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NONLINEAR PERTURBATIONS OF DYNAMICAL SYSTEMS AND NONLINEAR CONTROLLABILITY

by

S.P.Banks

University of Sheffield

Department of Control Engineering

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ABSTRACT

The nonlinear variation of constants formula is generalized to the case where the unperturbed operator has non-elliptic Fréchet derivation and is applied to nonlinear controllability.

1. Introduction

The relation between the solution of the nonlinear ordinary differential equation

$$\dot{x}(t) = f(x,t) \tag{1.1}$$

and that of the nonlinear perturbation

$$\dot{y}(t) = f(y,t) + g(y,t) \tag{1.2}$$

is known as the nonlinear variation of constants formula and can be expressed in the form

$$y(t;t_o,x_o) = x(t;t_o,x_o) + \int_{t_o}^{t} \Phi(t,s,y(s;t_o,x_o))g(y(s;t_o x_o),s)ds$$

where

$$\Phi(t;\ t_o,x_o) \triangleq \frac{\partial}{\partial x_o}\left[x(t;t_o,x_o)\right]$$

is the fundamental solution of the variational system

$$\dot{Z} = f_x\left[x(t;t_o,x_o),t\right] Z.$$

Here, $x(t;t_o,x_o)$, $y(t;t_o,x_o)$ are the solutions of (1.1) and (1.2) (which are assumed to exist) with initial condition x_o at t_o. This formula has been used successfully in stability studies (Brauer, 1966) and in nonlinear observer theory (Banks, 1981). Moreover, it has recently been generalized (Banks, 1984) to distributed systems of the form

$$\dot{\psi} = N\psi + M\psi \tag{1.3}$$

where N and M are nonlinear operators and $\mathcal{J}N(\phi(t))$ generates an evolution operator, where ϕ is the solution of the equation

$$\dot{\phi} = N\phi.$$

We wish to generalize this result to the case where $\mathcal{J}N(\phi(t))$ may only define a quasi-evolution operator (Curtain and Pritchard, 1978).

The nonlinear variation of constants formula will then be used to generalize the nonlinear controllability results of Magnusson et al (1980) to systems of the form

$$\dot{\psi} = N\psi + M\psi + Bu\ . \tag{1.4}$$

These results may be summatised in the following way. For the system

$$\dot{z} = Az + Nz + Bu \tag{1.5}$$

where A and B are linear operators and N is a nonlinear operator, we have the linear variation of constants formula

$$z(t) = \int_0^t T(t-s)Nz(s)ds + \int_0^t T(t-s)Bu(s)ds \tag{1.6}$$

if A generates a semigroup T(t). Define the linear operators L_t by

$$L_t x = \int_0^t T(t-s)x(s)ds \quad ,$$

and G by

$$Gu = \int_0^T T(t-s)Bu(s)ds \quad ,$$

on appropriate spaces. Then on the space $X = \mathcal{D}(G)/\mathrm{Ker}G$ we put the quotient norm and define $\tilde{G}$ by

$$\tilde{G}[u] = Gu \quad , \qquad u \in [u]$$

where $[u]$ is the image of u in $\mathcal{D}(G)/\mathrm{Ker}G$. It can then be shown that we can define a norm on the range of G ($\triangleq V$) by

$$\|v\|_V = \|\tilde{G}^{-1}v\|_X \quad .$$

If we wish to control from 0 to v, then we define the control

$$u(s) = \tilde{G}^{-1}\left[v - \int_0^T T(t-p)Nz(p)dp\right](s) \tag{1.7}$$

$$= \tilde{G}^{-1}\left[v - L_T Nz\right](s) \quad .$$

By (1.6), we have

$$z(T) = L_T Nz + L_T B\tilde{G}^{-1}\left[v - L_T Nz\right]$$

$$= L_T Nz + G\tilde{G}^{-1}\left[v - L_T N_z\right]$$

$$= v$$

The only difficulty now is to prove the existence of a solution of (1.6) with the control given by (1.7). This can be done using a variety of fixed point theorems.

In this paper, we shall illustrate this method using the contraction mapping principle applied to the nonlinear variation of constants formula. The only difficulty here is that the operator

$$G_\psi u = \int_0^T \Phi(t-s;\psi)Bu(s)\,ds$$

now depends on the solution ψ on (1.4) and so we must show that the various norms on Φ are independent of ψ.

2. Notation and Terminology

Throughout this paper, H will denote a Hilbert space while X, B_1, B_2 etc denote Banach spaces. In particular $H^p(\Omega)$, $H_o^{\ p}(\Omega)$, $L^p(\Omega)$ will denote the usual Sobolev and Lebesgue spaces although we shall normally not include reference to the domain Ω and write, simply, H^p, $H_o^{\ p}$, L^p etc.

