W}}_{\Phi 10}$, and consequently $\psi 0 \delta_n \in \mathfrak{W}$. Since $\sup\{\psi 0 \delta_n \mid n < \omega\} = \psi 0 \varepsilon_{(\Phi 10)+1}$ the proof is completed. $\square$

We now come to the well-ordering proof for Π^1_1-**TR**. Since (BI) is not available in this theory, 13.1 cannot be exploited to prove that $\mathfrak{I}(\mathrm{M}^{\mathfrak{W}}_{\Phi 10}, \Phi 10)$ holds. However, Π^1_1-**TR** proves $(\forall \alpha < \Phi 10)(\exists k < \omega)(\alpha < f(k) \wedge f(k) \in \mathfrak{W})$ (where f was defined in 13.3), establishing $\psi 0((\Phi 10) \cdot \varepsilon_0)$ as a lower bound for this theory.

Convention: For the remainder of this section we let $\mathfrak{f} := \Phi 10$.

Lemma 13.10 *Multiplication $\alpha \cdot \beta$ of ordinals from* $\mathrm{OT}(\Phi)$ *can be easily defined via the normal forms of α and β. For $\alpha \leq \varepsilon_0$ we have:*

(i) $\mathrm{K}_0(\mathfrak{f} \cdot \alpha) = \emptyset$.

(ii) $\nu < \mathfrak{f} \rightarrow \mathrm{SC}_\nu(\mathfrak{f} \cdot \alpha) = \emptyset$.

(iii) $\beta < \mathfrak{f} \cdot \alpha \rightarrow (\exists \xi < \alpha)(\exists \delta < \mathfrak{f})(\beta = \mathfrak{f} \cdot \xi + \delta)$.

(iv) $\beta < \psi 0(\mathfrak{f} \cdot \varepsilon_0) \rightarrow (\exists \xi < \varepsilon_0)\, \beta < \psi 0(\mathfrak{f} \cdot \xi)$.

Proof. The proofs consist of simple calculations and in (iii) and (iv) involve inductions on $\mathrm{G}\beta$. $\square$

Definition 13.11

$$\begin{aligned}
\mathfrak{H}(\delta) \quad &:= \quad \delta \leq \varepsilon_0 \wedge \\
&\qquad (\forall \mu \in \mathfrak{W} \cap \mathfrak{f})(\forall \eta, \nu \in \mathfrak{W} \cap \mu^+)[\mathrm{K}_\nu \eta < \mathfrak{f} \cdot \delta + \eta \rightarrow \psi\nu(\mathfrak{f} \cdot \delta + \eta) \in \mathfrak{W}], \\
\mathfrak{A}^\delta(\alpha, \mu, \nu) \quad &:= \quad \delta < \varepsilon_0 \wedge \mu \in \mathfrak{W} \cap \mathfrak{f} \wedge \alpha, \nu \in \mathfrak{W} \cap \mu^+ \wedge \mathrm{K}_\nu \alpha < \mathfrak{f} \cdot \delta + \alpha \wedge \\
&\qquad (\forall \eta \in \mathfrak{W} \cap \alpha)(\forall \tau' \in \mathfrak{W} \cap \mu^+)[\mathrm{K}_{\tau'} \eta < \mathfrak{f} \cdot \delta + \eta \rightarrow \psi\tau'(\mathfrak{f} \cdot \delta + \eta) \in \mathfrak{W}].
\end{aligned}$$

Lemma 13.12 Π^1_1-**TR** $\vdash (\forall \xi < \delta)\mathfrak{H}(\xi) \wedge \mathfrak{A}^\delta(\alpha, \mu, \nu) \rightarrow (\forall \gamma \in \mathrm{M}^{\mathfrak{W}}_\nu \cap \psi\nu(\mathfrak{f}\delta + \alpha))(\gamma \in \mathfrak{W})$.

Proof. Assume the antecedent of the implication we have to verify. Let $\gamma \in \mathrm{M}^{\mathfrak{W}}_\nu \cap \psi\nu(\mathfrak{f}\delta + \alpha)$. We shall carry out an induction on $\mathrm{G}\gamma$ in order to show $\gamma \in \mathfrak{W}$, by distinguishing between the different shapes γ might assume. We shall write $\mathfrak{f}\delta$ for $\mathfrak{f} \cdot \delta$.

1. $\gamma \leq \nu$: Then $\gamma \in \mathfrak{W}$ follows from 11.17(ii). Henceforth assume $\gamma > \nu$.

