ON NONLINEAR SYSTEMS AND ALGEBRAIC GEOMETRY

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1. Introduction

The theory of linear systems has been developed over many years into a unified collection of results based on the application of linear mathematics. In the state space theory the properties of linear operators have been used to obtain results in controllability, stability etc., and in the frequency domain the spectral representation of such operators can be used to generalise classical s-domain methods (see Banks, 1983). When we come to study nonlinear systems then we are faced with many types of problems which appear to have no unifying threads, and consequently different methods of approach have been developed for different types of systems. Why are nonlinear systems so difficult? Is there a frequency-domain theory of nonlinear systems and can we even define the spectrum of a nonlinear system?

In this paper we shall attempt to answer these questions (at least partially) for homogeneous systems of degree 2 (defined by the second term of a Volterra series). It will turn out that the natural generalisations of the classical concepts of poles and zeros of linear systems are varieties in two-dimensional projective space. We shall consider the structure of such varieties and obtain a nonlinear 'root locus' which begins on 'poles' and ends on 'zeros'. One of the main objects of this paper is to demonstrate a strong connection between nonlinear systems theory and algebraic geometry.

Algebraic geometry and algebraic topology have been applied to linear systems theory; see Brockett, 1982 and DeCarlo et al 1977. Moreover, the theory of two dimensional systems in image processing has been developed using two dimensional Laplace and z-transforms, as in Goodman, 1977, Fornasini and Marchesini, 1980, for example. In this paper we shall be concerned with a quotient of polynomials $p(s_1,s_2)$

and $q(s_1,s_2)$ in two variables and we shall call the zeros and poles of the system $K = p/q$ the zero sets of the irreducible factors of p and q, respectively. This is natural since these objects are irreducible in a geometric sense, just as points in the complex plane are irreducible.

In this paper $\mathbb{C}$ will denote the complex plane and if a complex 1-space has coordinate s we shall write $\mathbb{C}_s$. The ring of polynonials is two variables with coefficients in $\mathbb{C}$ will be denoted $\mathbb{C}[s_1,s_2]$ or $\mathbb{C}[X_1,X_2]$ depending on the context. n-dimensional (affine) complex space is denoted $\mathbb{C}^n$ and n-dimensional projective space is denoted $\mathbb{P}^n(\mathbb{C})$. Recall that an ideal in a commutative ring R is a subset $I \subseteq R$ such that

$$RI = IR \subseteq I.$$

Other notations will be introduced as we proceed.

2. Homogeneous degree 2 systems

In this paper we shall be concerned with a scalar input and output bilinear system of the form

$$\dot{x} = Ax + bu + uNx$$
$$y = cx \tag{2.1}$$

where $A, N \in \mathbb{R}^{nxn}$, $b,c \in \mathbb{R}^n$ and u is a scalar control. Then, as is well-known (Bruni et al 1974), the input-output relation of the equations (2.1) may be written

$$y(t) = ce^{At}\hat{x} + \sum_{i=1}^{\infty} \int_o^t \int_o^{t_1} \ldots \int_o^{t_{i-1}} ce^{A(t-t_1)} N(e^{A(t_1-t_2)} N(\ldots\ldots e^{A(t_{i-1}-t_i)} N(e^{At_i}\hat{x})u(t_i))\ldots u(t_1)dt_i \ldots dt_1$$

$$+ \sum_{i=1}^{\infty} \int_o^t \int_o^{t_1} \ldots \int_o^{t_{i-1}} ce^{A(t-t_1)} N(e^{A(t_1-t_2)} N(\ldots\ldots e^{A(t_{i-1}-t_i)} b\, u(t_i))\ldots u(t_i)dt_i \ldots dt_1 , \tag{2.2}$$

where $\hat{x}$ is the initial value of $x(t)$.

We shall consider the special case of homogeneous systems of degree 2, with zero initial values $\hat{x} = o$ i.e. where all terms except for i=1 in the second sum of (2.2) are zero. Then, we have

$$y(t) = \int_o^t \int_o^{t_1} ce^{A(t-t_1)} N(e^{A(t_1-t_2)} bu(t_2)) u(t_1) dt_2 dt_1 \qquad (2.3)$$

It is easy to show that this may be written in the form

$$y(t) = \int_o^t \int_o^t ce^{A\tau_1} N e^{A\tau_2} b\, u(t-\tau_1)u(t-\tau_2)d\tau_1\, d\tau_2 \qquad (2.4)$$

and so we may define a two-dimensional system by

$$y_2(t_1,t_2) = \int_o^{t_1} \int_o^{t_2} ce^{A\tau_1} Ne^{A\tau_2} bu(t_1-\tau_1)u(t_2-\tau_2)d\tau_1\, d\tau_2 \qquad (2.5)$$

so that (2.4) and (2.5) are related by

$$y(t) = y_2(t,t) \quad . \qquad (2.6)$$

We can now define the two-dimensional Laplace transform of (2.5) in the usual way and we obtain

$$Y_2(s_1,s_2) = c(s_1I-A)^{-1}N(s_2I-A)^{-1}b\ U(s_1)U(s_2)$$

and we define the transfer function of the system to be

$$H_2(s_1,s_2) \triangleq \frac{Y_2(s_1,s_2)}{U(s_1)U(s_2)} = c(s_1I-A)^{-1}N(s_2I-A)^{-1}b$$

Conversely, given a function $H_2(s_1,s_2)$ which is recognisable in the sense that there exist polynomials P,Q_1,Q_2 such that

$$H_2(s_1,s_2) = \frac{P(s_1,s_2)}{Q_1(s_1)Q_2(s_2)} \qquad (2.7)$$

(i.e. the denominator is separable), then it can be shown (Mitzel et al, 1979) that H has a bilinear realisation of the form (2.1). In this case the input-output stability of the system is simple and one can easily invert the Laplace transform in (2.7).

However, if H_2 is not recognisable then the study of a transfer function of the general form

$$H_2(s_1,s_2) = \frac{P(s_1,s_2)}{Q(s_1,s_2)} \qquad (2.8)$$

is not such a simple matter, although many stability results have been discovered in the theory of two-dimensional filters (see, for example Reddy et al, 1981). In this paper we shall take a different view and relate the general transfer function H_2 in (2.8) to elementary algebraic geometry by studying the 'singularities' of H_2 given by

$$Q(s_1,s_2) = 0 \qquad (2.9)$$

Such an equation defines curve C in $\mathbb{C}^2$ and we shall classify our systems in the next section according to the topological nature of the 'irreducible' components of C.

Before considering systems from this abstract viewpoint let us first indicate a general feedback system which is not, in general, recognisable. Suppose that the system (2.1) is included in the feedback system shown in fig. 2.1. Then we have

$$y(t) = \int_0^t \int_0^t ce^{A\tau_1} Ne^{A\tau_2} bu(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2$$

$$= \int_0^t \int_0^t k^2 ce^{A\tau_1} Ne^{A\tau_2} b(G*e^{\frac{1}{2}})(t-\tau_1)(G*e^{\frac{1}{2}})(t-\tau_2)d\tau_1 d\tau_2 \qquad (2.10)$$

where we regard kG as a (linear) compensator. Then, as above

$$Y_2(s_1,s_2) = k^2 c(s_1 I-A)^{-1} N(s_2 I-A)^{-1} b\ G(s_1)G(s_2)E^{\frac{1}{2}}(s_1)E^{\frac{1}{2}}(s_2)$$

where $E^{\frac{1}{2}}(s_1) = \mathcal{L}(e^{\frac{1}{2}})(s_1)$. ($\mathcal{L}$ denotes Laplace transform)

However,

$$e(t) = r(t) - y(t)$$

and we may write

$$e^{\frac{1}{2}}(t)e^{\frac{1}{2}}(t) = r(t) - y(t)$$

or $$e^{\frac{1}{2}}(t_1)e^{\frac{1}{2}}(t_2) = r_2(t_1,t_2) - y_2(t_1,t_2)$$

where $r_2(t,t) = r(t)$.