An operator N defined on a subset D of a Hilbert space H is called dissipative on D if

$$\langle Nx_1 - Nx_2, x_1 - x_2\rangle_H \leq 0 \quad , \quad \forall\ x_1, x_2 \in D$$

and strictly dissipative if

$$\langle Nx_1 - Nx_2, x_1 - x_2\rangle_H \leq -\varepsilon \|x_1 - x_2\|_H^2 \quad , \quad \forall\ x_1, x_2 \in D$$

for some $\varepsilon>0$. It often happens that, although N is a nonlinear operator, it is defined on a linear subspace $\mathcal{D}(N)$ of H. However, the subset of $\mathcal{D}(N)$ on which N is dissipative is usually strictly smaller than $\mathcal{D}(N)$. For this reason we shall denote by $\mathcal{D}_d(N)$ the dissipative domain of N; i.e. the set where N is dissipative.

The space of n times continuously differentiable functions defined on the region Ω is denoted by $C^n(\Omega)$, and the space of bounded operators from a Banach space X_1 to a Banach space X_2 is denoted $\mathcal{L}(X_1,X_2)$ or $\mathcal{L}(X)$ if $X_1 = X_2 = X$.

Finally if $N : B_1 \to B_2$ is a differentiable nonlinear operator we denote by $\mathfrak{Y}N(x)$ the (Fréchet) derivative of N at x. Note that, for each x,

$$\mathfrak{Y}N(x) \in \mathcal{L}(B_1,B_2) \quad .$$

The remaining notation which we use is either standard or is introduced as we proceed.

3. Nonlinear Systems and Variational Equations

In the theory of finite-dimensional differential equations (Coddington and Levinson, 1959) it is well known that if we consider a system

$$\dot{x} = f(x,t) \quad , \quad x(t_o) = x_o \quad , \qquad (3.1)$$

and we know that $x(t;t_o,x_o)$ satisfies (3.1), then the function

$$\Phi(t;t_o,x_o) \triangleq \frac{\partial}{\partial x_o}\left[x(t;t_o,x_o)\right] \qquad (3.2)$$

is the fundamental solution of the 'variational equation'

$$\dot{Z} = f_x\left[x(t;t_o,x_o),t\right] Z \qquad (3.3)$$

For simplicity, we shall consider autonomous systems in this paper. This result has been generalized recently by Banks (1984), to the case of an infinite dimensional nonlinear system of the form

$$\dot{\phi} = N\phi \qquad (3.4)$$

where N generates a (nonlinear) semigroup and $\mathcal{F}N(\bar{\phi})$ ($\in \mathcal{L}(\mathcal{D}(N),H)$) exists uniformly for each $\bar{\phi} \in \mathcal{D}_d(N)$ (the subset of $\mathcal{D}(N)$ on which N is a dissipative operator). Note that it is assumed that N is defined on a <u>linear</u> space $\mathcal{D}(N)$. In order to prove the generalization of (3.3) we assumed, essentially, that along a solution $S(t)\phi_o$ of (3.4), $\mathcal{F}N(S(t)\phi_o) : D(N) \to L^2$ is an elliptic operator.

In this section we wish to generalize this result to the case where $\mathcal{F}N(S(t)\phi_o)$ is not an elliptic operator, but can be expressed as a sum of an elliptic operator and another suitable operator. We shall need the following simple lemma, whose proof is elementary.

<u>Lemma 3.1</u>: The value of the integral $I_n(t) = \int_o^t \frac{s^{n/2}}{(t-s)^{\frac{1}{2}}} ds$ is given by

$$I_n(t) = 2\left(\frac{n(n-2)(n-4)\dots 2}{1.3.5\dots(n-1)(n+1)}\right) t^{\frac{n+1}{2}} \quad , \text{ n even}$$

and

$$I_n(t) = \frac{\pi}{2}\left(\frac{n(n-2)(n-4)\dots 1}{(n+1)(n-1)\dots 2}\right) t^{\frac{n+1}{2}} \quad , \text{ n odd} \ . \ \square$$

Let $\gamma_n = I_n(t)t^{-(\frac{n+1}{2})}$.

<u>Theorem 3.2</u>. Suppose that the operator A(t) is given by

$$A(t)\phi = A\phi + B(t)\phi$$

where A is linear and $\mathcal{D}(A) = \mathcal{D}(A(t)) = H^2 \cap H_o^1$ for each t and

$$B(\cdot) \in L^\infty(\mathbb{R}^+;\mathcal{L}(L^2,L^1)) \ .$$

Assume also that A generates a semigroup T_t for which

$$||T_t||_{\mathcal{L}(L^2)} \leq Me^{-\alpha t}$$

and

$$\|T_t\|_{\mathcal{L}(L^{1+\varepsilon},L^2)} \leq \frac{Me^{-\alpha t}}{t^{\frac{1}{2}}} \quad , \text{ for some } \varepsilon>0 .$$

Then the equation

$$U(t,\tau)\phi = T_t\phi + \int_\tau^t T_{t-s} B(s)U(s,\tau)\phi ds \tag{3.5}$$

is soluble for $U(t,\tau)\phi$ for all $\phi \in L^2$.