2. $\gamma =_{nf} \gamma_1 + \ldots + \gamma_n$ or $\gamma =_{nf} \varphi\gamma_1\gamma_2$. Then $\gamma_j \in \mathrm{M}^{\mathfrak{W}}_\nu \cap \psi\nu(\mathfrak{f}\delta + \alpha)$ and therefore, by the inductive assumption, $\gamma_j \in \mathfrak{W}$, thus $\gamma \in \mathfrak{W}$ by 11.13 and 11.15, respectively.

3. $\gamma =_{nf} \psi\nu(\mathfrak{f}\delta + \alpha')$ and $\alpha' < \alpha$: Let $\gamma' := \mathfrak{f}\delta + \alpha'$. Since $\gamma \in \mathrm{M}^{\mathfrak{W}}_\nu$, 11.7(i) entails that

$$(\forall \tau \in \mathfrak{W} \cap \nu)(\forall \xi \in \mathrm{SC}_\nu(\gamma'))\, \mathrm{SC}_\tau(\xi) \subseteq \mathfrak{W}.$$

The latter implies $\mathrm{SC}_\nu(\gamma') \subseteq \mathrm{M}^{\mathfrak{W}}_\nu \cap \psi\nu\gamma' \subseteq \mathrm{M}^{\mathfrak{W}}_\nu \cap \psi\nu(\mathfrak{f}\delta + \alpha)$. Thus

$$\mathrm{SC}_\nu(\gamma') \subseteq \mathfrak{W} \tag{53}$$

by the induction hypothesis. Next we show via a subsidiary induction on Q that for every distinguished set Q with $\mu \in Q$,

$$(\forall \tau \in Q \cap \mu^+)\,\mathrm{SC}_\tau(\gamma') \subseteq Q. \tag{54}$$

We shall frequently use the fact that $\mathfrak{W} \cap \mu^+ = Q \cap \mu^+$ holds (by 12.11). If $\tau = \nu$ then this follows from (53). If $\tau < \nu$ then $\mathrm{SC}_\tau(\gamma') \subseteq \mathrm{SC}_\tau(\gamma) \subseteq \mathfrak{W} \cap \mu^+ \subseteq Q$ since $\gamma \in \mathrm{M}_\nu^{\mathfrak{W}}$.

Now assume that $\nu < \tau \le \mu$. Since $\mathrm{SC}_\tau(\gamma') < \psi\tau\gamma' < \psi\tau(\mathfrak{f}\delta + \alpha)$, the subsidiary induction hypothesis yields $\mathrm{SC}_\tau(\gamma') \subseteq \mathrm{M}_\tau^{\mathfrak{W}} \cap \psi\tau(\mathfrak{f}\delta + \alpha)$. Moreover, $\mathrm{K}_\tau\alpha \subseteq \mathrm{K}_\nu\alpha < \mathfrak{f}\delta + \alpha$. Therefore $\mathfrak{A}^\delta(\alpha, \mu, \nu)$ and consequently, by the main induction hypothesis, $\mathrm{SC}_\tau(\gamma') \subseteq \mathfrak{W} \cap \mu^+ \subseteq Q$. This completes the proof of (54). As a result, $(\forall \tau \in \mathfrak{W} \cap \mu^+)\,\mathrm{SC}_\tau(\gamma') \subseteq \mathfrak{W}$. In combination with 11.16 the latter entails $\alpha' \in \mathfrak{W}$. Finally, $\mathfrak{A}^\delta(\alpha, \mu, \nu)$ and $\alpha' \in \mathfrak{W} \cap \alpha$ imply $\gamma \in \mathfrak{W}$.

4. $\gamma =_{nf} \psi\nu(\mathfrak{f}\delta' + \alpha')$, $\delta' < \delta$ and $\alpha' < \mathfrak{f}$: Let $\gamma' := \mathfrak{f}\delta' + \alpha'$. Let Q be a distinguished set. Via a subsidiary induction on Q we shall show that

$$(\forall \tau \in Q \cap \mathfrak{f})\,\mathrm{SC}_\tau(\gamma') \subseteq Q. \tag{55}$$

For $\tau \le \nu$ this follows as in the previous case. Let $\nu < \tau < \mathfrak{f}$. Since $\mathrm{SC}_\tau(\gamma') < \psi\tau\gamma'$ and $\psi\tau\gamma' < \psi\tau(\mathfrak{f}\delta)$ the subsidiary induction hypothesis yields $\mathrm{SC}_\tau(\gamma') \subseteq \mathrm{M}_\tau^{\mathfrak{W}} \cap \psi\tau(\mathfrak{f}\delta)$, so that, owing to $\mathfrak{A}^\delta(0, \tau, \tau)$ and the main induction hypothesis, we arrive at $\mathrm{SC}_\tau(\gamma') \subseteq \mathfrak{W} \cap \tau^+ \subseteq Q$. This concludes the proof of (55).