Hence

$$E^{\frac{1}{2}}(s_1)E^{\frac{1}{2}}(s_2) = R_2(s_1,s_2) - Y_2(s_1,s_2)$$

and so, denoting

$$K_2(s_1,s_2) = c(s_1I-A)^{-1}N(s_2I-A)^{-1}bG(s_1)G(s_2)$$

we have

$$Y_2(s_1,s_2) = k^2K_2(s_1,s_2)E^{\frac{1}{2}}(s_1)E^{\frac{1}{2}}(s_2)$$

$$= k^2K_2(s_1,s_2)(R_2(s_1,s_2) - Y_2(s_1,s_2)).$$

The feedback system therefore has the transfer function

$$\bar{K}_2(s_1,s_2) \triangleq \frac{Y_2(s_1,s_2)}{R_2(s_1,s_2)} = \frac{k^2K_2(s_1,s_2)}{1+k^2K_2(s_1,s_2)} \tag{2.11}$$

Note, however, that the system in fig. 2.1 is not a real system if $e < o$, although it is easy to see that the real system in fig. 2.2 has the same input-output relation from zero initial conditions provided e does not change sign. (For, if e is always positive the systems are trivially identical and if e is always negative, then from (2.10)

$$y(t) = \int_o^t \int_o^t ce^{A\tau_1}Ne^{A\tau_2}bi(G^*|e|^{\frac{1}{2}})(t-\tau_1)i(G^*|e|^{\frac{1}{2}})(t-\tau_2)d\tau_1d\tau_2$$

$$= -\int_o^t \int_o^t c\, e^{A\tau_1}Ne^{A\tau_2}b(G^*|e|^{\frac{1}{2}})(t-\tau_1)(G^*|e|^{\frac{1}{2}})(t-\tau_2)d\tau_1d\tau_2$$

which is the output of the system in fig. 2.2). If e does change sign, then we can use the translation invariance of the systems to make them equivalent on interval of constant sign although we must now consider non-zero initial conditions. This will make the transfer function (2.11) depend on the initial values.

In this paper we shall study the simplest case of a general transfer function such as (2.11) which we assume is independent of the initial values. In order to do this we shall devote the next section to an introduction to the algebraic geometry of curves in the projective space $\mathbb{P}^2(\mathbb{C})$ which we shall need in the sequel.

3. Algebraic Geometry of Projective Curves

In this section we shall digress somewhat to summarise the basic results of the algebraic geometry of projective curves (see, for example, Kendig 1977). This will be necessary, since it will turn out that the 'poles' and 'zeros' of our two-dimensional system are just such irreducible curves. These objects are then the natural generalisations of the eigenvalues or spectra of linear operators. We begin by reminding the reader of the definition of the complex projective 2-space $\mathbb{P}^2(\mathbb{C})$.

Definition 3.1 The projective space $\mathbb{P}^2(\mathbb{C})$ is defined by

$$\mathbb{C}^3/\sim$$

where $\sim$ is the equivalence relation on $\mathbb{C}^3$ given by

$$x = [x_1, x_2, x_3] \sim y = [y_1, y_2, y_3] \text{ iff } x = cy$$

for some $c \in \mathbb{C}\backslash\{o\}$. $\mathbb{P}^2(\mathbb{C})$ has the induced topology.

Alternatively, we can visualise $\mathbb{P}^2(\mathbb{C})$ as the complex 2 space $\mathbb{C}^2$ with a 'point at infinity' added to each complex line through $(0,0)$. Note that each such complex line with the one-point compactification is just the Riemann sphere or $\mathbb{P}^1(\mathbb{C})$.

Linear systems are defined naturally on $\mathbb{P}^1(\mathbb{C})$ (see Banks and Abhassi-Ghelmansarai, 1983) and this space has just a single point 'at infinity' as pointed out above. We are now going to study our two-dimensional system on $\mathbb{P}^2(\mathbb{C})$ which has a complete sphere $\mathbb{P}^1(\mathbb{C})$ 'at infinity'. This makes the behaviour of 'poles' and 'zeros' at infinity much more difficult than in the linear case as we shall see.

Let $\mathbb{C}[X_1,X_2]$ denote the ring of polynomials in the indeterminates X_1 and X_2. We define for $p_i \in \mathbb{C}[X_1,X_2]$, $i=1,\ldots,k$ the affine variety

$$V(\{p_i\}) = \{(x_1,x_2) \in \mathbb{C}^2 : \text{for each } i,\ p_i(x_1,x_2) = 0\}$$

If k=1 we have the affine hypersurface $V(p_1)$.

The projective variety (or projective curve) affined by p is the closure of the affine variety V(p) in $\mathbb{P}^2(\mathbb{C})$. We shall now describe the basic structure theory for projective curves; the proofs of all the following statements may be found, for example, in Kendig (1977). First recall that $\mathbb{C}[X_1,X_2]$ is a unique factorisation domain, so any $p \in \mathbb{C}[X_1,X_2]$ may be factorised in the form

$$p = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m} \tag{3.1}$$

where each p_i is irreducible (i.e. $p_i = q_i \cdot r_i$, for $q_i, r_i \in \mathbb{C}[X_1,X_2]$ implies $q_i \in \mathbb{C}$ or $r_i \in \mathbb{C}$) and the expression (3.1) is unique up to the order of factors and multiplication by elements of $\mathbb{C}$. The multiplicity of the factor p_i is α_i.

Note that

$$V(p) = \bigcup_{i=1}^{m} V(p_i) \tag{3.2}$$

and a variety V is irreducible if $V = V_1 \cup V_2$ implies $V = V_1$ or $V = V_2$. V(q) is irreducible iff $q \in \mathbb{C}[X_1,X_2]$ is irreducible. Hence, by (3.2) we may study irreducible varieties. The topological nature of each irreducible variety is then given by

Theorem 3.2 If $p \in \mathbb{C}[X_1, X_2] \setminus \mathbb{C}$ is irreducible and of degree n (i.e. the maximal monomial degree of p), then $V(p) \subseteq \mathbb{P}^2(\mathbb{C})$ is topologically a compact, connected, orientable manifold of some genus g with a finite number of points (possibly zero) identified with a finite number of points. □

Typically, an irreducible projective curve will have the topological structure shown in fig. 3.1. (Here the genus g = 2 and there are three pairs of identified points.) A curve V(p) defined by the polynomial $p \in \mathbb{C}[X_1, X_2]$ is said to be nonsingular at the point $P \in V(p)$ if

$$\frac{\partial p}{\partial X_1}(P) \neq 0 \text{ or } \frac{\partial p}{\partial X_2}(P) \neq 0$$

For simplicity, we shall assume that all irreducible projective curves in this paper are nonsingular. We shall consider singular varieties in a future paper. In the case of nonsingular varieties we can prove the genus formula

$$g = \frac{(n-1)(n-2)}{2} \tag{3.3}$$

for the curve V(p) where deg p = n.