Proof. For simplicity we shall take $\tau = 0$; the general case is similar. Define

$$U_o(t,0)\phi = T_t\phi$$

$$U_n(t,0)\phi = \int_o^t T_{t-s}B(s)U_{n-1}(s,0)\phi ds \quad , \qquad n\geq 1$$

Then,

$$\|U_o(t,0)\phi\|_{L^2} \leq Me^{-\alpha t} \|\phi\|_{L^2}$$

$$\|U_o(t,0)\phi\|_{L^2} \leq \frac{Me^{-\alpha t}}{t^{\frac{1}{2}}} \|\phi\|_{L^{1+\varepsilon}} \quad .$$

Hence,

$$\|U_1(t,0)\phi\|_{L^2} \leq \int_o^t \|T_{t-s}B(s)U_o(s,0)\phi\|_{L^2} ds$$

$$\leq \int_o^t \|T_{t-s}\|_{\mathcal{L}(L^1,L^2)} \|B(s)\|_{\mathcal{L}(L^2,L^1)} \|U_o(s,0)\phi\|_{L^2} ds$$

$$\leq \int_o^t \frac{Me^{-\alpha(t-s)}}{(t-s)^{\frac{1}{2}}} \beta Me^{-\alpha s} \|\phi\|_{L^2} ds$$

$$= e^{-\alpha t}\beta M^2 \int_o^t \frac{1}{(t-s)^{\frac{1}{2}}} ds \cdot \|\phi\|_{L^2} \quad ,$$

$$= 2e^{-\alpha t}\beta M^2 t^{\frac{1}{2}} \,||\phi||_{L^2}$$

where

$$\beta = ||B(\cdot)||_{L^\infty(\mathcal{L}(L^2,L^{1+\varepsilon}))} .$$

Similarly,

$$||U_2(t,0)\phi||_{L^2} \leq \int_o^t ||T_{t-s}B(s)U_1(s,0)\phi||_{L^2}\, ds$$

$$\leq \int_o^t \frac{Me^{-\alpha(t-s)}}{(t-s)^{\frac{1}{2}}}\, \beta\, 2e^{-\alpha s}\beta M^2 s^{\frac{1}{2}}\, ||\phi||_{L^2}\, ds$$

$$= 2M^3\beta^2 e^{-\alpha t} \int_o^t \frac{s^{\frac{1}{2}}}{(t-s)^{\frac{1}{2}}}\, ds \cdot ||\phi||_{L^2}$$

$$= 2M^3\beta^2 e^{-\alpha t}\gamma_1 t\, ||\phi||_{L^2} .$$

Generally, we have

$$||U_m(t,0)\phi||_{L^2} \leq 2M^{m+1}\beta^m e^{-\alpha t}(\gamma_1\cdots\gamma_{m-1})\, t^{m/2}\, ||\phi||_{L^2} .$$

It is easily shown that

$$\gamma_1\cdots\gamma_{m-1} = \begin{cases} \dfrac{\pi^{(m-1)/2}}{3.5.7...m} & , \text{ m odd} \\ \dfrac{\pi^{(m-1)/2}}{2.4.6...m} & , \text{ m even} \end{cases}$$

and so

$$\gamma_1\cdots\gamma_{m-1} \leq \begin{cases} \{(\frac{\pi}{2})^{(m-1)/2}\} \;/(\frac{m-1}{2})! & , \quad \text{m odd} \\ \pi^{-\frac{1}{2}}\, (\frac{\pi}{2})^{m/2} \;/\; (\frac{m}{2})! & , \quad \text{m even} \end{cases}$$

Now we define

$$U(t,0)\phi = \sum_{m=o}^{\infty} U_m(t,0)\phi \quad ,$$

and note that

$$||U(t,0)\phi||_{L^2} \leq \sum_{m=o}^{\infty} ||U_m(t,0)\phi||_{L^2}$$

$$\leq Me^{-\alpha t} ||\phi||_{L^2}$$

$$+ \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} 2e^{-\alpha t}\beta^m M^{m+1} \left(\frac{\pi}{2}\right)^{(m-1)/2} \frac{t^{m/2}}{((m-1)/2)!} ||\phi||_{L^2}$$

$$+ \sum_{\substack{m=2 \\ m \text{ even}}}^{\infty} 2e^{-\alpha t}\beta^m M^{m+1}\pi^{-\frac{1}{2}} \left(\frac{\pi}{2}\right)^{m/2} \frac{t^{m/2}}{(m/2)!} ||\phi||_{L^2}$$

$$= \left\{\left(1 - \frac{2}{\pi^{\frac{1}{2}}}\right)Me^{-\alpha t} + 2t^{\frac{1}{2}}e^{-\alpha t}\beta M^2 \exp\left(M^2\beta^2 \frac{\pi t}{2}\right)\right.$$

$$\left. + 2\pi^{-\frac{1}{2}}Me^{-\alpha t}\exp\left(M^2\frac{\beta^2}{2}\pi t\right)\right\}||\phi||_{L^2}$$

$$= Me^{-\alpha t}\left\{\left(1 - \frac{2}{\pi^{\frac{1}{2}}}\right) + 2(t^{\frac{1}{2}}\beta M + \pi^{-\frac{1}{2}})\exp\left(\frac{M^2\beta^2\pi t}{2}\right)\right\}||\phi||_{L^2}$$

Hence, $U(t,0)$ is a bounded operator for each $t \geq 0$. Similarly, we could show that $U(t,s)$, defined in the obvious way, is a bounded operator for each $t \geq s$. It is now a simple matter to show that $U(t,s)$ satisfies (3.5). □

It is easy to show that theorem 3.2 may be generalised as follows:

Corollary 3.3. Suppose that

$$A(t)\phi = A_1(t)\phi + B(t)\phi$$

where $B(\cdot)$ is as in theorem 3.2 and $A_1(t)$ is now time-dependent and generates an evolution operator $V(t,s)$ such that

$$||V(t,s)||_{\mathcal{L}(L^2)} \leq Me^{-\alpha(t-s)}$$

$$||V(t,s)||_{\mathcal{L}(L^{1+\varepsilon},L^2)} \leq \frac{Me^{-\alpha(t-s)}}{(t-s)^{\frac{1}{2}}} \quad , \quad \text{for some } \varepsilon>0 .$$

Then the equation

$$U(t,\tau)\phi = V(t,\tau)\phi + \int_\tau^t V(t,s)B(s)U(s,\tau)\phi \, ds$$

is soluble for $U(t,\tau)\phi$ for all $\phi \in L^2$. □

Example 3.4. Consider the nonlinear heat equation

$$\frac{\partial\phi}{\partial t} = N\phi \triangleq \mathcal{K}(\phi) \frac{\partial^2\phi}{\partial x^2} \quad , \quad 0 \leq x \leq 1 \tag{3.6}$$

where the coefficient $\mathcal{K}$ depends on the temperature. It is well known that this equation has a solution. Suppose that $\mathcal{K}$ satisfies the condition

$$\left| \frac{\mathcal{K}(x_1)-\mathcal{K}(x_2)}{x_1-x_2} \right| \leq K_d < \infty \quad , \quad |\mathcal{K}(x_1)| \leq K_m \quad \text{for all } x_1,x_2 .$$

Then, for $\phi_1,\phi_2 \in H_o^1 \cap H^2 \cap C^2$, we have

$$<\mathcal{K}(\phi_1) \frac{\partial^2\phi_1}{\partial x^2} - \mathcal{K}(\phi_2) \frac{\partial^2\phi_2}{\partial x^2} \ , \ \phi_1-\phi_2>_{L^2}$$

$$= \int_o^1 \mathcal{K}(\phi_1) \left(\frac{\partial^2\phi_1}{\partial x^2} - \frac{\partial^2\phi_2}{\partial x^2} \right) (\phi_1-\phi_2)\,dx + \int_o^1 (\mathcal{K}(\phi_1)-\mathcal{K}(\phi_2)) \left(\frac{\partial^2\phi_2}{\partial x^2} \right) (\phi_1-\phi_2)\,dx$$

$$= - \int_o^1 \mathcal{K}'(\phi_1) \frac{\partial\phi_1}{\partial x} \left(\frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial x} \right) (\phi_1-\phi_2)\,dx - \int_o^1 \mathcal{K}(\phi_1) \left(\frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial x} \right)^2 dx$$

$$+ \int_o^1 \frac{\mathcal{K}(\phi_1)-\mathcal{K}(\phi_2)}{\phi_1-\phi_2} \left(\frac{\partial^2\phi_2}{\partial x^2} \right) (\phi_1-\phi_2)^2 dx$$

$$\leq K_d \max_{x\in[0,1]} \left|\frac{\partial\phi_1}{\partial x}\right| \left\|\frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial x}\right\|_{L^2} \|\phi_1-\phi_2\|_{L^2}$$

$$- K_m \left\|\frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial x}\right\|^2_{L^2} + K_d \max_{x\in[0,1]} \left|\frac{\partial^2\phi_2}{\partial x^2}\right| \|\phi_1-\phi_2\|^2_{L^2}$$

$$\leq \frac{K_d}{\pi}\left[\max_{x\in[0,1]}\left|\frac{\partial\phi_1}{\partial x}\right| + \frac{1}{\pi}\max_{x\in[0,1]}\left|\frac{\partial^2\phi_2}{\partial x^2}\right|\right] \left\|\frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial x}\right\|^2_{L^2}$$

$$- K_m \left\|\frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial x}\right\|^2_{L^2}$$

(Since $\pi^2 \|\phi_1-\phi_2\|^2_{L^2} \leq \left\|\frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial x}\right\|^2_{L^2}$)

Now, if $C_\varepsilon^2[0,1]$ denotes the set of $\phi \in C^2[0,1]$ such that

$$K_m - \frac{K_d}{\pi}\left(\max_{x\in[0,1]}\left|\frac{\partial\phi}{\partial x}\right| + \frac{1}{\pi}\max_{x\in[0,1]}\left|\frac{\partial^2\phi}{\partial x^2}\right|\right) \triangleq \varepsilon > 0$$

then,

$$\left\langle \kappa(\phi_1)\frac{\partial^2\phi_1}{\partial x^2} - \kappa(\phi_2)\frac{\partial^2\phi_2}{\partial x^2}, \phi_1-\phi_2\right\rangle_{L^2} \leq -\varepsilon\pi^2 \|\phi_1-\phi_2\|^2_{L^2}$$

for $\phi_1, \phi_2 \in H_o^1 \cap H^2 \cap C_\varepsilon^2$.