(55) implies $(\forall \tau \in \mathfrak{W} \cap \mathfrak{f})\,\mathrm{SC}_\tau(\alpha') \subseteq \mathfrak{W}$, thence $\alpha' \in \mathrm{M}_{\mathfrak{f}}^{\mathfrak{W}} \cap \mathfrak{f}$. Via 13.3 we thus infer $\alpha' \in \mathfrak{W}$. Since $\delta' < \delta$ we also have $\mathfrak{H}(\delta')$ and therefore $\gamma \in \mathfrak{W}$.

□

Lemma 13.13 Π_1^1-**TR** $\vdash \delta < \varepsilon_0 \wedge (\forall \xi < \delta)\mathfrak{H}(\xi) \to \mathfrak{H}(\delta)$.

Proof. Assume $\delta < \varepsilon_0$ and $(\forall \xi < \delta)\mathfrak{H}(\xi)$. From $\mathfrak{A}^\delta(\alpha, \mu, \nu)$ and $\alpha, \nu \in Q$ we can infer $\psi\nu(\mathfrak{f}\delta + \alpha) \in \mathrm{M}_\nu^Q$ and with the help of 13.12 also $\mathrm{M}_\nu^Q \cap \psi\nu(\mathfrak{f}\delta + \alpha) \subseteq Q$, and hence $\psi\nu(\mathfrak{f}\delta + \alpha) \in Q$ by 11.4(viii). This shows

$$\mathfrak{A}^\delta(\alpha, \mu, \nu) \to \psi\nu(\mathfrak{f}\delta + \alpha) \in \mathfrak{W}. \tag{56}$$

Let $\mu \in \mathfrak{W} \cap \mathfrak{f}$. We want to show

$$(\forall \alpha, \nu \in \mathfrak{W} \cap \mu^+)\,[\mathrm{K}_\nu\alpha < \mathfrak{f}\delta + \alpha \to \psi\nu(\mathfrak{f}\delta + \alpha) \in \mathfrak{W}]. \tag{57}$$

So let Q be a distinguished set with $\mu \in Q$. Since $\mathfrak{W} \cap \mu^+ = Q \cap \mu^+$ it suffices to show that if $\alpha, \nu \in Q \cap \mu^+$ and $\mathrm{K}_\nu\alpha < \mathfrak{f}\delta + \alpha$ hold true then $\psi\nu(\mathfrak{f}\delta + \alpha) \in Q$. We use induction on Q with α being the variable of induction. By induction hypothesis we then have

$$(\forall \eta \in Q \cap \alpha)(\forall \tau \in Q \cap \mu^+)[\mathrm{K}_\tau\eta < \mathfrak{f}\delta + \eta \to \psi\tau(\mathfrak{f}\delta + \eta) \in Q].$$

But then (56) implies $\psi\nu(\mathfrak{f}\delta + \alpha) \in Q$. □

Theorem 13.14 $\psi 0((\Phi 10)\cdot\varepsilon_0) \le |\Pi^1_1\text{-}\mathbf{TR}|$.

Proof. Given $\beta < \psi 0((\Phi 10)\cdot\varepsilon_0)$ there exists (by 13.10(iv)) ω_n such that $\beta < \psi 0((\Phi 10)\cdot\omega_n)$. Since in Π^1_1-**TR** we have full transfinite induction on the initial segment of ordinals $\le \omega_n$, Lemma 13.13 yields $\Pi^1_1\text{-}\mathbf{TR} \vdash \mathfrak{H}(\omega_n)$. Thus, using 11.21 and 12.8, we obtain

$$\Pi^1_1\text{-}\mathbf{TR} \vdash \mathrm{WO}(\psi 0((\Phi 10)\cdot\omega_n)),$$

which implies $\psi 0((\Phi 10)\cdot\varepsilon_0) \le |\Pi^1_1\text{-}\mathbf{TR}|$. □

14 Well-ordering proofs in Σ^1_2-TRDC$_0$ and Σ^1_2-TRDC.

We start with the key lemma for all of the remaining well-ordering proofs.