Now, to obtain the topological structure of a general variety $V(p) = \bigcup_{i=1}^{m} V(p_i)$ (by 3.2)) where the p_i are irreducible and relatively prime we use Bézout's theorem which states that

$$\deg(V(p_1) \cap \ldots \cap V(p_m)) = \prod_{i=1}^{m} \deg p_i$$

where the value on the left is the total number of points of intersections of the $V(p_i)$'s in $\mathbb{P}^2(\mathbb{C})$ counted with multiplicity. Hence V(p) consists of a set of m compact orientable manifolds which touch each other at a finite number of points . (Recall that we are assuming that each $V(p_i)$ is nonsingular.)

Example 3.3 It is easy to see that

$$V(p_1) \triangleq V(X_2^2 - X_1(X_1^2 - 1)) \subseteq \mathbb{P}^2(\mathbb{C})$$

is topologically a torus and that

$$V(p_2) \triangleq V(X_1)$$

is a sphere. Now deg $p_1 = 3$ and deg $p_2 = 1$ so that

$$V(p) \triangleq V(p_1p_2) = V(p_1) \cup V(p_2)$$

is topologically a torus and a sphere which touch at three points (by Bézout's theorem). V(p) is shown in fig. 3.2.

We have now summarised the elementary theory of projective varieties in $\mathbb{P}^2(\mathbb{C})$ and for varieties with nonsingular irreducible components we see that there is a particularly nice topological classification in terms of Riemann surfaces whose genus satisfy (3.3). When we return in the next section to nonlinear systems theory, we shall interpret the irreducible components of p and q ($\in \mathbb{C}[X_1,X_2]$) in the transfer function $H_2 = p/q$ as the 'zeros' and 'poles' of the system. In the case of linear systems theory, when we have a transfer function of the form $G(s) = p(s)/q(s)$ where deg p = m, deg q = n ($n > m$), then we say that there are n-m zeros at infinity; i.e. when we extend G to $\mathbb{P}^1(\mathbb{C})$ (which is topologically a sphere) then G(s) has a zero of order n-m at $\infty \in \mathbb{P}^1(\mathbb{C})$. We are faced with the same problem in two-dimensions with $H_2 = p(s_1,s_2)/q(s_1,s_2)$; however, in this case the part of $\mathbb{P}^2(\mathbb{C})$ 'at infinity' corresponds to a projective one-space $\mathbb{P}^1(\mathbb{C})$. Hence, to determine the behaviour of H_2 at infinity it will turn out that we must look at sphere coverings of $\mathbb{P}^1(\mathbb{C})$.

Definition 3.4 An m-sheeted sphere covering of $\mathbb{P}^1(\mathbb{C})$ is a triple $(M,\mathbb{P}^1(\mathbb{C}),\pi)$ where M is a locally compact space and π: $M \to \mathbb{P}^1(\mathbb{C})$ is a continuous surjection such that

(i) If $p \in M$, $\exists$ a disk $D \subseteq M$ such that $p \in D$ and each of the m components of $\pi^{-1}(D)$ is (homeomorphic) to a disk D_α, $1 \le \alpha \le m$

(ii) $\pi|_{D_\alpha} : D_\alpha \to D$ is a homeomorphism.

π is called the <u>covering map</u>.

The triple $(\bar{M}, \mathbb{P}^1(\mathbb{C}), \bar{\pi})$ is called a <u>near m sheeted cover</u> of $\mathbb{P}^1(\mathbb{C})$ if these exist a finite number of points $P_1,..,P_k \in \mathbb{P}^1(\mathbb{C})$ such that

$$(\bar{M}\backslash f^{-1}(\{P_1,...,P_k\}), \mathbb{P}^1(\mathbb{C})\backslash(\{P_1,...,P_k\}), f|\bar{M}\backslash f^{-1}(\{P_1,...,P_k\}))$$

is an m-sheeted cover of $\mathbb{P}^1(\mathbb{C})/(\{P_1,...,P_k\})$.

Then it can be shown that if C is a curve in $\mathbb{P}^2(\mathbb{C})$, C is either a near m-sheeted covering of $\mathbb{P}^1(\mathbb{C})$ or the one-point compactification of such a covering.

A final result which we shall need is the fact that the set $\mathbb{C}(X_1)^*$ of all fractional-power series of the form $\sum_{i=i_0}^{\infty} a_i X_1^{i/n}$ is an algebraically closed field (i.e. any polynomial in the ring $\mathbb{C}(X_1)^* [X_2]$ splits completely in $\mathbb{C}(X_1)^*$) and so we have

$$p(X_1,X_2) = \prod_{k=1}^{n} (X_2 - (\sum_i a_{ij}(X_1 - x_0)^{i/m_k})) \tag{3.4}$$

for any polynomial $p \in \mathbb{C}[X_1,X_2]$ of degree n and any $x_0 \in \mathbb{C}$, where we have assumed that p has been written in the form

$$p(X_1,X_2) = X_2^n + a_1(X_1)X_2^{n-1} +...+ a_n(X_1)$$

with deg $a_i(X_1) \le i$ or $a_i \equiv 0$. Moreover, each of the series in (3.4) converges in a neighbourhood of x_0.

4. <u>On the inversion of the Laplace Transform</u>

We shall now return to the frequency domain study of homogeneous degree 2 systems and suppose that we have a transfer function of the form

$$H_2(X_1,X_2) = \frac{p(X_1,X_2)}{q(X_1,X_2)} \tag{4.1}$$

Let us write p and q as products of irreducible factors; then we have

$$H_2 = \frac{p_1^{i_1}\dots p_m^{i_m}}{q_1^{j_1}\dots q_n^{j_n}} \tag{4.2}$$

As we mentioned earlier, we shall assume that each of the polynomials p_i, q_j is nonsingular and so the zero set of each one in $\mathbb{P}^2(\mathbb{C})$ is a compact manifold of some genus depending on the polynomial.

Definition 4.1 Let

$$\mathcal{Z}_k = \overline{\{(X_1,X_2) \in C^2 : p_k(X_1,X_2) = 0\}}, \; 1 \leq k \leq m$$

$$\mathcal{P}_\ell = \overline{\{(X_1,X_2) \in \mathbb{C}^2 : q_\ell(X_1,X_2) = 0\}}, \; 1 \leq \ell \leq n ,$$

where the bar denotes projective completion (i.e. closure of the set in $\mathbb{P}^2(\mathbb{C})$). Then we say that $\mathcal{Z}_k$ $(1 \leq k \leq m)$, P_ℓ $(1 \leq \ell \leq n)$ are, respectively, the open loop zeros and poles of the system (4.2), with multiplicities i_k, j_ℓ.

