APPLICATION OF TIME-DOMAIN ANALYSIS IN PROCESS CONTROL USING APPROXIMATE MODELS

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1. Introduction

Controller design for multivariable systems whose models are unknown or highly complex are frequently based upon the use of a simple, approximate and very often rough-and-ready model. References (1)(2)(3) give both frequency domain and time domain design techniques using approximate models. These techniques require only a knowledge of a plant's open-loop step response $Y_o(t)$ from any source.

This report continues the theme of Ref.(4) by using the time domain design method. The frequency domain techniques suffer from the general problem of the frequency domain methods, i.e. it is difficult to make predictions about the details of the closed-loop transient performance if the real plant model are not available. The use of time domain methods in stability assessment is unusual but it may have some advantages over frequency domain method, particularly in the multivariable case. In the examples considered in this paper it gives less conservative designs and only requires simpler calculations than that in the frequency domain. A more important benefit of time domain methed is, however, the possibility of providing bounds on the deterioration in predicted transient performance to be expected due to the approximation used.

For convenience's sake we summarize the techniques of time domain described by Ref. (2) and (3) as following:

Suppose that the controller K stabilizes the model G_A (Fig. 1b) and that simulations are undertaken to reliably calculate the matrix

$$W_A(t) = |W_A^{(1)}(t),\ldots\ldots W_A^{(\ell)}(t)|$$

Where $W_A^{(j)}(t)$ is the response from zero initial conditions of the system $(1+KFG_A)^{-1}KF$ to the input vector $E^{(j)}(t)$ defined by

$$E(t) = Y_0(t) - Y_{A0}(t) = |E^{(1)}(t)\ldots.E^{(\ell)}(t)| \qquad (1)$$

Where $Y_0(t)$ and $Y_{A0}(t)$ are open-loop unit step responses of real plant and approximate model respectively.

Then the controller K will stabilize the real plant G (Fig. 1a) if

(a) The composite system KF is both controllable and observable and

(b) The following inequality holds.

$$\gamma<1 \tag{2}$$

where γ is any available upper bound for the spectral radius $\gamma(N^P_\infty(W_A))$ and $N^P_\infty(f)$ is a norm defined by

$$N^P_\infty(f) = \sup_{T\geq 0}\left(|f(0^+)| + \sum_{k=1}^{k^*(T)} |f(t_k)-f(t_{k-1})| + |f(T)-f(t^*_k)|\right) \tag{3}$$

where $0 = t_o<t_1<t_2< \ldots$ are the local maxima and minima of f and $k^*(T)$ is the largest integer satisfying $t_k\leq T$.

Moreover an alternative result can be derived if the following inequality holds

$$\gamma(N^P_\infty(V_A))<1 \tag{4}$$

where

$$V_A(t) = \int_0^t H_E(t-t') H_{KF}(t'))dt' \tag{5}$$

and

(a) the demand signal R is the response from zero initial conditions of a mxm stable system H_o to the step $\hat{\gamma}(t)=\alpha$, $t>o$

(b) $Y^{(o)}(t)$ is the response from initial conditions of an mxm stable, proper system H_1 to the step $\hat{y}(t)=\beta$, $t>o$, and

(c) $\eta(t)$ is the mx1 vector defined by the convolution integral

$$\eta(t) = \int_0^t H_E(t-t')\{H_{KHo}(t')\alpha - H_{KFH_1}(t')\beta\}\, dt' \tag{6}$$

Under these conditions for all $t\geq o$, the response y(t) of the real feedback scheme Fig. 1a from the zero initial conditions to the demand R(t) satisfies the bound $|Y_j(t) - Y^{(1)}_j(t)| \leq \varepsilon_j(t) \quad 1\leq j\leq m$ (7)

where

$$\varepsilon(t) = \begin{bmatrix} \varepsilon_1(t) \\ \vdots \\ \varepsilon_m(t) \end{bmatrix} \triangleq (I_m - N_t^p(V_A))^{-1} N_t^p(V_A) \sup_{o \le t' \le t} \| y^{(1)}(t') - y^{(o)}(t') \|_p \tag{8}$$

and $Y^{(1)}(t) = Y_A(t) + \eta(t)$ where $Y_A(t)$ is the known response of Fig. 1(b) from zero initial conditions to the demand R.

In formulas (5) and (6), $H_E(t)$ has the form

$$H_E(t) = \left[H_E^{(1)}(t) \dots . H_E^{(\ell)}(t) \right] \tag{9}$$

where $H_E^{(j)}(t)$ is the response from zero initial conditions of the proper system $(I+G_A KF)^{-1}$ to the j^{th} column $E^{(j)}(t)$ of the error matrix $E(t)$.

$H_{KF}(t)$ is the ℓxm impulse response matrix of the composite system KF, $H_{KFH_1}(t)$ and $H_{KHo}(t)$ are the impulse response matrices of KFH_1 and KH_o respectively.

If we choose $H_o = I_m$

$$H_1 = (I_m + G_A KF)^{-1} G_A K \quad \text{(i.e. feedback system Fig. 1(b))}$$

$$\alpha = \beta$$

then $Y^{(o)}(t) = Y_A(t)$ (10)

So $\| Y^{(1)}(t) - Y^{(o)}(t) \|_p$ in (8) is replaced by $\| \eta(t) \|_p$ and hence

$$\varepsilon(t) = (I_m - N_t^p(V_A))^{-1} N_t^p(V_A) \sup_{o \le t' \le t} \| \eta(t) \|_p \tag{11}$$

Throughout this report we shall assume that this is the case unless otherwise stated.

In this report, we first consider some examples to illustrate the application of these design techniques. Also, we will point out that in some cases, a less conservative result can be obtained by using time domain method than that by using frequency domain method. In section 3.1 we will give an improved bound, the use of which provide a more accurate prediction on the transient performance. Moreover, a less conservative necessary condition for including integral action in the controller is derived in section 2.3.1.