Lemma 14.1

$$\Sigma^1_2\text{-}\mathbf{TRDC}_0 \vdash \eta\in\mathfrak{W} \wedge (\forall\xi\in\mathfrak{W}\cap\eta)(\forall\alpha\in\mathfrak{W})(\Phi\xi\alpha\in\mathfrak{W}) \to (\forall\beta\in\mathfrak{W})(\Phi\eta\beta\in\mathfrak{W}).$$

Proof. We shall argue on the basis of $\Sigma^1_2\text{-}\mathbf{TRDC}_0$. Suppose $\eta\in\mathfrak{W}$ and

$$(\forall\xi\in\mathfrak{W}\cap\eta)(\forall\alpha\in\mathfrak{W})(\Phi\xi\alpha\in\mathfrak{W}).$$

Let $\beta\in\mathfrak{W}$. Pick a distinguished set Q with $\eta,\beta\in Q$. For every distinguished set X we then have

$$(\forall\xi\in Q\cap\eta)(\forall\alpha\in X)\exists Y[\mathrm{Ds}(Y) \wedge \Phi\xi\alpha\in Y].$$

Thus, with the help of (Σ^1_2-AC) we find a set U such that

$$(\forall\xi\in Q\cap\eta)(\forall\alpha\in X)[\mathrm{Ds}(U_{\langle\xi,\alpha\rangle}) \wedge \Phi\xi\alpha\in U_{\langle\xi,\alpha\rangle}].$$

Letting

$$U^* := \bigcup\{U_{\langle\xi,\alpha\rangle} \mid \xi\in Q\cap\eta \wedge \alpha\in X\}$$

we have $\mathrm{Ds}(U^*)$ (by 11.22) and also $(\forall\xi\in Q\cap\eta)(\forall\alpha\in X)(\Phi\xi\alpha\in U^*)$. For an arbitrary distinguished set P the foregoing considerations imply that

$$(\forall i<\omega)\forall X\exists Y[(i=0\to Y=P) \wedge \\ (i>0 \wedge \mathrm{Ds}(X) \to [\mathrm{Ds}(Y) \wedge (\forall\xi\in Q\cap\eta)(\forall\alpha\in X)(\Phi\xi\alpha\in Y)])]. \tag{58}$$

By applying (Σ^1_2-TRDC) (in actuality (Σ^1_2-DC) suffices) to (58) we can draw the existence of a set Z satisfying $Z_0 = P$ and for all $i>0$,

$$\mathrm{Ds}(\bigcup\{Z_j \mid j<i\}) \to \mathrm{Ds}(Z_i) \wedge (\forall\xi\in Q\cap\eta)(\forall\alpha\in\bigcup\{Z_j\mid j<i\})\,\Phi\xi\alpha\in Z_i.$$

Induction on i in conjunction with 11.22 yields $\mathrm{Ds}(Z_i)$ for all i. Note that this induction is permissible in our background theory since $\{i<\omega \mid \mathrm{Ds}(Z_i)\}$ is a set by (Δ^1_2-CA). Letting $P^* := \bigcup\{Z_i \mid i<\omega\}$ we have

$$\mathrm{Ds}(P^*) \wedge P\subseteq P^* \wedge (\forall\xi\in Q\cap\eta)(\forall\alpha\in P^*)\,\Phi\xi\alpha\in P^*.$$

Thus we showed that for all $\gamma \in Q$ and for all X there exists Y such that

$$\mathrm{Ds}(X) \quad \rightarrow \quad \exists Z[\mathrm{Ds}(Z) \wedge Q \cup X \subseteq Z \wedge (\forall \xi \in Q \cap \eta)(\forall \alpha \in Z)(\Phi\xi\alpha \in Z) \wedge Y = \mathrm{W}^Z_{\Phi\eta\gamma}].$$

The latter formula is equivalent to a Σ^1_2 formula (using (Σ^1_2-AC)), hence an via an application of (Σ^1_2-TRDC), with $< \cap (Q \times Q)$ being the well-ordering, there exists a set R such that

$$(\forall \gamma \in Q)[\mathrm{Ds}(R_{Q\gamma}) \rightarrow \exists Z[\mathrm{Ds}(Z) \wedge Q \cup R_{Q\gamma} \subseteq Z \wedge \\ (\forall \xi \in Q \cap \eta)(\forall \alpha \in Z)(\Phi\xi\alpha \in Z) \wedge R_\gamma = \mathrm{W}^Z_{\Phi\eta\gamma}]], \tag{59}$$

where $R_{Q\gamma} := \bigcup\{R_\delta \mid \delta \in Q \cap \gamma\}$. By induction on Q we shall show that

$$(\forall \gamma \in Q)[\mathrm{Ds}(R_\gamma) \wedge \Phi\eta\gamma \in R_\gamma]. \tag{60}$$

So assume inductively that $(\forall \delta \in Q \cap \gamma)[\mathrm{Ds}(R_\delta) \wedge \Phi\eta\gamma \in R_\delta]$. This implies $\mathrm{Ds}(R_{Q\gamma})$ and, in view of (59), there exists a set Z satisfying the following:

(a) $\mathrm{Ds}(Z)$;

(b) $Q \cup R_{Q\gamma} \subseteq Z$;

(c) $(\forall \xi \in Q \cap \eta)(\forall \alpha \in Z)(\Phi\xi\alpha \in Z)$;

(d) $(\forall \delta \in Q \cap \gamma)\Phi\eta\delta \in Z$;

(e) $R_\gamma = \mathrm{W}^Z_{\Phi\eta\gamma}$.