Hence, A is a dissipative operator on $\mathcal{D}_d(N) \triangleq H_o^1 \cap H^2 \cap C_\varepsilon^2$ and it follows that A generates a nonlinear contraction semigroup S(t) on $\overline{\mathcal{D}_d(N)}$ (cf. Barbu, 1976). Moreover, we have

$$\|S(t)\phi\| \leq e^{-\varepsilon\pi^2 t}\|\phi\| \quad \text{for all } \phi \in \overline{\mathcal{D}_d(N)} .$$

Now, the Fréchet derivative of N is given by

$$\mathcal{F}N(\psi)(\phi) = \kappa(\psi)\frac{\partial^2\phi}{\partial x^2} + \kappa'(\psi)\frac{\partial^2\psi}{\partial x^2}\cdot\phi$$

for $\psi \in \mathcal{D}_d(N)$. Consider the linear operator

$$A(t)\phi = \kappa(\psi(t)) \frac{\partial^2 \phi}{\partial x^2} + \kappa'(\psi(t)) \frac{\partial^2 \psi(t)}{\partial x^2} \phi$$

for $\psi(t) \in \mathcal{D}_d(N)$ for all $t \geq 0$. Then

$$A = \frac{\partial^2}{\partial x^2}$$

satisfies the condition of theorem 3.2; for it is well known that

$$\|T_t\|_{\mathcal{L}(H^{-1},L^2)} \leq \frac{M_1 e^{-\alpha t}}{t^{\frac{1}{2}}} \quad , \qquad \text{some } M_1 > 0$$

and since $H^1 \subseteq L^{(1+\varepsilon)/\varepsilon}$ (by the Sobolev embedding theorem) we have, by duality, $L^{(1+\varepsilon)} \subseteq H^{-1}$. A simple limiting argument now shows that $A_1(t) \triangleq \kappa(\psi(t)) \frac{\partial^2}{\partial x^2}$ satisfies the condition of corollary 3.3.

Also,

$$B(t)\phi \triangleq \kappa'(\psi(t)) \frac{\partial^2 \psi(t)}{\partial x^2} \phi$$

satisfies

$$\|B(t)\phi\|_{L^{1+\varepsilon}} \leq K_d \cdot \left\| \frac{\partial^2 \psi(t)}{\partial x^2} \right\|_{L^{2\bar{\varepsilon}}} \left\{ \|\phi\|_{L^2} \right\},$$

where $\bar{\varepsilon} = (1+\varepsilon)/(1-\varepsilon)$. However, if $\psi(t) = S(t)\psi(o)$ for some $\psi(o) \in \mathcal{D}_d(N)$, then $S(t)\psi(o) \in \overline{\mathcal{D}_d(N)}$ for all t and so $\left\| \frac{\partial^2}{\partial x^2} \psi(t) \right\|_{L^{2\bar{\varepsilon}}} \in L^\infty(0,\infty)$.

Hence,

$$B(\cdot) \in L^\infty(\mathbb{R}^+, \mathcal{L}(L^2, L^1)) .$$

Returning to the general system

$\dot{\phi} = N\phi$

(cf. (3.4)) we can now easily generalise theorem 3.2.4 in Banks (1984) to obtain

<u>Theorem 3.5.</u> Let $N : \mathcal{D}(N) \to H$ be a nonlinear operator which generates a contraction semigroup $S(t)\phi$ on $\mathcal{D}_d(N)$, and suppose that $\mathcal{F}N(\phi)$ $(\in \mathcal{L}(\mathcal{D}(N),H))$ exists uniformly for each $\phi \in \mathcal{D}_d(N)$. Suppose that $\mathcal{F}N(S(t)\phi)$ satisfies the conditions of corollary 3.3, i.e. it is of the form $A_1(t)+B(t)$. Then $S(t)\phi$ is differentiable with respect to ϕ and satisfies the relation

$$\mathcal{F}_{\phi o} S(t)\phi_o = U(t,0)$$

where U is the quasi-evolution operator which is the solution of the equation

$$U(t,\tau)\phi = V(t,\tau) \quad + \int_\tau^t V(t,s)B(s)U(s,\tau)\phi \, ds \tag{3.7}$$

and V is the evolution operator generated by $A_1(t)$. □

We now define

$$\Phi(t,\phi_o) = \mathcal{F}_{\phi o}\left[S(t)\phi_o\right] \in \mathcal{L}(\mathcal{D}(N),H)$$

Note that we have

$$\frac{\partial}{\partial t_o} S(t-t_o)\phi_o = -\Phi(t-t_o,\phi_o)N\phi_o \qquad \text{a.e.} \tag{3.8}$$