2. Stability Assessment

2.1 Example Plant and Their Approximate Model

As ref. (4), we consider a multivariable plant that has an "unknown" transfer function matrix (TFM)

$$G \left[\frac{g_{ij}}{T_{ij}S + 1} e^{-\tau_{ij}S} \right]_{\ell xm} \tag{12}$$

and the two particular examples

$$G_1(s) = \begin{bmatrix} \frac{119.3}{812,8S+1} e^{-18.6S} & -\frac{62.3}{904s+1} e^{-14.6S} \\ \frac{55.3}{766.3S+1} e^{-114S} & -\frac{109.7}{715S+1} e^{-17.5S} \end{bmatrix} \tag{13}$$

$$G_2(s) = \begin{bmatrix} \frac{119.3}{812.8S+1} e^{-3.7S} & -\frac{124.6}{9045+1} e^{-2.92s} \\ \frac{110.7}{766.3S+1} e^{-22.8S} & -\frac{109.7}{715S+1} e^{-3.5s} \end{bmatrix} \tag{14}$$

(G_1 is of less open-loop interaction but longer time delay, and G_2 is of more open-loop interaction but with smaller time delay.)

We wish to design forward path controller K using a simple approximate model to guarantee the real feedback system is stable and has certain performance specification. In this section we focuss our attention to the stability problem and leave the performance assessment to section 3.

As stated previously, the open-loop responses are assumed given from plant trials or model simulations. Throughout this report we also assume that G(s) is both controllable and observable and G(0) or $Y_o(\infty)$ is nonsingular.

Following Ref. (4), we can choose approximate model as table (1).

Table (1)

No.	pseudo-diagonal model	No.	First order model
1	$G_A = \text{diag} \{\frac{1}{1+\alpha s}\} P$ where α is a scalar constant P is a constant matrix	1	$G_A = (A_o S + A_1)^{-1}$ $A_o^{-1} = \lim_{s\to\infty} s\, G(s)$ (or $= \left.\frac{dY_o}{dt}\right\vert_{t=o}$) $A_1^{-1} = \lim_{s\to o} G(s)$ (or $= Y_o(\infty)$)
2	$G_A = \text{diag} \{\frac{1}{1+\alpha_j s}\} G(o)$ where α_j are scalar constants	2	$G_A = (A_o S + A_1)^{-1}$ Real $(G^{-1}(i\omega)) \longrightarrow A_1$ (see Ref.4) Imag $(G^{-1}(i\omega) \longrightarrow A_o\omega$
		3	$A_1 = G(o)^{-1}$ (or $= Y_o(\infty)$) $A_o = \begin{bmatrix} \alpha_1 & o \\ o & \alpha_2 \end{bmatrix} \cdot A_1$
		4	$A_1 = G(o)^{-1}$ (or $= Y_o(\infty)$) A_o by iterative choice

2.2 Proportional Control

As an example, for the real plant $G_1(s)$, we will choose the first order model No. 1 (see Table 1) as the approximate model. Which is

$$A_o^{-1} = \begin{bmatrix} \frac{119.3}{812.8} & -\frac{62.3}{904} \\ \frac{55.3}{766.3} & -\frac{109.7}{715} \end{bmatrix}, \qquad A_o = \begin{bmatrix} 8.79 & -3.95 \\ 4.128 & -8.37 \end{bmatrix}$$

$$A_1^{-1} = \begin{bmatrix} 119.3 & -62.3 \\ 55.3 & -109.7 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0.01137, & -0.00646 \\ 0.00573, & -0.01237 \end{bmatrix}$$

The open-loop response error E(t) are shown in Fig. 2.

Choose the controller to be of the form[5].

$$K = kA_o - A_1 \tag{15}$$

and supposing that F = I. Then we can calculate

$$\begin{aligned} W_A(t) &= \mathcal{L}^{-1}\ (I+KFG_A)^{-1}KFE \\ &= \mathcal{L}^{-1}\ ((G_A^{-1}+K)G_A)^{-1}KE \\ &= \mathcal{L}^{-1}\ G_A^{-1}(G_A^{-1}+K)^{-1}KE \\ &= \mathcal{L}^{-1}\ \frac{(SI+A_1A_o^{-1})}{S+k}\ (kA_o-A_1)E \end{aligned} \tag{16}$$

Because we just know E(t) in date form in general rather than in transfer function form, so E can be considered as a sequence of impulse in calculating.

We then check condition (2) to decide the largest gain k that can be used. In general, the smaller the spectral radius γ is, the more robust the design will be. The bigger the value of γ and hence the higher the gain, the faster the response and smaller the steady-state error. For simplicity, we choose our design to ensure that $\gamma \leq 0.8$ (choosing $\gamma = \gamma\ (N_\infty^P(W_A))$. By this means we get the largest gain that can be used to be approximately k = 0.013. The response W_A is shown in Fig. 3 with the norm :

$$N_\infty^P(W_A) = \begin{bmatrix} 0.659 & 0.056 \\ 0.809 & 0.405 \end{bmatrix}, \quad \text{spectral radius } \gamma(N_\infty^P(W_A)) = 0.78$$

The closed-loop response of both real system and approximate system are shown in Fig. 4. The response bend, i.e. the bend between

$$Y_A + \eta \pm \varepsilon$$

can be calculated from formulas (5) - (11) and are shown by dotted line in the same figure. From there we can see that the bend is too wide to give a usefully accurate prediction of response. We will discuss this problem in section 2.4.

2.3 Proportional Plus Integral Control

In most cases the pure proportional control system generates steady-state error particularly in low gain control. To offset the steady-state errors, integral action should be included in the controller.