If $\gamma = \Phi\eta\gamma$ we have $\Phi\eta\gamma = \gamma \in Z \cap \gamma^+ = \mathrm{W}^Z_\gamma = R_\gamma$, which implies $\mathrm{Ds}(R_\gamma)$ and $\Phi\eta\gamma \in R_\gamma$.

Next assume that $\gamma < \Phi\eta\gamma$. If $\nu \in Z \cap \Phi\eta\gamma$ then $\mathrm{SC}_\nu(\Phi\eta\gamma) \subseteq \mathrm{SC}_\nu(\eta) \cup \mathrm{SC}_\nu(\gamma) \subseteq Z$ by 11.7(ii) since $\eta, \gamma \in Q \subseteq Z$. Therefore we have

$$\Phi\eta\gamma \in \mathrm{M}^Z_{\Phi\eta\gamma}. \tag{61}$$

We will also show that

$$\mathrm{M}^Z_{\Phi\eta\gamma} \cap \Phi\eta\gamma \subseteq Z. \tag{62}$$

Let $\rho \in \mathrm{M}^Z_{\Phi\eta\gamma} \cap \Phi\eta\gamma$. We shall employ induction on $\mathrm{G}\rho$ to show that $\rho \in Z$. If $\rho \notin \mathrm{SC}$ then $\rho \in Z$ follows from the inductive assumption by means of 11.13 and 11.15. Now suppose $\rho \in \mathrm{SC}$. If there exists $\nu \in Z \cap \Phi\eta\gamma$ with $\mathrm{S}\rho \leq \nu$ then $\mathrm{SC}_\nu(\rho) = \{\rho\} \subseteq Z$. Thus, in addition, we may assume that

$$\rho \in \mathrm{SC} \wedge (\forall \nu \in Z \cap \Phi\eta\gamma)(\nu < \mathrm{S}\rho). \tag{63}$$

We will distinguish several cases.

1. $\rho =_{nf} \psi\mu\zeta$: Then we have $\mu \in \mathrm{M}^Z_{\Phi\eta\gamma} \cap \Phi\eta\gamma$ by (63) since $\mu = \mathrm{S}\rho$. Applying the induction hypothesis we obtain $\mu \in Z$ which contradicts (63). Thus this case is ruled out.

2. $\rho =_{nf} \Phi\zeta\sigma$: Then (63) in conjunction with the induction hypothesis yields $\zeta, \sigma \in Z$.

   (i) $\zeta < \eta$: Then we have $\zeta \in \mathfrak{W} \cap \eta = Q \cap \eta$ by 11.12 since $\eta \in Q$. Whence $\rho \in Z$ holds owing to (c).

   (ii) $\zeta = \eta$ and $\sigma < \gamma$: Then, using (d), from $\sigma \in \mathfrak{W} \cap \gamma = Q \cap \gamma$ we obtain $\rho \in Z$.

   (iii) $\eta < \zeta$: In this case $\rho < \gamma$ must hold. Since $\gamma < \Phi\eta\gamma$ holds by assumption, $\rho \in Z$ follows with the aid of 11.16(i) since in this case we have $\mathrm{S}\gamma \in Q \cap \Phi\eta\gamma \subseteq Z \cap \Phi\eta\gamma$.

This completes the proof of (62). Applying (61), (62) and (e) in conjunction with 11.9, we conclude that $\mathrm{Ds}(R_\gamma) \wedge \Phi\eta\gamma \in R_\gamma$, thereby finishing the poof of (60). Finally, since $\beta \in Q$, (60) enables us to conclude that $\Phi\eta\beta \in \mathfrak{W}$. □

Corollary 14.2 *For any (meta) n, Σ^1_2-$\mathbf{TRDC}_0 \vdash (\forall\alpha \in \mathfrak{W})\, \Phi n\alpha \in \mathfrak{W}$.*

Proof. Use meta-induction on n. 12.2 yields the induction base while 14.1 provides the induction step. □