For, we have

$$\left(\frac{d}{dt}\right)^+ S(t-t_o)\phi_o = NS(t-t_o)\phi_o$$

where $(d/dt)^+$ denotes the right derivative. However $NS(t-t_o)\phi_o$ is differentiable from the right with respect to t_o, since

$$\left(\frac{\partial}{\partial t_o}\right)^+ NS(t-t_o)\phi_o = \mathcal{F}N(S(t-t_o)\phi_o)\ \frac{\partial^+ S(t-t_o)}{\partial t_o}\ \phi_o$$

and so $\left(\frac{d}{dt}\right)^+ S(t-t_o)\phi_o$ is also differentiable from the right with respect to t_o and

$$\left(\frac{\partial}{\partial t_o}\right)^+ \left(\frac{d}{dt}\right)^+ S(t-t_o)\phi_o = \left(\frac{d}{dt}\right)^+ \left(\frac{\partial}{\partial t_o}\right)^+ S(t-t_o)\phi_o$$

$$= \mathcal{F}N(S(t-t_o)\phi_o)\ \frac{\partial^+}{\partial t_o}\ S(t-t_o)\phi_o$$

$$= A_1(t-t_o)\ \frac{\partial^+}{\partial t_o}\ S(t-t_o)\phi_o + B(t-t_o)\ \frac{\partial^+}{\partial t}\ S(t-t_o)\phi_o \quad .$$

Thus,

$$\left(\frac{\partial}{\partial t_o}\right)^+ S(t-t_o)\phi_o = -V(t-t_o,0)N\phi_o + \int_{t_o}^{t} V(t-t_o,s-t_o)B(s-t_o) \times \frac{\partial^+}{\partial t}\ S(s-t_o)\phi_o ds$$

since $\frac{\partial^+}{\partial t_o}\ S(t-t_o)\phi_o = -N\phi_o$ when $t = t_o$. The asserted relation (3.8) now follows by uniqueness of the solution of (3.7).

We may now consider the perturbation

$$\dot{\psi} = N\psi + M(t)\psi \qquad (3.9)$$

of equation (3.4) for some nonlinear operator $M(t)$. Suppose that the equations (3.4) and (3.9) both have solutions which exist for all t for initial conditions in some subset $C \subseteq H$, and that these solutions, say $\phi(t;\phi_o)$, $\psi(t;\psi_o)$ satisfy

$$\phi(t;\phi_o), \psi(t;\psi_o) \in C \quad \text{for } \phi_o, \psi_o \in C.$$

Then we can state the following result which says that the solutions of (3.4) and (3.9), starting from the same initial condition ϕ_o are related by the nonlinear variation of constants formula. (The proof is the same as in the finite-dimensional case if we take account of (3.8).)

Theorem 3.6. Under the above assumptions we have

$$\psi(t;\phi_o) - \phi(t;\phi_o) = \int_o^t \Phi(t-s;\psi(s;\phi_o))M(s)\psi(s;\phi_o)ds \ . \square$$

Example 3.7. Returning to example 3.4, we consider the nonlinear perturbation

$$\frac{\partial\psi}{\partial t} = \mathcal{K}(\psi)\frac{\partial^2\psi}{\partial x^2} - \psi^2 \ , \quad 0 \le x \le 1 \ , \ \psi(o) = \psi_o \tag{3.10}$$

of equation (3.6), on the time interval $[0,T]$. For each positive integer n consider the sequence of approximations of equation (3.10) defined as follows:

First subdivide the interval $I_1 \triangleq [0,T]$ into n equal parts, i.e.

$$I_{nj} = \left[(j-1)\frac{T}{n}, j\frac{T}{n}\right] \quad , \quad 1 \le j \le n$$

Now consider the system defined on I_{nj} by

$$\frac{\partial\psi_{nj}(x,t)}{\partial t} = \mathcal{K}\left(\psi_{nj}(x,(j-1)\frac{T}{n})\right)\frac{\partial^2\psi_{nj}(x,t)}{\partial x^2} - \psi_{nj}{}^2(x,t) \quad , \tag{3.11}$$

with initial condition $\psi_{nj}(x,(j-1)\frac{T}{n}) = \psi_{n(j-1)}(x,(j-1)\frac{T}{n})$ for j>1 and $\psi_{n1}(x,0) = \psi_o(x)$. We can write each equation (3.11) in the form

$$\frac{\partial\psi_{nj}(x,t)}{\partial t} = A_{nj}\,\psi_{nj}(x,t) - \psi_{nj}^2(x,t) \tag{3.12}$$

where each operator A_{nj} is time-invariant and is clearly strongly elliptic. If $A = \partial^2/\partial x^2$, then $\mathcal{D}(A_{nj}) = \mathcal{D}(A) = H_o^1 \cap H^2$ for all n and j. Let $T_{nj}(t)$ denote the semigroup generated by A_{nj}. Then it is well-known (Henry, 1981) that the equations (3.12) have a unique solution (for $\psi_o \in \mathcal{D}(A)$) given by

$$\psi_{nj}(x,t) = T_{nj}(t)\psi_{nj}(x,(j-1)T/n) - \int_{(j-1)\frac{T}{n}}^{t} T_{nj}(t-s)\psi_{nj}^2(x,s)ds \ , \quad (3.13)$$

for $t \in I_{nj}$. However, using the above notation it is easy to see from example 3.4 that

$$\langle A_{nj}\psi_{nj}^1 - A_{nj}\psi_{nj}^2 , \psi_{nj}^1 - \psi_{nj}^2 \rangle_{L^2([0,1])}$$

$$\leq \left(\frac{K_d}{\pi} \max_{x\in[0,1]} \left| \frac{\partial\psi}{\partial x}{}_{n(j-1)}(x,(j-1)T/n) \right| - K_m \right) \left\| \frac{\partial\psi^1}{\partial x}{}_{nj} - \frac{\partial\psi^2}{\partial x}{}_{nj} \right\|_{L^2}^2$$

$$\leq - \varepsilon\pi^2 \| \psi_{nj}^1 - \psi_{nj}^2 \| \ ,$$

for $\psi_{nj}^1, \psi_{nj}^2 \in H_o^1 \cap H^2 \cap C_\varepsilon^2 \cap C_p \triangleq \mathcal{D}_d(A_{nj})$ where ε is as before and

$$C_p = \{\psi \in H_o^1([0,1]) : \psi(x) \geq 0 \ , \ \forall \ x \in [0,1]\} \ .$$