2.3.1 Necessary Condition for Inclusion of Integral Action.

The necessary condition for the theory to enable the inclusion of integral action in the frequency domain approach is given as following (Ref. 4).

$$\gamma(\| G_A^{-1}(o)\|_p \; \| E_\infty\|_p) < 1 \tag{17}$$

Here we consider this problem from time domain view-point. When choosing $\gamma = \gamma(N_\infty^P(W_A))$, condition (2) becomes

$$\gamma(N_\infty^P(W_A)) < 1 \tag{18}$$

It is clear that

$$N_\infty^P(W_A) \geq \| W_A(\infty)\|_p$$

and hence

$$\gamma(N_\infty^P(W_A)) \geq \gamma(\| W_A(\infty)\|_p \tag{19}$$

Then the necessary condition for satisfying condition (18) is that

$$\gamma(\| W_A(\infty)\|_p) < 1 \tag{20}$$

By definition,

$$W_A(t) = \mathcal{L}^{-1}(I+KG_A)^{-1}K(G(s)-G_A(s))\,\frac{1}{S} \tag{21}$$

So the final value of W_A is

$$W_A(\infty) = \lim_{t\to\infty} W_A(t) = \lim_{s\to o} s.\ (I+KG_A)^{-1}K(G(s)-G_A(s))\ \frac{1}{S}$$

$$= (I+K(o)G_A(o))^{-1}K(o)(G(o)-G_A(o)) \qquad (22)$$

But in the other hand

$$E(\infty) = \lim_{t\to\infty} E(t) = \lim_{s\to o} s.\ (G(s)-G_A(s)\ \frac{1}{S} = G(o)-G_A(o) \qquad (23)$$

substituting (23) into (22), we get

$$W_A(\infty) = (I+K(o)G_A(o))^{-1}K(o)E(\infty) \qquad (24)$$

or

$$W_A(\infty) = U(o)E(\infty) \qquad \text{(for brevity)}$$

where

$$U(s) = (I+K(s)G_A(s))^{-1}K(s)$$

The necessary condition (20) is then

$$\gamma(\|U(o)E(\infty)\|_p)<1 \qquad (25)$$

when the controller includes integral action, $\lim_{s\to o} K(s) = \infty$

then

$$U(o) = G_A(o)^{-1}$$

so the condition (25) replaced by

$$\gamma(\|G_A(o)^{-1}.E(\infty)\|_p)<1 \qquad (26)$$

This is the necessary condition for including integral action in the controller, because it is clear that

$$\gamma(\|G_A^{-1}(o).E(\infty)\|_p)\leq \gamma(\|G_A^{-1}(o)\|_p.\|E(\infty)\|_p)$$

and hence the condition(26) is less conservative than (17). In other words, using the time domain method the integral action is more easily included in the controller than it is in the frequency domain method. If we choose G_A such that $G(o) = G_A(o)$, then $E(\infty) = 0$ and hence (26) can always be satisfied, i.e., integral action can always be included in the controller. This point is the same with ref. (4).

2.3.2 For Pseudo-Diagonal Model

As an example, we choose the pseudo-diagonal model No.2 to approach the real plant $G_1(s)$. i.e.

$$G_A = \text{diag}\ \{\frac{1}{1+\alpha_j s}\}\ G(o) \tag{27}$$

where

$$G(o) = \begin{bmatrix} 119.3 & -62.3 \\ 55.3 & -109.7 \end{bmatrix}, \qquad \begin{matrix} \alpha_1 = 850 \\ \alpha_2 = 750 \end{matrix}$$

The open-loop response error are shown in Fig. 5.

Choose the controller to be of the form

$$K = G(o)^{-1}\ \text{diag}\ \{k_j + \frac{c_j}{S}\} \tag{28}$$

and hence obtain

$$W_A(t) = \mathcal{L}^{-1}(I+KG_A)^{-1}KE$$

$$= \mathcal{L}^{-1}(I+G(o)^{-1}\text{diag}\{k_j+\frac{c_j}{S}\}\ \text{diag}\ \{\frac{1}{1+\alpha_j S}\}\ G(o))^{-1}\ G(o)^{-1}\ \text{diag}\{k_j + \frac{c_j}{S}\}\ E$$

$$= \quad G(o)^{-1}\mathcal{L}^{-1}\ \text{diag}\ \{\frac{(1+\alpha_j S)(k_j S+c_j)}{\alpha_j S^2+(1+k_j)S+c_j}\}\ E \tag{29}$$

As stated previously, E can be handled as a series of impulse in calculating. By the same means described in section 2.2, gains k_j and c_j can be obtained to satisfy $\gamma(N^P_\infty(W_A)) \le 0.8$, i.e.

$k_1 = k_2 = 5.8$

$c_1 = c_2 = 0.003$

The response $W_A(t)$ is shown in Fig. 6 and has norm

$$N_{\infty}^{P}(W_A) = \begin{bmatrix} 0.527 & 0.147 \\ 0.734 & 0.373 \end{bmatrix}$$

The spectral radius is

$$\gamma(N_{\infty}^{P}(W_A)) = 0.79$$

The closed-loop response of both real and approximate system and the response bend are shown in Fig. 7. Note that we again suffer from the problem of wide error bands.

2.4. Discussion

For purpose of comparison, we summarize the computational results in table (2), where the frequency result are quoted from Ref. (4).

From table (2) we can see that for G_1, (i.e. less open-loop interaction but larger time delay), the time domain method produces much less conservative result. The gain can be used is about twice of those obtained by using frequency domain method.

In contrast for G_2, the difference between the results of two domains is quite small. Clearly, frequency domain methods may sometimes procure less conservative results than those of the time domain techniques. It depends on the error matrix and the form of model and controller. Despite the general proof of the conclusion has not been presented, it is the authors' opinion that one should use both time domain and frequency domain as the frequency domain ideas are more familiar and give better insight into system robustness.