It is clear from the discussion above that these are the natural objects to be considered as the nonlinear generalisations of the poles and zeros of linear systems in $\mathbb{P}^1(\mathbb{C})$. As we proceed we hope to demonstrate that by considering projective algebraic curves in this way, we can extend much of the classical frequency domain theory for linear systems. Let us first consider the inversion problem of the two-dimensional Laplace transform $H_2(X_1,X_2)$ (we shall use X_1, X_2 rather than the traditional S_1, S_2). In the linear case, the easiest method is to use a partial fraction expansion of the transfer function. We shall now see how we might generalise this to H_2. The first thing to notice is that, for polynomials in a single variable $p(s)$, $q(s)$ of degrees m and n, respectively, we may write

$$\frac{p(s)}{q(s)} = \sum_{i=1}^{n} \frac{a_i}{q_i(s)} \triangleq \sum_{i=1}^{n} \frac{a_i}{s+s_i}$$

where each a_i is a constant and q_i is a linear irreducible component of q (assuming q has no repeated root). If we try the same idea with

$H_2 = p(X_1,X_2)/q(X_1,X_2)$ by writing $q = q_1 \cdots q_n$ in terms of irreducible factors (which we shall assume for simplicity are distinct) then it is not in general possible to write

$$H_2 = \frac{p(X_1,X_2)}{q(X_1,X_2)} = \sum_{i=1}^{n} \frac{a_i}{q_i}$$

even if we allow the a_i's to be polynomials. For example,

$$\frac{1}{X_1^2 - X_2^2} \neq \frac{a_1}{X_1 - X_2} + \frac{a_2}{X_1 + X_2} \tag{4.3}$$

for any $a_1, a_2 \in \mathbb{C}[X_1,X_2]$. In order to resolve this problem we must consider the ideals generated by p and $q_1,\ldots,q_n$.

<u>Definition 4.2</u> If $p \in \mathbb{C}[X_1,X_2]$, then the <u>principle ideal</u> generated by p is the set

$$\begin{aligned}(p) &\triangleq \{r : r = r_1 p \text{ for some } r_1 \in \mathbb{C}[X_1,X_2]\}\\ &= p.\mathbb{C}[X_1,X_2] \quad .\end{aligned}$$

Note that if p is irreducible then (p) is a prime ideal. Each prime ideal (p) will generate the affine variety V(p) given by

$$\begin{aligned}V(p) &= \{(X_1,X_2) \in \mathbb{C}^2 : \bar{p}(X_1,X_2) = 0 \text{ for all } \bar{p} \in (p)\}\\ &= \{(X_1,X_2) \in \mathbb{C}^2 : p(X_1,X_2) = 0\}\end{aligned}$$

We emphasize that we used the symbol V(p) earlier to be the <u>projective</u> variety defined by p. This should not cause any confusion.

It is easy to show that (Kendig, 1977)

$$\begin{array}{ll}\text{(i)} & (p_1 p_2) = (p_1).(p_2) = (p_1) \cap (p_2)\\ \text{(ii)} & V((p_1)+(p_2)) = V(p_1) \cap V(p_2)\\ \text{(iii)} & V((p_1) \cap (p_2)) = V(p_1) \cup V(p_2)\\ \text{(iv)} & (p_1) \subseteq (p_2) \Rightarrow V(p_2) \subseteq V(p_1)\\ \text{(v)} & V(p_1) = V(p_2) \Rightarrow (p_1) = (p_2)\end{array} \tag{4.4}$$

for any prime ideals (p_1), (p_2). We can now prove

Lemma 4.3 If $r(X_1,X_2) = \dfrac{p(X_1,X_2)}{q_1(X_1,X_2)\ldots q_n(X_1,X_2)}$ where each q_i is irreducible and $p,q_1,\ldots,q_n$ are relatively prime, then we can write

$$r(X_1,X_2) = \sum_{i=1}^{n} \frac{p_i(X_1,X_2)}{q_i(X_1,X_2)} \tag{4.5}$$

for some polynomials p_i, $1 \leq i \leq n$, if and only if

$$(p) \subseteq (q_1')+ \ldots . +(q_n')$$

where $q_i' = \pi_{j\neq i} q_j$. (If n=2 this is just $(p) \subseteq (q_1) + (q_2)$)

Proof If. This is trivial, since if $p \in (p) \subseteq (q_1')+\ldots.+(q_n')$ then there must exist $p_i \in \mathbb{C}[X_1,X_2]$ such that

$$p = \sum_{i=1}^{n} (\pi_{j\neq i} q_j)p_i .$$

only if. Suppose that we may write r in the form (4.5) and let $t(X_1,X_2) \in (p)$. Then $t = pt_1$ for some polynomial t_1 and so

$$t = \sum_{i=1}^{n} q_i'(t_1p_i) \in (q_1')+\ldots+(q_n') \ . \ \square$$

Corollary 4.4. Under the assumptions of lemma 4.3, if

$$\bigcap_{i=1}^{n} V(q_i) \subseteq V(p)$$

then we may write r in the form (4.5), and conversely.

Proof: This follows immediately from (4.4). $\square$

In other words, in order to write $r = p/(q_1\ldots q_n)$ as a partial fraction (4.5) it is necessary and sufficient that the affine variety defined by p contains the intersections of the affine varieties defined by the q_i's. We now see why we cannot have equality in (4.3). In fact, since $p = 1$, $V(p) = \emptyset$ and $V(X_1-X_2) \cap V(X_1 + X_2) = 0$. However, we have

Corollary 4.5 $1/(q_i\ldots.q_n) = \sum_{i=1}^{n} p_i/q_i$ for some $p_i \in \mathbb{C}[X_1,X_2]$ $(1 \leq i \leq n)$ if and only if

$$(1) = \sum_{i=1}^{n} (q_i')$$

i.e. if and only of the ideals (q_i') are comaximal. □

We can naturally ask how we should determine the polynomials p_i in (4.5) if they exist. Let us note first that they are not unique, for (in the case n=2) if

$$r = \frac{p}{q_1 q_2} = \frac{p_1}{q_i} + \frac{p_2}{q_2}$$

then

$$r = \frac{(p_1 + \bar{p}_1 q_1)}{q_1} + \frac{(p_2 - \bar{p}_1 q_2)}{q_2}$$

for any $\bar{p}_1 \in \mathbb{C}[X_1, X_2]$. However, if

$$r = \sum_{i=1}^{n} \frac{p_i}{q_i} \text{ and } r = \sum_{i=1}^{n} \frac{\bar{p}_i}{q_i} ,$$

then

$$\sum_{i=1}^{n} (p_i - \bar{p}_i) q_i' = 0 \quad . \tag{4.6}$$

Consider q_i. This occurs in all the terms in the sum of (4.6) except the i^{th}. Hence $(p_i - \bar{p}_i)$ is divisible by q_i, since the q_i' s are irreducible and relatively prime. Therefore

$$(p_i - \bar{p}_i) \in (q_i)$$

and so p_i is determined modulo the ideal (q_i). Hence

$$p_i \in \mathbb{C}[X_1, X_2]/(q_i) . \tag{4.7}$$

The quotient ring in (4.7) is called the coordinate ring of the (affine) variety $V(q_i)$ and is denoted by $\mathbb{C}_{q_i}[x_1, x_2]$, where

$$x_1 = X_1 + (q_i), \; x_2 = X_2 + (q_i).$$

Since q_i is determined by its values on $V(q_i)$, then if

$$r = \frac{p_1}{q_1} + \ldots + \frac{p_i}{q_i} + \ldots + \frac{p_n}{q_n} \tag{4.8}$$

we have

$$rq_i = \frac{p_1 q_i}{q_1} + \ldots\ldots + p_i + \ldots\ldots + \frac{p_n q_i}{q_n}$$

Hence if we make the substitution $X_1 \to x_1$, $X_2 \to x_2$ we have

$$\begin{aligned} r(x_i, x_2) q_i(x_1, x_2) &= p_i(x_1, x_2) \\ &= \frac{p(x_1, x_2)}{\prod_{j \neq i = 1}^{n} \{ q_j(x_i, x_2) \}} \end{aligned} \tag{4.9}$$

Note that if the partial fraction expansion (4.8) is valid, then the q must be divisible by $\prod_{j \neq i=1}^{n} q_j$ in $\mathbb{C}_{q_i}[x_1, x_2]$.