Corollary 14.3 *For any (meta) n, Σ^1_2-$\mathbf{TRDC} \vdash (\forall\xi \leq \omega_n)(\forall\alpha \in \mathfrak{W})\, \Phi\xi\alpha \in \mathfrak{W}$.*

Proof. In Σ^1_2-$\mathbf{TRDC}$ one has full induction for arbitrary formulae over any segment ω_n. Thus the assertion follows from 14.1. □

Theorem 14.4 *(i) $\psi 0(\Phi\omega 0) \leq |\Sigma^1_2$-$\mathbf{TRDC}_0|$.*

(ii) $\psi 0(\Phi\varepsilon_0 0) \leq |\Sigma^1_2$-$\mathbf{TRDC}|$.

Proof. (i) and (ii) are consequences of 14.2 and 14.3, respectively, by also enlisting the help of 11.21 and 12.8. □

15 Well-ordering proofs in Σ^1_2-TRDC + BR and Σ^1_2-TRDC + BR(impl-Σ^1_2).

Definition 15.1 Let $\boldsymbol{\vartheta}_0 := \Omega_1$, $\boldsymbol{\zeta}_0 := \psi 0\boldsymbol{\vartheta}_0$, $\boldsymbol{\vartheta}_{n+1} := \Phi\boldsymbol{\zeta}_n 0$, $\boldsymbol{\zeta}_{n+1} := \psi 0\boldsymbol{\vartheta}_{n+1}$.

Lemma 15.2 *(i) For all n: $\mathrm{K}_0\boldsymbol{\vartheta}_n < \boldsymbol{\vartheta}_n$, $\boldsymbol{\vartheta}_n < \boldsymbol{\vartheta}_{n+1}$ and $\boldsymbol{\zeta}_n =_{nf} \psi 0\boldsymbol{\vartheta}_n$.*

(ii) For every $\alpha < \Phi\Omega_1 0$ there exists n such that $\alpha < \boldsymbol{\vartheta}_n$.

(iii) For every $\beta < \psi 0(\Phi\Omega_1 0)$ there exists n such that $\beta < \boldsymbol{\zeta}_n$.

Proof. We show (i) by induction on n. This is obvious when $n = 0$. Let $n = m + 1$. By the induction hypothesis we have $\mathrm{K}_0\boldsymbol{\vartheta}_n = \mathrm{K}_0\boldsymbol{\zeta}_m = \{\boldsymbol{\vartheta}_m\} \cup \mathrm{K}_0\boldsymbol{\vartheta}_m < \boldsymbol{\vartheta}_n$, and consequently $\boldsymbol{\zeta}_n =_{nf} \psi 0\boldsymbol{\vartheta}_n$, $\boldsymbol{\zeta}_m < \boldsymbol{\zeta}_n$ and $\boldsymbol{\vartheta}_n = \Phi\boldsymbol{\zeta}_m 0 < \Phi\boldsymbol{\zeta}_n 0 = \boldsymbol{\vartheta}_{n+1}$.

(ii): We use induction on $\mathrm{G}\alpha$. First suppose $\alpha =_{nf} \Phi\xi\eta$. Then, by induction hypothesis, there exist $n, n' < \omega$ such that $\xi < \boldsymbol{\vartheta}_n$ and $\eta < \boldsymbol{\vartheta}_{n'}$. Letting $k := \max(n, n') + 1$ it follows by (i) that $\alpha < \boldsymbol{\vartheta}_k$. In all other cases the assertion follows directly from the induction hypothesis.

(iii) is easily shown by induction on $\mathrm{G}\beta$ making use of (ii). □

Lemma 15.3 *For all (meta) n, Σ^1_2-$\mathbf{TRDC}+\mathrm{BR}\vdash \boldsymbol{\zeta}\in\mathfrak{W}$.*

Proof. We use (meta) induction on n. For $n=0$ this is a consequence of 12.2 and 11.21. If $n=m+1$ then the induction hypothesis yields that $\boldsymbol{\zeta}_m\in\mathfrak{W}$ is deducible in the theory and therefore, by 12.8, $\mathrm{WO}(\boldsymbol{\zeta}_m)$ holds. The segment below $\boldsymbol{\zeta}_m$ is then a primitive recursive provable well-ordering, thus an application of BR yields $\Phi\boldsymbol{\zeta}_m 0=\boldsymbol{\vartheta}_n\in\mathfrak{W}$. Consequently, using 15.2 and 11.21, we have the derivability of $\psi 0\boldsymbol{\vartheta}_n=\boldsymbol{\zeta}_n\in\mathfrak{W}$. □