Now we give a short discussion of the robustness of the design. The design is said to be robust in the sense that if the plant change to a "new plant" $\hat{G}$ with step response $\hat{Y}_o$, stability of scheme Fig 1a will be retained if $(\hat{Y}_o - Y_o)$ is small enough.

Suppose that the error matrix $(\hat{Y}_o - Y_{Ao})$ for the new plant is $\hat{E}$, and

$$\hat{W}_A(t) = \mathcal{L}^{-1}(I+KFG_A)^{-1}K\hat{E} \tag{30}$$

No.	Model	Controller	Frequency domain	Time domain
		For G_1		
1	pseudo-diagonal $G_A=\begin{bmatrix}\frac{1}{1+800S} & 0 \\ 0 & \frac{1}{1+800S}\end{bmatrix}G(o)$	P control $K=G(o)^{-1}\{k\}$	$k=2.3$ $\gamma(s)=0.8$	$k=5.0$ $\gamma(N_\infty^P(W_A))=0.79$
2	$G_A=\begin{bmatrix}\frac{1}{1+850S} & 0 \\ 0 & \frac{1}{1+750S}\end{bmatrix}G(o)$	P + I control $K=G(o)^{-1}\mathrm{diag}\{k_j+\frac{c_j}{s}\}$	$k_1=k_2=2.8$ $c_1=c_2=0.002$ $\gamma(s)=0.8$	$k_1=k_2=5.8$ $c_1=c_2=0.003$ $\gamma(N_\infty^P(W_A))=0.79$
3	First order $G_A=(A_o s+A_1)^{-1}$ $A_o^{-1}=\lim_{s\to\infty} sG(s)$ $A_1^{-1}=G(o)$	P control $K=kA_o-A_1$	$k=0.006$ $\gamma(s)\simeq 0.8$	$k=0.013$ $\gamma(N_\infty^P(W_A))=0.78$
4	$G_A=(A_o s+A_1)^{-1}$ $A_o=\begin{bmatrix}\alpha_1 & 0 \\ 0 & \alpha_2\end{bmatrix}A_1$ $A_1^{-1}=G(o)$ $\alpha_1=770$, $\alpha_2=800$	p+I control $K=(k+C+\frac{kc}{s})A_o-A_1$	$k=0.004$ $c=0.0003$ $\gamma(s)=0.83$	$k=0.0085$ $c=0.0003$ $\gamma(N_\infty^P(W_A))=0.79$
		For G_2		
5	$G_A=(A_o S+A_1)^{-1}$ $A_o=\begin{bmatrix}-50 & 52 \\ -55 & 50\end{bmatrix}$ $A_1^{-1}=G(o)$	p+I control $K=(k+c+\frac{kc}{s})A_o-A_1$	$k=0.0037$ $c=0.0001$ $\gamma(s)=0.8$	$k=0.0040$ $c=0.0001$ $\gamma(N_\infty^P(W_A))=0.78$
6	$G_A=(A_o S+A_1)^{-1}$ $A_o=\begin{bmatrix}57.97 & -52.16 \\ 54.59 & -55.68\end{bmatrix}$ $A_1^{-1}=\begin{bmatrix}117.5 & -114.7 \\ 112.4 & -115.7\end{bmatrix}$	p control $K=kA_o-A_1$	$k=0.0037$ $\gamma(s)=0.8$	$k=0.0038$ $\gamma(N_\infty^P(W_A))=0.8$

The norm of $\hat{W}_A$ can be easily found and, for simplicity, represented as

$$N_\infty^P(\hat{W}_A) \leq \mu N_\infty^P(W_A) \tag{31}$$

Where μ is a scalar constant, the value of which indicates the magnitude of the change in the plant.

W_A is the response matrix to the "original" error matrix that is used in the controller design.

It follows that the spectral radius is

$$\gamma(N_\infty^P(\hat{W}_A)) \leq \mu\gamma(N_\infty^P(W_A)) \tag{32}$$

and that the largest value of μ permitted is given by

$$\mu \leq \frac{1}{\lambda_o} \tag{33}$$

where $\lambda_o=\gamma(N_\infty^P(W_A))$ is the spectral radius that is used in the controller design process.

Formulas (31) and (33) give in fact a numerical limit to the change in the plant that guarantee that the scheme of Fig. 1a is still stable.

For example, if we use the gain value obtained by using frequency domain (see Table 2) for plant G_1, but check the stability condition by time domain (The reason of choosing lower gain will be mentioned in section 3), the spectral radius $\gamma(N_\infty^P(W_A))$ is about 0.5. That means $N_\infty^P(\hat{W}_A)$ can be twice of $N_\infty^P(W_A)$ and the real system will still be stable. By checking the condition in both domains, we get more confidence about the robustness of the design.

3. Performance Assessment

3.1 Improved Bounds on Transient Performance Deterioration

As stated previously, (8) or (11) gives a bound to limit the performance deterioration due to the approximation. But from Fig. 4 and Fig. 7 we can see that the response band is too wide to give a usefully accurate prediction of the real system response.

The partial reason of the presence of wide band is because the gain is too big. It is trivially verified that when the gain increases, the $N_t^p(V_A)$ and $\eta(t)$ increases and hence $\varepsilon(t)$ increases (see formula (11)). In other words, the use of higher gains necessarily leads to wider response bands. So, in the following design (see section 3.3) we prefer to use a lower gain to ensure that the response band is rather narrow even at expense of transient performance.

We next note that $\varepsilon(t)$ monotonically increases as time increases. This is unfortunately not appropriate to the response in practice, because, for a stable system, the closed-loop response approaches its steady-state value when time approaches the setting time. In the particular case when integral action is included in the controller, the final value of both Y_A and Y are unity (for unit step input) and hence the value of deterioration due to the approximation appraoches zero as time approaches the setting time. That means that it should be easier to predict the performance when time is near to the setting time. On the other hand, it would be useful to get a accurate prediction to the final value of the response (because the steady-state error is one of the most important performance specification). Any way, there is the necessity and possibility of modifying the theory to decrease the size of ε and, in particular, to remove the property of monotonicity. In this direction, we have got a "improved bounds" which is stated as foolowing. (The proof of which will be given in a future paper).