<u>Example 4.6</u> Consider the transfer function

$$H_2(X_1, X_2) = \frac{X_1}{(X_1 - X_2^2)(X_1 + 2X_2^2)} = \frac{p}{q_1 q_2} \quad , \text{ say.}$$

Note first that $V(p) \supseteq \{0\} = V(q_1) \cap V(q_2)$ and so we may write

$$H_2 = \frac{p_1}{q_1} + \frac{p_2}{q_2}$$

Now, $$p_1 = \left. \frac{p}{q_2} \right|_{X_1 = X_2^2} = \frac{1}{3}$$

and similarly

$$p_2 = \frac{2}{3}. \quad \square$$

More generally we know from (3.4) that at almost all points (γ_1, γ_2) of a variety $V(p)$ we may write

$$X_2 = \gamma_2 + a_1(X_1 - \gamma_1) + a_2(X_1 - \gamma_1)^2 + \ldots\ldots,$$

for some coefficients a_i. Then in (4.9) we may take

$$x_2 = \gamma_2 + a_1(x_1 - \gamma_1) + a_2(x_1 - \gamma_1)^2 + \ldots\ldots$$

to obtain

$$p_i(x_1, \gamma_2 + \sum_{i=1}^{\infty} a_i(x_1 - \gamma_1)^i)$$

which defines $p_i(X_1, X_2)$ by analytic continuation .

We have now reduced the inversion problem to the inversion of terms of the form p/q where q is irreducible and $p \in \mathbb{C}_q[x_1,x_2]$. (Provided the conditions of lemma 4.3 hold. If they do not then we have

$$\frac{p}{q} = \frac{p}{q_1 \cdots q_n} ,$$

where $\bigcap_{i=1}^{n} V(q_i) \not\subseteq V(p)$. However, since the q_i's are irreducible and relatively prime we know that $\bigcap_{i=1}^{n} V(q_i)$ consists of a finite number of points, say $(\alpha_j,\beta_j) \in \mathbb{C}^2$ for $1 \leq j \leq N$. Hence, for example

$$\prod_{j=1}^{N} (X_1-\alpha_j)\frac{p}{q} = \prod_{j=1}^{N} (X_1 - \alpha_j)\; p/(q_1 \cdots q_n) \tag{4.10}$$

satisfies the hypothesis of lemma 4.3, and so we may write

$$\frac{p}{q} = \sum_{i=1}^{N} \frac{1}{\prod_{j=1}^{N} (X_1-\alpha_j)} \frac{p_i}{q_i} \tag{4.11}$$

where, as before, q_i is irreducible and $p_i \in \mathbb{C}_{q_i}[x_1,x_2]$. However, if we can invert p_i/q_i to give $f_i(t_1,t_2)$, say, then the inverse transform of

$\dfrac{1}{\prod_{j=1}^{N} (X_1-\alpha_j)} \dfrac{p_i}{q_i}$ is just

$$\{ \prod_{k=1}^{N} \left(\frac{1}{\prod_{j\neq k=1}^{N} (\alpha_k-\alpha_j)}\right) e^{\alpha_j t_1} \}*_{t_1} f_i(t_1,t_2) , \tag{4.12}$$

where $*_{t_1}$ denotes the t_1 convolution.)

The inversion problem of the quotient p/q, for q irreducible can be further reduced to the inversion of 1/q. For, if

$$p(X_1,X_2) = \sum a_{ij} X_1^i X_2^j ,$$

then

$$\mathcal{L}^{-1}_{X_1,X_2} p = \sum a_{ij} \delta^{(i)}(t_1)\delta^{(j)}(t_2) = F(t_1,t_2), \text{ say} \tag{4.13}$$

where $\delta^{(k)}$ denotes the k^{th} derivative of the delta function. (Of course, F is a distribution, cf Schwartz, 1950). Then,

$$\mathcal{L}^{-1}_{X_1,X_2}(p.U(X_1)U(X_2)) = F(t_1,t_2)*_{t_1 t_2}(u(t_1)\,u(t_2))$$

$$= v(t_1,t_2),$$

say, where $*_{t_1 t_2}$ denotes the double (t_1,t_2)- convolution and u is the input to the system. Now,

$$Y(X_1,X_2) = \frac{p(X_1,X_2)}{q(X_1,X_2)}\, U(X_1)U(X_2)$$

is equivalent to the partial differential equation

$$\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \frac{\partial^{i+j} y}{\partial t_1^i\, \partial t_2^j} = v(t_1,t_2) \tag{4.14}$$

with zero 'initial' conditions

$$\frac{\partial^i y}{\partial t_1^i}(o,t_2) = 0 \ , \text{ for } 0 \le i \le m-1$$

$$\frac{\partial^j y}{\partial t_2^j}(t_1,o) = 0 \ , \text{ for } 0 \le j \le n-1 \tag{4.15}$$

where
$$q(X_1,X_2) = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}\, X_1^i X_2^j$$

Let $G(t_1,\xi_1;t_2,\xi_2)$ be the Green's function for the equations (4.14) and (4.15); i.e.

$$\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \frac{\partial^{i+1}}{\partial t_1^i\, \partial t_2^j} G(t_1,\xi_1;t_2,\xi_2) = \delta(t_1-\xi_1)\ (t_2-\xi_2) \tag{4.16}$$

Then we have,

$$y(t_1,t_2) = \iint G(t_1,\xi_1;t_2,\xi_2)\ u(\xi_1,\xi_2)d\xi_1 d\xi_2$$

Hence the inversion problem can be solved completely if we can find a

Grein's function for any two-dimensional differential operator with zero initial conditions and an irreducible characteristic polynomial. Consider the following simple example.

Example 4.6 Suppose a system is defined by

$$Y(X_1,X_2) = \frac{U(X_1)U(X_2)}{X_1 - X_2^2}$$

Then we must solve

$$\frac{\partial G}{\partial t_1} - \frac{\partial^2 G}{\partial t_2^2} = \delta(t_1 - \xi_1)\delta(t_2 - \xi_2)$$