Lemma 15.4 *For all (meta) n, Σ^1_2-$\mathbf{TRDC}+\mathrm{BR}(\text{impl-}\Sigma^1_2)\vdash \boldsymbol{\rho}_n\in\mathfrak{W}$, where $\boldsymbol{\rho}_0:=\Phi 00$ and $\boldsymbol{\rho}_{m+1}:=\Phi\boldsymbol{\rho}_m 0$.*

Proof. We use (meta) induction on n. Let's denote the above theory by $\mathbf{T}$. The case $n=0$ follows from 12.2. Let $n=m+1$. The induction hypothesis yields $\mathbf{T}\vdash\boldsymbol{\rho}_m\in\mathfrak{W}$. Owing to 11.11, provably in $\mathbf{T}$ there exists a distinguished set Q such that $\boldsymbol{\rho}_m\in Q$ and $Q=Q\cap\boldsymbol{\rho}_m^+$. With the formula

$$F[U]\ :=\ \exists P\,[\mathrm{Ds}(P)\wedge\boldsymbol{\rho}_m\in P\wedge U=\{\langle\alpha,\beta\rangle\mid\alpha,\beta\in P\wedge\alpha<\beta\wedge\beta<\boldsymbol{\rho}_m^+\}]$$

it thus holds that

$$\mathbf{T}\ \vdash\ \exists! X(\mathrm{WO}(X)\wedge F[X]).\tag{64}$$

Let $G(\xi):=(\forall\alpha\in\mathfrak{W})\,\Phi\xi\alpha\in\mathfrak{W}$ and $\boldsymbol{\tau}:=\boldsymbol{\rho}_m^+$. Since $F[U]$ is (provably in $\mathbf{T}$) equivalent to a Σ^1_2 formula, via an application of $\mathrm{BR}(\text{impl-}\Sigma^1_2)$ to (64), $\mathbf{T}$ proves transfinite induction on $\mathfrak{W}\cap\boldsymbol{\tau}$. In particular,

$$\mathbf{T}\ \vdash\ \mathrm{Ds}(Q)\wedge\boldsymbol{\rho}_m\in Q\wedge(\forall\eta\in Q\cap\boldsymbol{\tau})[Q\cap\eta\subseteq G\to G(\eta)]\to(\forall\eta\in Q\cap\boldsymbol{\tau})G(\eta).\tag{65}$$

In conjunction with the induction hypothesis and 14.1, (65) implies $\mathbf{T}\vdash\boldsymbol{\rho}_n\in\mathfrak{W}$. □

Theorem 15.5 *(i) $\psi 0(\Phi\Omega_1 0)\le|\Sigma^1_2$-$\mathbf{TRDC}+\mathrm{BR}|$.*

(ii) $\psi 0\Gamma^\Phi_0\le|\Sigma^1_2$-$\mathbf{TRDC}+\mathrm{BR}(\text{impl-}\Sigma^1_2)|$, where $\psi 0\Gamma^\Phi_0:=\mathrm{OT}(\Phi)\cap\Omega_1$.

Proof. (i) follows from 15.3, 15.2(iii), and 12.8. (ii) follows from 15.4, 11.21, 12.8 and 9.8. □

16 Prospectus

The lower bounds for the proof-theoretic ordinals of theories considered in this article turn out to be sharp. Proofs of upper bounds, though, will only appear in the second part of this paper which is devoted to ordinal analysis. We will finish this paper by listing all theories and their proof-theoretic ordinals.

(i) $|\mathbf{ID}_{\prec^*}|\le|\mathbf{ID}^*+(\mathrm{BI})|\le|\mathbf{KPl}^*|\le\psi 0\varepsilon_{(\Phi 0\Omega_1)+1}$.

(ii) $|\Pi^1_1\text{-}\mathbf{TR}_0| = |\mathbf{AUT\text{-}ID}^{pos}_0| = |\mathbf{AUT\text{-}ID}^{mon}_0| = |\Pi^1_1\text{-}\mathbf{TR}_0 + \Delta^1_2\text{-}\mathbf{CA}_0| = |\mathbf{AUT\text{-}KPl}^r| = |\mathbf{AUT\text{-}KPl}^r + \mathbf{KPi}^r| = |\mathbf{KPi}^w + \text{FOUNDR(impl-}\Sigma)| = |\mathbf{KPi}^w + \text{FOUND(impl-}\Sigma)| = |\Delta^1_2\text{-}\mathbf{CA} + \text{BI(impl-}\Sigma^1_2)| = |\Delta^1_2\text{-}\mathbf{CA} + \text{BR(impl-}\Sigma^1_2)| = \psi 0(\Phi 10)$.