Under the condition stated in section 1, suppose that

$$\begin{aligned} H_o &= I \\ H_1 &= (I+G_AKF)^{-1}G_AK \\ \alpha &= \beta \end{aligned} \tag{34}$$

then the response bounds are given by

$$|Y_j(t) - Y_j^{(1)}(t)| \leq \varepsilon_j(t) \tag{35}$$

where

$$Y_j^{(1)}(t) = Y_A(t) + \eta(t) \tag{36}$$

and

$$\varepsilon(t) = B^{-1}(I-N_t^p(BV_A))^{-1}N_t^p(BV_A). \sup_{o\le t'\le t} |B\eta(t')| \tag{37}$$

In formula (37), B is a diagonal exponential function matrix. i.e.

$$B = \text{diag}\left\{e^{\alpha_j t}\right\} \tag{38}$$

where α_j can be any real value.

The only condition to use (35) -(38) is that

$$\gamma(N_t^p(BV_A)) < 1 \tag{39}$$

3.2 Some Analysis about $\eta(t)$

We note that $\eta(t)$ plays an important role in the response band. In fact

$$Y_1 = Y_A + \eta$$

is the central line of the response band. So η represents the difference between Y_A and the central line of response band. Here we give a simple analysis about $\eta(t)$, in particular, its final value $\eta(\infty)$.

From (6), we can immediately get

$$\eta(s) = H_E(s) * H_c(s) \tag{40}$$

where

$$\begin{aligned} H_E(s) &= (I+G_A(s)K(s))^{-1}E(s) \\ &= \frac{1}{s}(I+G_A(s)K(s))^{-1}(G(s)-G_A(s)) \end{aligned} \tag{41}$$

$H_c(s)$ is Laplace transform of $\{H_{KH_o}\alpha - H_{KFH_1}\beta\}$

when we choose H_o, H_1, α and β as in (34) we obtain

$$\begin{aligned} H_c(s) &= K(s)(I-H_1(s))\alpha \\ &= K(s)(I-(I+G_A(s)K(s))^{-1}G_A(s)K(s))\alpha \end{aligned} \tag{42}$$

where α is a constant vector.

It is trivially verified that the final value of $\eta(t)$ is

$$\eta(\infty) = \lim_{s\to o} S\eta(s) = (I+G_A(o)K(o))^{-1}(G(o)-G_A(o))\cdot H_c(o)$$

$$= (I+G_A(o)K(o))^{-1}(G(o)-G_A(o))\cdot K(o)(I-Y_A(\infty))\alpha \tag{43}$$

where $Y_A(\infty)=(I+G_A(o)K(o))^{-1}G_A(o)K(o)$, it is the final value of closed-loop response for unit step input of Fig. 1(b).

We then analyse the following cases:

(i) If choose approximate model such that

$$G(o) = G_A(o)$$

it is then clearly true that

$$\eta(\infty) = 0$$

(ii) When $G(o) \neq G_A(o)$, but integral action is included in the controller, then $Y_A(\infty) = I$ and hence

$$\eta(\infty) = 0$$

(iii) When $G(o) \neq G_A(o)$ and no integral action is included in the controller, then $\eta(\infty)$ is a non zero vector, that is

$$\eta(\infty) = (I+G_A(o)K(o))^{-1}(G(o)-G_A(o))K(o)(I-Y_A(\infty))\alpha \tag{44}$$

3.3 Procedure of Application

Suppose the stability assessment has been done by means of section 2. Then turn our attention to find the response band. As as example, choose a first order model as the approximate model and use proportional plus integral control. The form of model and controller is exactly the same as No 4 of Table 2, but the values of the gains are chosen to be the same as values obtained by using frequency domain analysis, i.e. $k = 0.004$, $C = 0.0003$. The open-loop response errors are illustrated in Fig. 8. These gain values, the spectral radius is

$$\gamma(N_\infty^P(W_A)) = 0.4317$$

Obviously the gain can be somewhat increased from stability view-point, but we prefer to use these value for obtaining a rather narrow response band. The procedure is as follows:

(i) Calculate $V_A(t)$ by formula (5)

$$V_A(t) = \int_o^t H_E(t-t')H_{KF}(t')dt' \tag{45}$$

where

$$H_E(t) = \mathcal{L}^{-1}(1+G_AK)^{-1}E(s) \tag{46}$$

or equivalently

$$H_E(t) = \mathcal{L}^{-1}(G_A{}^{-1}+K)^{-1}G_A{}^{-1}E$$

$$= \mathcal{L}^{-1}(A_oS+(k+c+\frac{kc}{S})A_o)^{-1}(A_oS+A_1)E$$

$$= \mathcal{L}^{-1}\quad \frac{S}{(S+k)(S+c)}\quad (SI+A_o{}^{-1}A_1)E \tag{47}$$

In practical calculating E can be handled as a sequence of impulse.

When F = I

$$H_{KF} = \mathcal{L}^{-1}((k+c)A_o-A_1 + \frac{kc\cdot A_o}{S}$$

$$= ((k+c)A_o-A_1)\cdot\delta(t) + kc\cdot A_o\cdot H(t) \tag{48}$$

where $\delta(t)$ is unit impulse function ($\delta(t)$ may be considered a square wave, the area under which is equal to unity,) $H(t)$ is unit step function.