Hence, on $\mathcal{D}_d(A_{nj}) \triangleq \mathcal{D}_d(A)$ (which is independent of n and j), (3.12) defines a stable dynamical system with solution in $C([o,\infty); \mathcal{D}_d(A_{nj}))$. It is now easy to see that the sequence $\Psi_n(x,t)$ defined by

$$\Psi_n(x,t) = \psi_{nj}(x,t) \quad , \ t \in I_{nj} \quad , \quad 1 \leq j \leq n \ \text{ and } \ n \geq 1$$

is uniformly bounded and equicontinuous in $C([o,T];L^2)$. Hence, by the Arzela-Ascoli theorem $\{\Psi_n(x,t)\}$ is precompact and therefore has a limit point in $C([o,T]; L^2([o,1]))$. The limit is clearly a solution of (3.10), for any T.

Since equation (3.10) has a solution for $\psi(o) \in \mathcal{D}_d(A)$ it follows from theorem 3.6 and example 3.4 that the solutions $\psi(t)$ of (3.10) and $\phi(t)$ of (3.6) are related by

$$\psi(t;\phi_o) - \phi(t;\phi_o) = -\int_o^t \Phi(t-s;\psi(s;\phi_o))\cdot\psi^2(s;\phi_o)ds$$

where

$$\Phi(t;\xi) = \mathcal{F}_\xi(\phi(t;\xi)) \quad .$$

(4). Application to Nonlinear Controllability

In this section we shall consider the controllability of a nonlinear system of the form (defined on a Hilbert space H)

$$\dot{\psi}(t) = N\psi(t) + M\psi(t) + Bu(t) \tag{4.1}$$

where N and M are nonlinear operators and B is a linear operator defined on a Hilbert space U. We shall use the techniques developed by Magnusson et al (1980). Consider the system

$$\dot{\phi}(t) = N\phi(t) \quad , \quad \phi(o) = \phi_o \tag{4.2}$$

Then if $\phi(t;\phi_o)$ is the solution of this equation we define, as above

$$\Phi(t;\phi_o) = \mathcal{F}_{\phi_o}(\phi(t;\phi_o)) \quad .$$

Using the nonlinear variation of constants formula, we may relate the solutions of (4.1) and (4.2) by

$$\psi(t) = \phi(t;\phi_o) + \int_o^t \Phi(t-s;\psi(s))M\psi(s)ds$$

$$+ \int_o^t \Phi(t-s;\psi(s))Bu(s)\,ds \tag{4.3}$$

Now the range of $\Phi(t-s;\psi(s))$ will depend, in general, on $\psi(s)$. However, we can define the operator $G_\psi : L^p[o,T;U] \to H (1<p<\infty)$

by

$$G_\psi u = \int_o^T \Phi(t-s;\psi(s))Bu(s)\,ds \tag{4.4}$$

Let V_ψ = Range G_ψ. As in Magnusson et al (1980), we define a space

$$X_\psi = L^p[o,T;U] \;/\; \ker G_\psi \tag{4.5}$$

with the quotient norm. Then we define the 1-1 map $\tilde{G}_\psi : X_\psi \to H$ by

$$\tilde{G}_\psi[u] = G_\psi u \quad , \qquad u \in [u] \tag{4.6}$$

and so

$$\|\tilde{G}_\psi[u]\|_H \leq \|G_\psi\| \cdot \|[u]\|_{X_\psi} \tag{4.7}$$

Moreover, we can define a norm on Range $(\tilde{G})$ by $\|v\|_{V_\psi} = \|\tilde{G}_\psi^{-1} v\|_{X_\psi}$, and we may show that

$$\|G_\psi u\|_{V_\psi} \leq \|u\|_U \tag{4.8}$$

so that $G_\psi \in \mathcal{L}[L^p[0,T;U],V_\psi]$. Finally, we can define a control

$$u_\psi = \tilde{G}_\psi^{-1} v$$

which will drive the origin to v for the system

$$\dot{\psi}(t) = N\psi(t) + Bu(t) \quad .$$

For the perturbed system

$$\dot{\psi}(t) = N\psi(t) + M\psi(t) + Bu(t) \quad , \tag{4.9}$$

the control

$$u(s) = \tilde{G}_\psi^{-1} \left[v - \int_0^T \Phi(t-\tau;\psi(\tau))M\psi(\tau)d\tau\right](s) \tag{4.10}$$

will drive the origin to v in time T.