(iii) $|\Pi^1_1\text{-}\mathbf{TR}| = |\mathbf{AUT\text{-}ID}^{pos}| = |\mathbf{AUT\text{-}ID}^{mon}| = |\mathbf{AUT\text{-}KPl}^w| = \psi 0((\Phi 10) \cdot \varepsilon_0)$.

(iv) $|\Pi^1_1\text{-}\mathbf{TR} + (\text{BI})| = |\mathbf{AUT\text{-}ID}^{pos}_2| = |\mathbf{AUT\text{-}ID}^{mon}_2| = |\mathbf{AUT\text{-}KPl}| = \psi 0 \varepsilon_{(\Phi 10)+1}$.

(v) $|\Pi^1_1\text{-}\mathbf{TR} + \Delta^1_2\text{-}\mathbf{CA}| = |\Pi^1_1\text{-}\mathbf{TR} + \Sigma^1_2\text{-}\mathbf{AC}| = |\mathbf{AUT\text{-}KPl}^w + \mathbf{KPi}^w| = \psi 0(\Phi 1 \varepsilon_0)$.

(vi) $|\Delta^1_2\text{-}\mathbf{TR}_0| = |\Sigma^1_2\text{-}\mathbf{TRDC}_0| = |\Delta^1_2\text{-}\mathbf{CA}_0 + (\Sigma^1_2\text{-BI})| = |\mathbf{KPi}^r + (\Sigma\text{-FOUND})| = |\mathbf{KPi}^r + (\Sigma\text{-REC})| = \psi 0(\Phi \omega 0)$.

(vii) $|\Delta^1_2\text{-}\mathbf{TR}| = |\Sigma^1_2\text{-}\mathbf{TRDC}| = |\Delta^1_2\text{-}\mathbf{CA} + (\Sigma^1_2\text{-BI})| = |\mathbf{KPi}^w + (\Sigma\text{-FOUND})| = |\mathbf{KPi}^w + (\Sigma\text{-REC})| = \psi 0(\Phi \varepsilon_0 0)$.

(viii) $|\Delta^1_2\text{-}\mathbf{TR} + \text{BR(impl-}\Sigma^1_2)| = |\Delta^1_2\text{-}\mathbf{TR} + \text{BI(impl-}\Sigma^1_2)| = |\Sigma^1_2\text{-}\mathbf{TRDC} + \text{BR(impl-}\Sigma^1_2)| = |\Sigma^1_2\text{-}\mathbf{TRDC} + \text{BI(impl-}\Sigma^1_2)| = |\mathbf{KPi}^w + (\Sigma\text{-REC}) + \text{FOUNDR(impl-}\Sigma)| = |\mathbf{KPi}^w + (\Sigma\text{-REC}) + \text{FOUND(impl-}\Sigma)| = \psi 0 \Gamma^\Phi_0$.

(ix) $|\Delta^1_2\text{-}\mathbf{TR} + \text{BR}| = |\Sigma^1_2\text{-}\mathbf{TRDC} + \text{BR}| = |\mathbf{KPi}^w + (\Sigma\text{-REC}) + \text{FOUNDR(impl-}\Sigma(\text{M}))| = \psi 0(\Phi \Omega_1 0)$.

(x) $|\Pi^1_1\text{-}\mathbf{TR} + \text{BR}| = |\mathbf{AUT\text{-}KPl}^w + \text{FOUNDR(impl-}\Sigma(\text{M}))| = \psi 0((\Phi 10) \cdot \Omega_1)$.

(xi) $|\Pi^1_1\text{-}\mathbf{TR} + \text{BR(impl-}\Sigma^1_2)| = |\mathbf{AUT\text{-}KPl}^w + \text{FOUNDR(impl-}\Sigma)| = \psi 0 \omega^{(\Phi 10)+(\Phi 10)}$.

(xii) $|\Pi^1_1\text{-}\mathbf{TR} + \Delta^1_2\text{-}\mathbf{CA} + \text{BR}| = |\mathbf{AUT\text{-}KPl}^w + \mathbf{KPi}^w + \text{FOUNDR(impl-}\Sigma(\text{M}))| = \psi 0(\Phi 1 \Omega_1)$.

(xiii) $|\Pi^1_1\text{-}\mathbf{TR} + \Delta^1_2\text{-}\mathbf{CA} + \text{BR(impl-}\Sigma^1_2)| = |\mathbf{AUT\text{-}KPl}^w + \mathbf{KPi}^w + \text{FOUNDR(impl-}\Sigma)| = \psi 0(\Phi 20)$.

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