The convolution integral can be approximately calculated as follows:

$$V_A(t) = \sum_{i=o}^{N-1} H_E(t-i\Delta t)H_{KF}(i\Delta t)\Delta t \tag{49}$$

where $\Delta t = \frac{t_1}{N}$, it is sufficiently small compared with the smallest time constant, t_1 is the largest integral time.

$$H_E(\tau) = o \quad \text{for} \quad \tau<o.$$

By this way, we get $V_A(t)$ which are shown in Fig. 9. From these the norm $N^P_\infty(V_A)$ and spectral radius $\gamma(N^P_\infty(V_A))$ can be calculated as following (It is not necessary in this stage)

$$N^P_\infty(V_A) = \begin{bmatrix} 0.116 & 0.068 \\ 0.336 & 0.276 \end{bmatrix} \qquad \gamma(N^P_\infty(V_A)) = 0.368$$

The value of $\gamma(N_\infty^P(V_A))$ is near to the values of $(N_\infty^P(W_A))$.

(ii) Calculate $\eta(t)$. Its general form is

$$\eta(t) = \int_o^t H_E(t-t')\left\{H_{KH_o}(t')\alpha - H_{KFH_1}(t')\beta\right\}dt \tag{50}$$

For the case of $H_o = I$, $F=I$, $\alpha=\beta$, and $H_1=(I+G_AK)^{-1}G_AK$

$H_{KH_o} = H_K(t)$ which is found by (48)

$$H_{KH_1} = \mathcal{L}^{-1}K(I+G_AK)^{-1}G_AK$$

$$= \mathcal{L}^{-1}K(G_A(G_A^{-1}+K))^{-1}G_AK$$

$$= \mathcal{L}^{-1}\left(\frac{(k+c)s+kc}{s}A_o-A_1\right)\frac{s}{(s+k)(s+c)}\left(\frac{(k+c)s+kc}{s}-A_o^{-1}A_1\right) \tag{51}$$

For our example, $\eta(t)$ is shown in Fig. 10.

(iii) Calculate response bound ε

(a) We first take $\alpha_j = o$ and calculate ε by (37) (in this case it is the same with (11)) to get the "original bound" ε_o(to distinguish the "improved bound")

(b) We next give a real value of α_j(which can be either positive or negative) and calculate $\varepsilon_1(t)$ by (37). Meanwhile check if

$$\gamma(N_t^P(\mathrm{diag}(e^{\alpha_j t})V_A))<1 \tag{52}$$

holds. If it holds, then ε_1 is one of the bound. If not, ignore that part of ε_1. More precisely, if

$$\gamma(N_{t'}^P(\mathrm{diag}(e^{\alpha_j t'})V_A(t'))) = 1 \tag{53}$$

then ignore $\varepsilon_1(t)$ for $t>t'$

(c) Repeat procedure (b) to get $\varepsilon_i(t)$ for a variety of choices of α_j. The value of α_j should be chosen such that ε_i becomes as small as possible under the condition (52)

(d) The improved response bound ε is

$$\varepsilon = \inf \varepsilon_i$$

for all i

By this means, the improved bound can be found. For our example, the original and improved bound are shown in Fig. 11. We can see from there that the size of the bound has been evidently decreased, particularly when the time approaches the settling time.

(iv) Calculate the closed-loop response and their bound.

For this example, the closed loop response of approximate system is

$$Y_A = \mathcal{L}^{-1} \frac{1}{(s+k)(s+c)} \{kcI + s((k+c)I - A_o^{-1}A_1)\} \frac{1}{s} \tag{54}$$

The upper bound y_u is

$$y_u = y_A + \eta + \varepsilon \tag{55}$$

and the lower bound y_ℓ is

$$y_\ell = y_A + \eta - \varepsilon \tag{56}$$

When the TFM of the real plant is known, the closed-loop response of the real system Fig. 1(a) y can be found by any means.

Fig. 12 shows the closed-loop response y_A, y and the response bound y_u and y_ℓ.

By the same means, we can apply the same calculations to system No 2 and No 3 of Table 2. The value of gains are the same as those obtained by frequency domain methods, which are represented in 4th column of Table 2. (These two systems were calculated in section 2.3 and 2.2 respectively. When we use the higher gain value, their closed-loop response are shown in Fig. 7 and 4 respectively). The closed-loop response and response bound are shown in Fig. 13 and 14 respectively.

Comparing Fig. 13 with Fig. 7 and Fig. 14 with Fig. 4, we can see the response bound has been evidently improved by using the improved bound method and decrease the gain value.

From Fig. 12-14, we can say that the theory is a useful aid in stability and performance assessment. The controller design using time domain method is successful which is also indicated by the similar stability and overall dynamics characteristics of the real and approximate schemes and the rather accurate prediction on the response.

3.4 A Note to The Improved Bound

Even though we can evidently decrease the response bound by using the procedures (a)-(d) described in section 3.3 in most cases, sometimes difficulties may occur. In that case, the response bound can not be decreased very much. For example, using No. 5 approximate system of Table 2 for G_2, we do the same calculation with 3.3. The original and improved bound are shown in Fig. 15. That is the best improvement that we can do. The reason may be because that $G_2(o)^{-1}$ is bad conditioned (the eigenvectors of $G_2(o)^{-1}$ are more skew than those of $G_1(o)^{-1}$. But a useful general analysis of this problem has not been achieved yet. The closed-loop response and this bound are shown in Fig. 16. We can see the band is much wider than those in Fig.12-14.

4. Summary and Discussion

(1) In this report, controllers for multivariable process plants have been designed by using a time domain method. The choosing of the approximate model and the form of controller are same as Ref.(4). In some cases the use of time domain method may produce less conservative results than those using frequency domain methods. So the authors suggest that one should parallelly use the two domains method to check the stability criterion and hence to know how robust the design is.

(2) The more important benefit of time domain method is that it can provide a response bound to predict errors in the response of the real system. In section 3.1 an improved bound method has been outlined. By using the improved method a more accurate bound can be found in most cases.