We must now impose conditions so that the system (4.9) with the control (4.10) is well-defined, i.e. soluble for $\psi(t)$ in the mild sense. Hence we require that the operator

$$\Xi(\psi) = \int_0^t \Phi(t-s;\psi(s))M\psi(s)ds + \int_0^t \Phi(t-s;\psi(s))Bu(s)ds \quad , \tag{4.11}$$

with u given by (4.10) has a fixed point. Following Magnusson et al

(1980) we shall illustrate this by using the contraction mapping principle. The next theorem is proved in exactly the same way as in Magnusson (op. cit.).

Theorem 4.1. Let B_1, B_2 be Banach spaces and let $p_1, p_2, q, r, s, a, \omega, R, K$ be positive real numbers with

$$r \geq 1,\ p_1 \geq q \geq 1,\ \frac{1}{r} = \frac{1}{q} + \frac{1}{s} - 1\ ,\ \frac{1}{s} + \frac{1}{\omega} = 1\ ,\ p_2 > \omega\ .$$

Assume that

(a) $\Phi(t;\psi) \in \mathcal{L}(B_2, B_1) \cap \mathcal{L}(B_2, V)\ ,\quad t>0,\quad \psi \in \mathcal{D}_d(N)$

with

$$\|\Phi(t;\psi)x\|_{B_1} \leq g_1(t)\cdot \|x\|_{B_2}\ ,\quad g_1 \in L^{p_1}[0,T]\ .$$

$$\|\Phi(t;\psi)x\|_{V} \leq g_2(t)\cdot \|x\|_{B_2}\ ,\quad g_2 \in L^{p_2}[0,T]$$

(independent of ψ).

(b) There exists a continuous mapping $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$k(s_1, s_2) = k(s_2, s_1)\ ,\quad k(s_1, 0) \to 0 \text{ as } s_1 \to 0$$

and $M : L^r[0,T;B_1] \to L^2[0,T;B_2]$ satisfies

$$\|Mx_1 - Mx_2\|_{L^s[0,t;B_2]} \leq k(\|x_1\|_{L^r[0,T;B_1]}, \|x_2\|_{L^r[0,T;B_1]})\ \|x_1 - x_2\|_{L^r[0,T;B_1]}$$

(c) If $\|x_1\|_{L^r[0,t;B_1]} \leq a\ ,\quad \|x_2\|_{L^r[0,t;B_1]} \leq a\ ,$

then

$$(R\|g_2\|_{L^\omega[0,T]} + \|g_1\|_{L^q[0,T]})k(\|x_1\|_{L^r[0,T;B_1]}, \|x_2\|_{L^r[0,T;B_1]}) \leq K < 1$$

where

$$\|L_\psi(\cdot)B_u\|_{L^r[0,T;B_1]} \leq R\,\|u\|_{L^p[0,T;U]}$$

and

$$L_\psi(t)x = \int_0^t \Phi(t-s;\psi(s))x(s)\,ds \quad .$$

(d) $\|v\|_V \leq \frac{1-K}{R}\, a \quad .$

Then the operator (4.11) has a unique fixed point in $L^r[0,T;B_1]$ inside the ball of radius a. □

Example 4.2. Consider the system

$$\frac{\partial\psi}{\partial t} = \kappa(\psi)\frac{\partial^2\psi}{\partial x^2} - \psi^2 + u \quad , \qquad \psi(0,t) = \psi(1,t) = 0 \quad ,$$

defined on the spatial interval $[0,1]$. We can apply theorem 4.1 with $B_1 = L^4[0,1]$, $B_2 = L^2[0,1]$. The only problem remaining is to show that $\Phi(t;\psi)$ $(\psi \in \mathcal{D}_d(A))$ may be bounded independently of ψ. Now $\Phi(t;\psi)$ is the fundamental solution of the system

$$\frac{\partial\phi}{\partial t} = \kappa(\psi)\frac{\partial^2\phi}{\partial x^2} + \kappa'(\psi)\,\frac{\partial^2\psi}{\partial x^2}\,\phi \quad , \tag{4.12}$$

where $\psi \in \mathcal{D}_d(A)$ satisfies

$$\frac{\partial\psi}{\partial t} = \kappa(\psi)\,\frac{\partial^2\psi}{\partial x^2}$$

However, by differentiating (4.12) with respect to x, it is easy to see that the range R of Φ is certainly contained in $H_0^1[0,1]$. Moreover, using the limit argument above in proving existence of (3.10), we can express the solution of (4.12) as a sequence $\Phi_n(t;\psi)$ each having range $H_0^1[0,1]$. Hence the range of $\Phi(t;\psi)$ is $H_0^1[0,1]$ independent of $\psi \in \mathcal{D}_d(A)$. The appropriate bounds on $\|\Phi(t;\psi)x\|_{H_0^1[0,1]}$ now follow easily from corollary 3.3.

5. Conclusions

In this paper we have extended the nonlinear variation of constants formula to systems with an unperturbed operator has non elliptic Fréchet derivative. Throughout the paper we have illustrated the results using the heat conduction equation with temperature dependent thermal conductivity. We have also extended a nonlinear controllability result of Magnusson et al (1980) to such systems. Other fixed point theorems have been applied by the latter authors and we could equally well apply these here. However the general method is well illustrated using the contraction mapping theorem and so we have only described this method in detail.

6. References

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