However, it is well known (Friedman, 1956) that this equation has the solution

$$G(t_1,\xi_1;t_2,\xi_2) = (4\pi(t_1 - \xi_1))^{-\frac{1}{2}} \exp\left[-(4(t_1-\xi_1))^{-1}(t_2-\xi_2)^2\right]$$

and we have

$$y(t_1,t_2) = G *_{t_1 t_2} u(t_1)u(t_2)$$

$$= \int_0^{t_1} \int_0^{t_2} (t_1,\xi_1;t_2,\xi_2)u(\xi_1)u(\xi_2)d\xi_1 d\xi_2$$

and so

$$y(t) = \int_0^t \int_0^t G(t,\xi_1;t,\xi_2)u(\xi_1)u(\xi_2)d\xi_1 d\xi_2 \ .$$

5. Spectral Theory of Homogeneous degree 2 Systems

We have shown above that for systems defined by a quotient of two polynomials $p(X_1,X_2)$ and $q(X_1,X_2)$, the natural generalization of poles and zeros of a linear system are the irreducible varieties defined in $\mathbb{P}^2(\mathbb{C})$ by the irreducible components of q and p, respectively. An obvious question is to ask how such objects change under changes in the system structure. In particular we shall consider the feedback system in section 2 (cf.(2.11)) given by

$$\bar{K}_2 = \frac{k\ K_2(s_1,s_2)}{1+k\ K_2(s_1,s_2)} \tag{5.1}$$

(We shall again use the symbols s_1, s_2 instead of X_1, X_2 to conform with the usual system theory notation, and we have replaced k^2 by k which we can clearly do). Suppose that

$$K_2(s_1,s_2) = \frac{p(s_1,s_2)}{q(s_1,s_2)}$$

$$= \frac{p_1(s_1,s_2)\dots p_m(s_1,s_2)}{q_1(s_1,s_2)\dots q_n(s_1,s_2)} \tag{5.2}$$

where $\{p_i\}$ and $\{q_j\}$ are the irreducible factors of p and q, respectively, which we are assuming are distinct, for simplicity. Then

$$\bar{K}_2 = \frac{k\ p(s_1,s_2)}{q(s_1,s_2)+kp(s_1,s_2)} \quad . \tag{5.3}$$

Let us recall the basic topological structure of the root locus of a linear system defined by the transfer function

$$G(s) = \frac{p(s)}{q(s)} = \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{n}(s-p_i)} \tag{5.4}$$

where $n > m$. Then interpreting

$$\bar{G} = \frac{kp}{q+kp}$$

On $\mathbb{P}^1(\mathbb{C})$ (i.e. the Rienann sphere) we know that the locus of $q + kp = 0$ for $k \in [0,\infty]$ is a connected directed graph in $\mathbb{P}^1(\mathbb{C})$ with n branches starting at the open loop poles and n branches ending on the open loop zeros (with n-m open loop zeros at infinity). A typical situation is shown in Fig. 5.1 when n=4, m=2. As we have shown, some of the n branches may coalesce at finite isolated points, when q + kp has multiple roots.

However, the number of poles is always equal to the number of zeros on the compact surface (counting multiplicities) and there are never less than n poles (again counting multiplicities). The next example shows that such simple behaviour will not generally occur for degree 2 systems.

Example 5.1 Suppose we have the system

$$K_2 = \frac{s_1^2 + s_2}{s_1 s_2} \tag{5.5}$$

Then

$$p = s_1^2 + s^2 \quad , q_1 = s_1, \; q_2 = s_2,$$

and the 'root locus' is specified by

$$s_1 s_2 + k\,(s_1^2 + s_2) = 0 \tag{5.6}$$

Clearly the root locus begins on open loop poles and ends on open loop zeros. Now each of the polynomials p, q_1, q_2 is irreducible and clearly each defines a variety in $\mathbb{P}^2(\mathbb{C})$ which is topologically a sphere. Moreover, these three 'spheres' intersect each other at 0. (The intersections of p, q_1 and p, q_2 at 0 is of multiplicity 2). Now consider (5.5) for positive k. Then $s_1 s_2 + k(s_1^2 + s_2)$ $(k > o)$ is clearly irreducible and is also nonsingular.

Hence, by the genus formula (3.3), the irreducible variety in $\mathbb{P}^2(\mathbb{C})$ defined by this polynomial is of genus 0; i.e. it is also a sphere. Hence our system has two open loop poles but only one closed loop pole for $k > o$. As $k \to \infty$ this pole tends to the open loop zero. The root locus is shown in fig. 5.2. (We can only show the topological structure of the poles on the root locus, of course; picturing (even visualising!) their embedding in $\mathbb{P}^2(\mathbb{C})$ is much more difficult). Note that the three dimensional embeddings of the projective curves in $\mathbb{R}^3$ must intersect at points other than zero. The sphere corresponding to the closed loop pole can only tend towards the open loop zero sphere with intersection just at 0 in four dimensions.

We may naturally ask 'what happens to the system (5.5) at infinity?' As we remarked earlier 'infinity' in $\mathbb{P}^2(\mathbb{C})$ is a one dimensional projective space (i.e. $P^1(\mathbb{C})$) which we add to $\mathbb{C}^2$ to form the completion $\mathbb{P}^2(\mathbb{C})$. In order to determine the number of 'poles' and 'zeros' there are at infinity we shall count the sphere coverings of $P^1(\mathbb{C})$ generated by p and q. We mentioned earlier that $\mathbb{P}^2(\mathbb{C})$ can be defined as the union of $\mathbb{C}^2$ and the set of points added to each complex 1-space of $\mathbb{C}^2$ to form a $P^1(\mathbb{C})$, together with the obvious topology. Now each 1-subspace of $\mathbb{C}^2$ is given by

$$s_1 - \alpha s_2 = 0 \quad , \quad \alpha \in \mathbb{C}\setminus\{0\}$$

(apart from the s_1-axis) and so the projective 1-space $P^1(\mathbb{C})$ at infinity in $\mathbb{P}^2(\mathbb{C})$ is parameterised by $\alpha = s_1/s_2$, apart from the point $s_2 = 0$. Hence a system given by

$$K_2(s_1,s_2) = \frac{p(s_1,s_2)}{q(s_1,s_2)}$$

behaves at infinity like the system

$$K_2'(\alpha,s_2) = \frac{p(\alpha s_2,s_2)}{q(\alpha s_2,s_2)} \triangleq \frac{p'(\alpha,s_2)}{q'(\alpha,s_2)}$$

Let $p_1',\ldots,p_{m'}'$ and $q_1',\ldots,q_{m'}'$ be the irreducible components of p' and q', respectively. Then

$$K_2'(\alpha,s_2) = \frac{p_1'\cdots p_m'}{q_1'\cdots q_m'}$$

Consider any factor, say p_1'. Then if we project onto $\mathbb{C}_\alpha$ along $\mathbb{C}_{s_2}$ we obtain either a trivial point (if $p_1'(\alpha,s_2)$ is independent of s_2) or a sphere covering of $\mathbb{C}_\alpha$ which corresponds to a 'pole' at infinity. The topological properties of the pole are specified by the nature of the covering.

Returning to example 5.1 we have

$$K_2'(\alpha,s_2) = \frac{\alpha^2 s_2^2+s_2^2}{\alpha s_2 . s_2} = \frac{(\alpha^2 s_2+1)}{\alpha . s_2}$$

Now, $p'(\alpha,s_2) = \alpha^2 s_2+1$ is a one-sheeted covering of $\mathbb{C}_\alpha$ and is therefore a sphere. $q_1'(\alpha,s_2) = \alpha$ is trivial and does not represent a pole at infinity and $q_2'(\alpha,s_2) = s_2$ is again a sphere covering of $\mathbb{C}_\alpha$. Hence the system (5.5) has a pole and a zero at infinity which 'cancel' so that no branch of the root locus approaches the projective 1-subspace $\mathbb{P}^1(\mathbb{C})$ at infinity which we have already seen above. Incidentally, if we parameterise $\mathbb{P}^1(\mathbb{C})$ at infinity by $\beta = s_2/s_1$ instead of α then we obtain

$$K_2''(s_1,\beta) = \frac{s_1^2 + \beta s_1}{\beta s_1^2} = \frac{s_1 + \beta}{\beta s_1}$$

and we arrive at the same conclusion as before.