(3) In section 2.3 of this report, a less conservative necessary condition for including integral action in controller is derived. In other words, using time domains method, the integral action is easier to be included in controller than it is by using frequency domain method.

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(4) Owens, D.H. and Wang, H.M. and Chotai, A.: 'Some case study in approximate models in multivariable process control', ibid, Research Report No. 203, November 1982.

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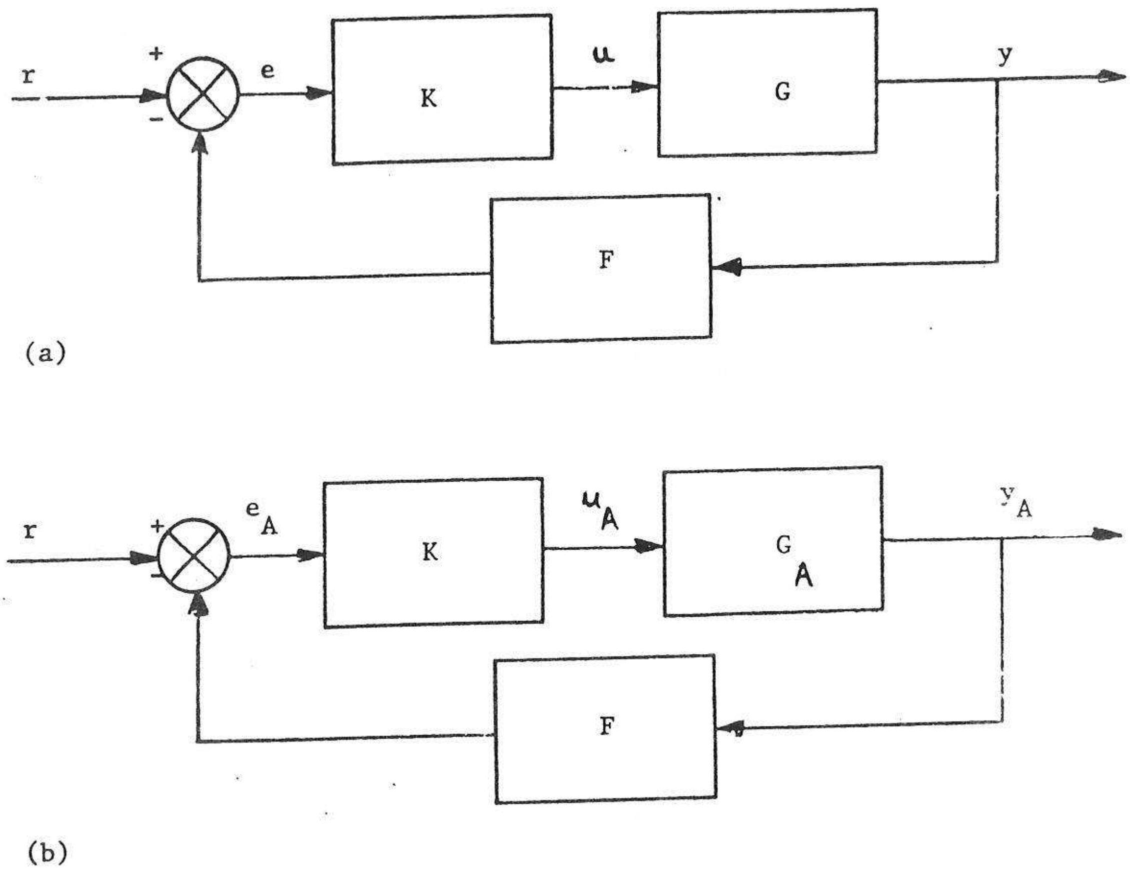