We may now state the following result

<u>Theorem 5.2</u> Let the homogeneous degree 2 system be defined by

$$K_2(s_1,s_2) = \frac{p(s_1,s_2)}{q(s_1,s_2)} \tag{5.7}$$

and substitute $s_1 = \alpha s_2$. Then write $K_2(\alpha s_2,s_2)$ in the form

$$K_2'(\alpha,s_2) = \frac{p_1'(\alpha,s_2) \;....\; p_{m'}'(\alpha,s_2)}{q_1'(\alpha,s_2) \;....\; q_{n'}'(\alpha,s_2)}$$

where each p_i' and q_j' is irreducible has a polynomial in $(\alpha,s_2))$ and the p_i''s and q_j''s are relatively prime. Suppose that no p_i or q_j is independent of s_2. Then there are m' poles and n' zeros of the system (5.7) at infinity.

(Note that the factors p_i' correspond to poles at infinity while the factors q_j' correspond to zeros. For example, if $K_2 = {}^1/s_1$ then at infinity $K_2' = {}^1/\alpha s_2$ which gives a sphere covering of the X_2 axis and corrresponds to the zero of K_2 when $s_1 = \infty$).

Example 5.3 Consider

$$K_2(s_1,s_2) = \frac{s_1}{(s_1+1)s_2}$$

This system has two poles and one zero in the finite (affine) space $\mathbb{C}^2$ and, putting $s_1 = \alpha s_2$, we see that

$$K_2'(\alpha,s_2) = \frac{\alpha\, s_2}{(\alpha s_2+1)s_2}$$

and so it has one zero at infinity. The root locus starts at the two spheres given by

$$s_1 + 1 = 0,\ s_2 = 0$$

and tends to the zero $s_1 = 0$ and the sphere covering of $\mathbb{C}_{s_1}$ as $k \to \infty$. To see this intuitively, note that the root locus is given by

$$(s_1+1)s_2 + ks_1 = 0$$

and so

$$s_2 = -\frac{ks_1}{s_1+1}$$

Now $s_1/(s_1+1)$ can take on any complex value for $s_1 \in \mathbb{P}^1(C)$ and so $-k\, s_1/(s_1+1)$ can take on any arbitrarily large complex value as $k \to \infty$. Hence we obtain a simple covering of $\mathbb{C}_{s_1}$ at $s_2 = \infty$.

6. Conclusions

In this paper we have studied the spectral theory of homogeneous degree 2 systems and we have seen that the classical notions of poles and zeros generalise to irreducible subvarieties of $\mathbb{P}^2(\mathbb{C})$. This is reasonable since the irreducible subvarieties of $P^1(\mathbb{C})$ are just points and these correspond to irreducible components of polynomials of one variable.
We have shown that the inversion problem for the two-dimensional Laplace transform is equivalent to the problem of finding Green's functions for irreducible linear partial differential operators with constant coefficients (when the denominator has no repeated factors). The elementary

theory of projective curves in $\mathbb{P}^2(\mathbb{C})$ has been applied to obtain a root locus theory for homogeneous degree 2 systems.

We may, of course, ask to what extent we may generalise the theory to systems of higher degree. For degree three systems we immediately arrive at a problem, for the classification of the projective subvarieties of $\mathbb{P}^3(\mathbb{C})$ is not even known. Many topological invariants for projective varieties have been discovered, but a complete classification of the topollogy of subvarieties of $\mathbb{P}^n(\mathbb{C})$ is at the heart of modern algebraic geometry (see Iitaka, 1982 or Hartshorne, 1977), and is one of the fundamental problems of contemporary mathematics.

In view of the above remarks, we can 'reduce' the general theory of nonlinear systems to the study of partial differential operators and algebraic geometry, which gives some further insight into why nonlinear systems are so difficult to deal with in general. For, a classification of nonlinear systems according to their spectral types is equivalent to a classification of all subvarieties of $\mathbb{P}^n(\mathbb{C})$, which, as we have said, is equivalent (essentially) to algebraic geometry.

7. References

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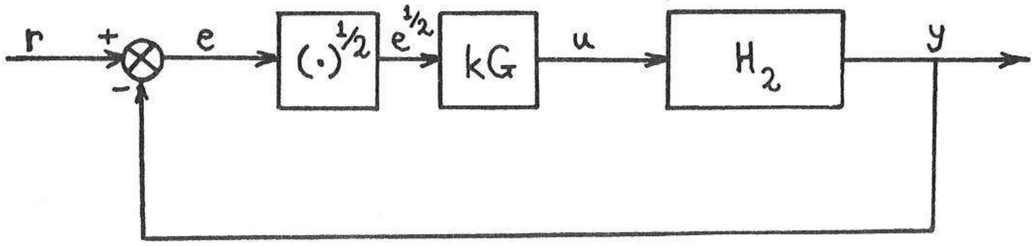


Fig 2.1

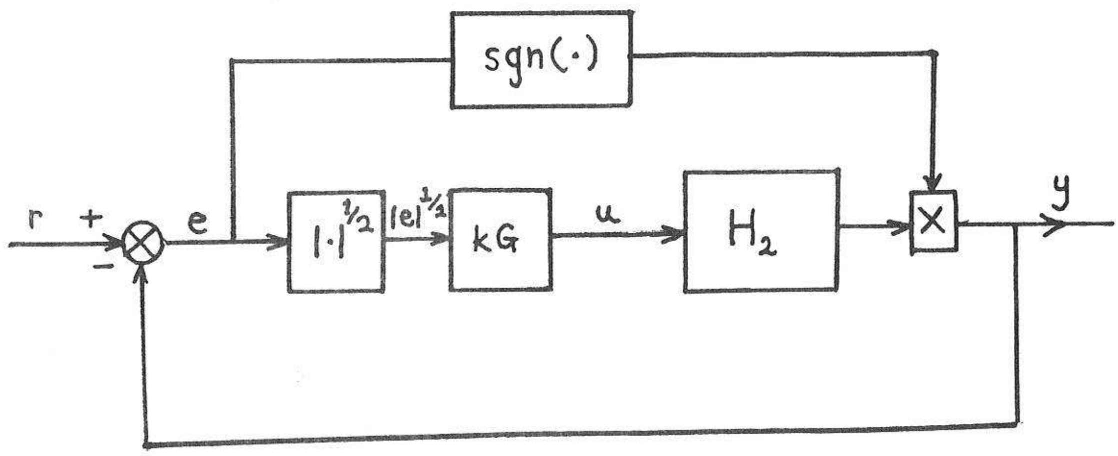


Fig 2.2

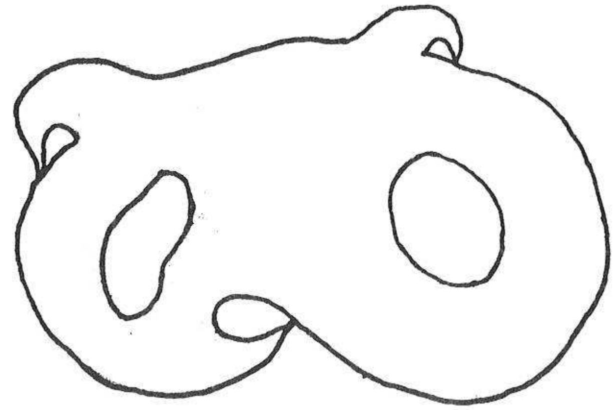


fig. 3.1.

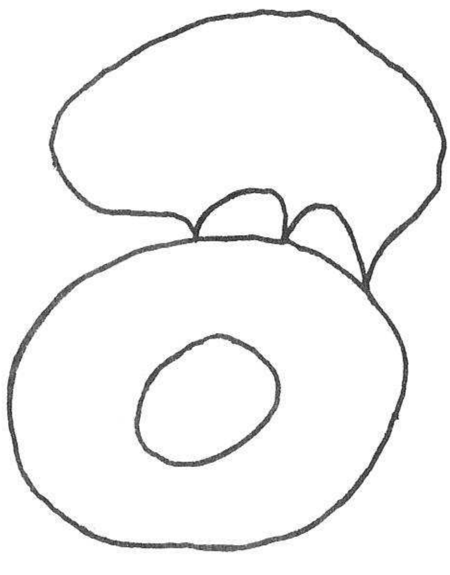
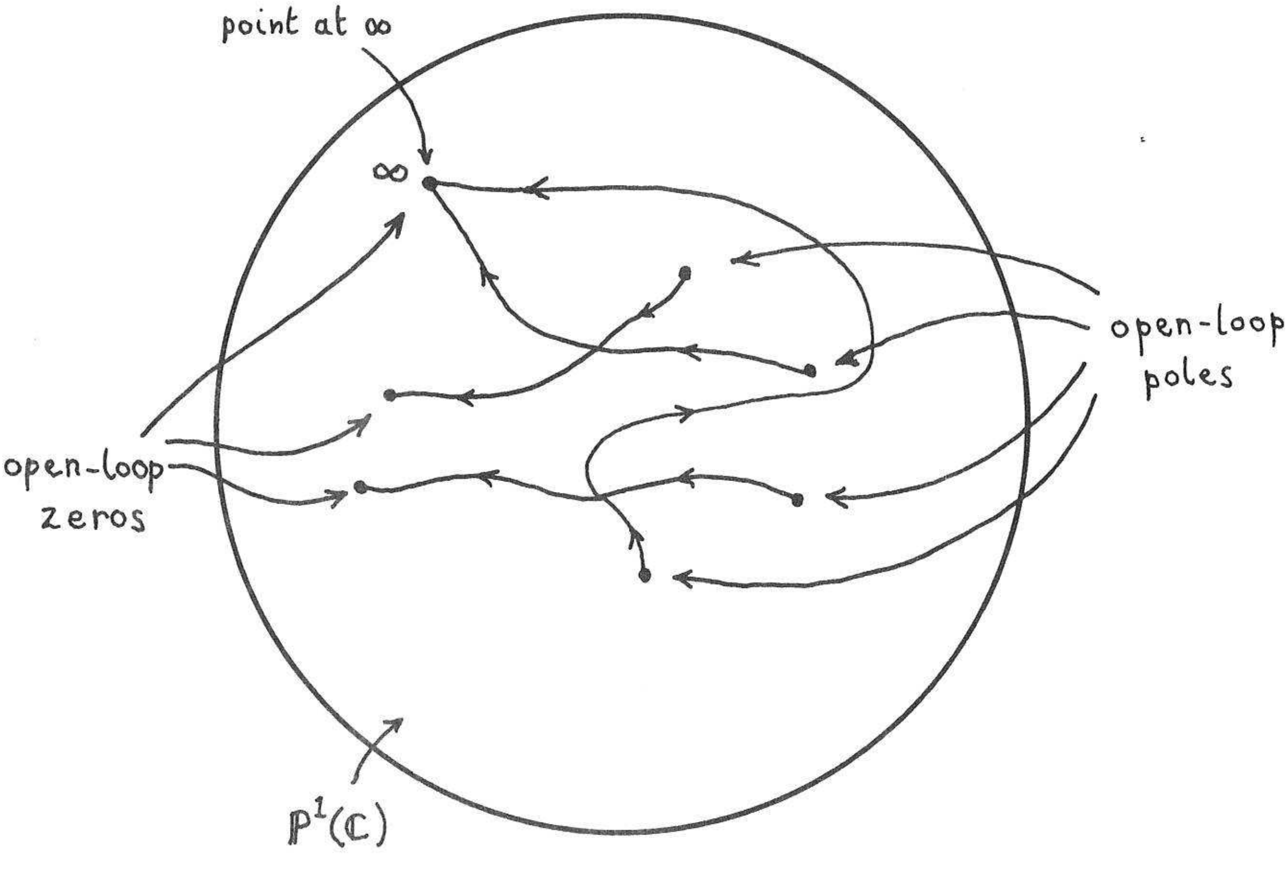
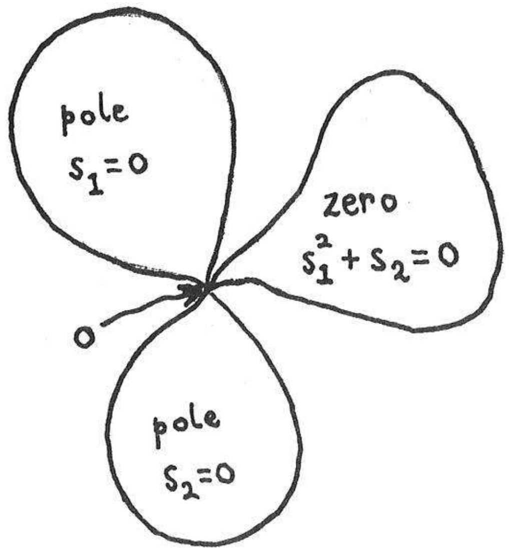
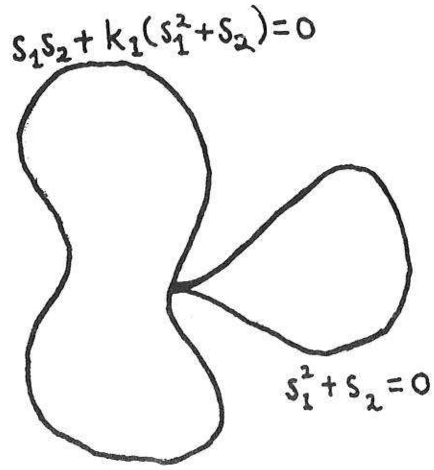


fig. 3.2.

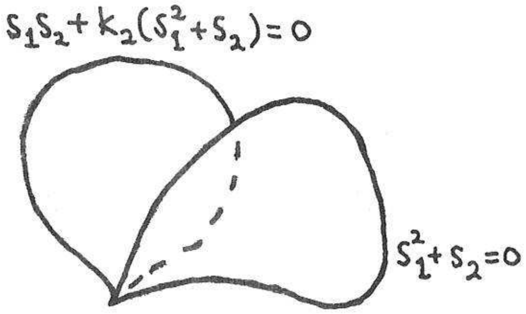

point at ∞
∞
open-loop poles
open-loop zeros
$\mathbb{P}^1(\mathbb{C})$



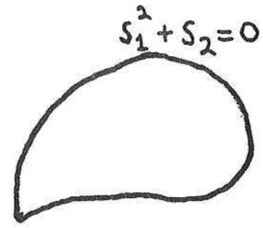
(a). $k=0$



(b). $k=k_1>0$



(c). $k=k_2>k_1>0$



(d). $k=\infty$