Fig. 1 (a) Real and (b) Approxmating Feedback Systems

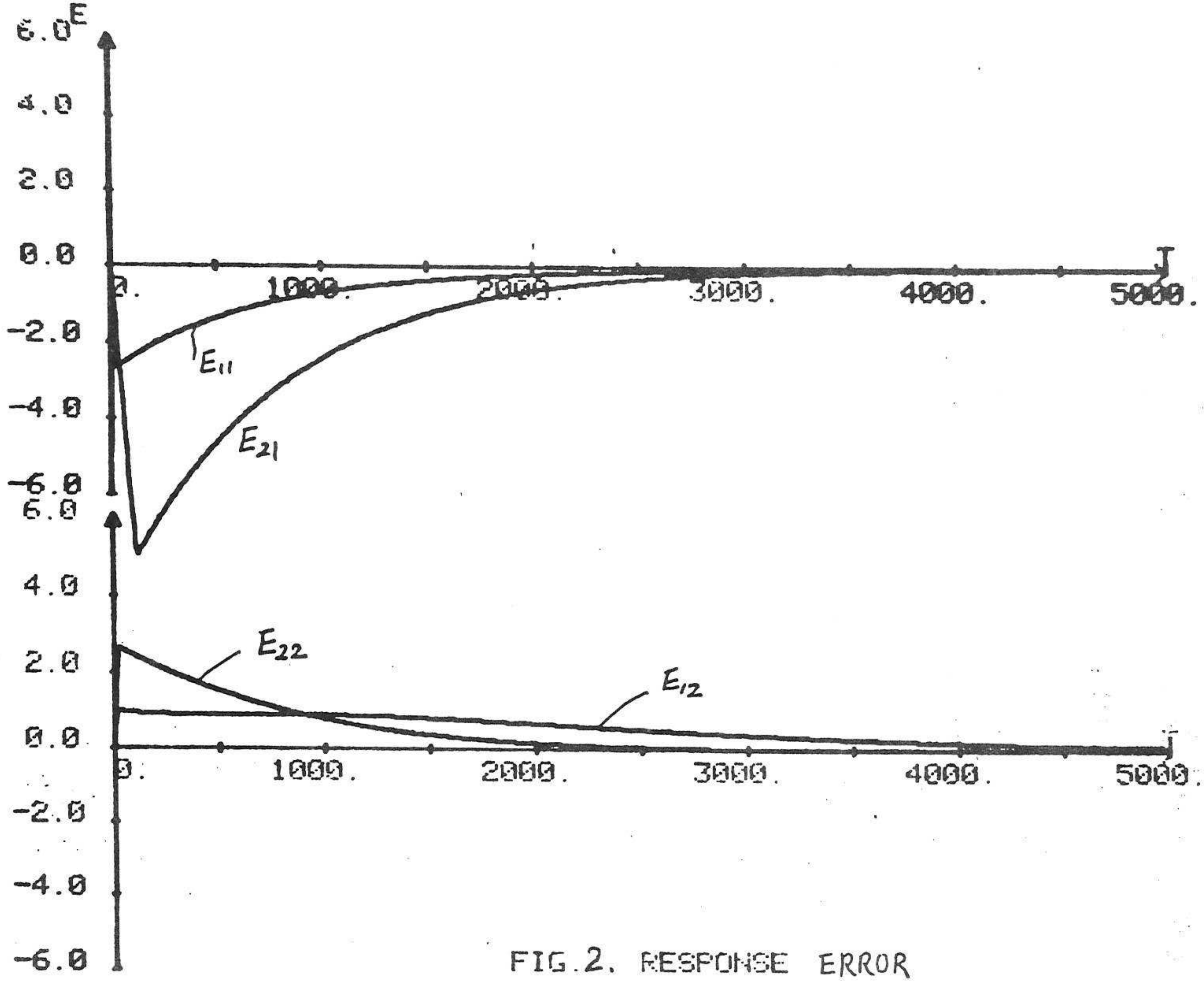


FIG.2. RESPONSE ERROR

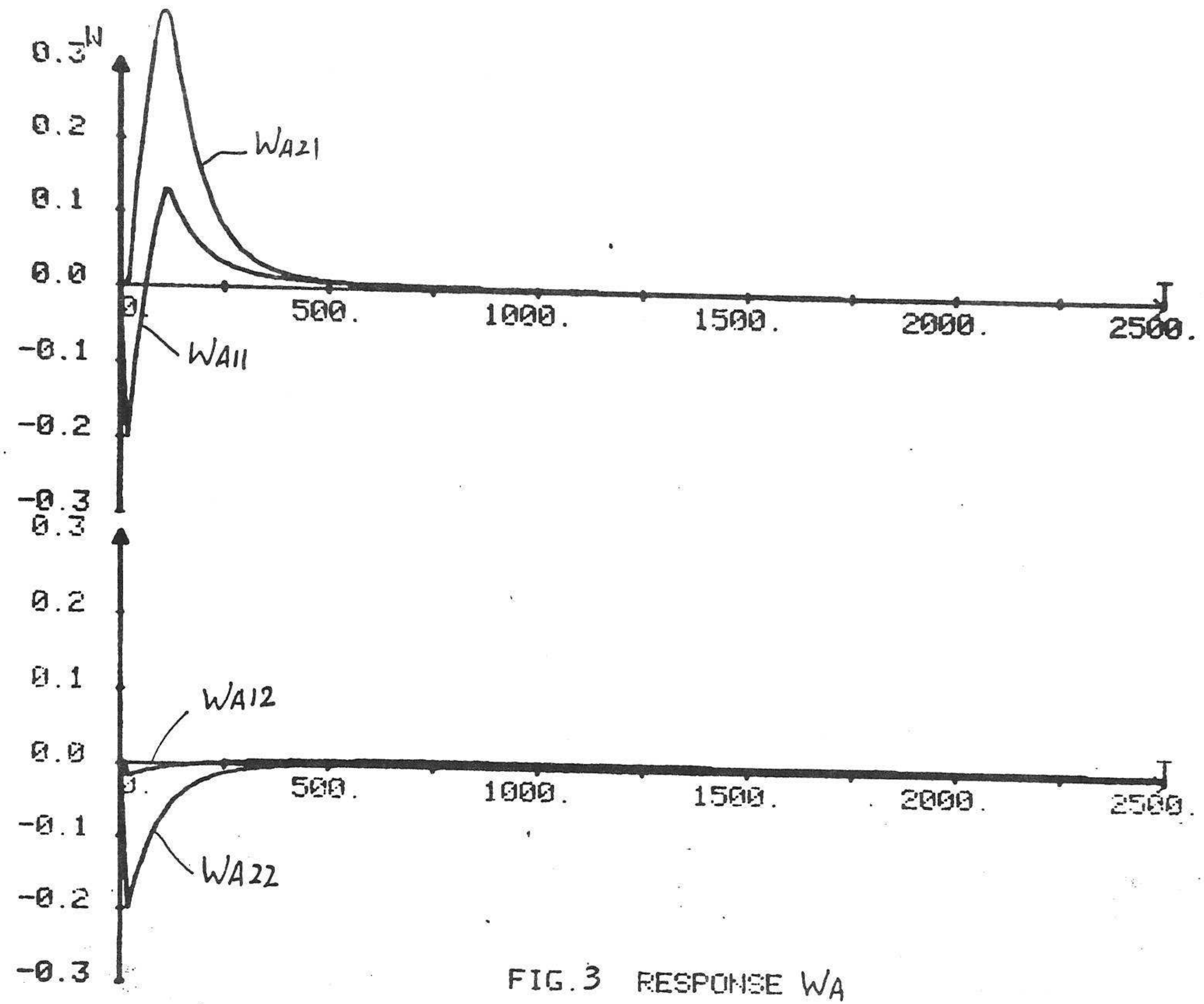


FIG. 3 RESPONSE W_A

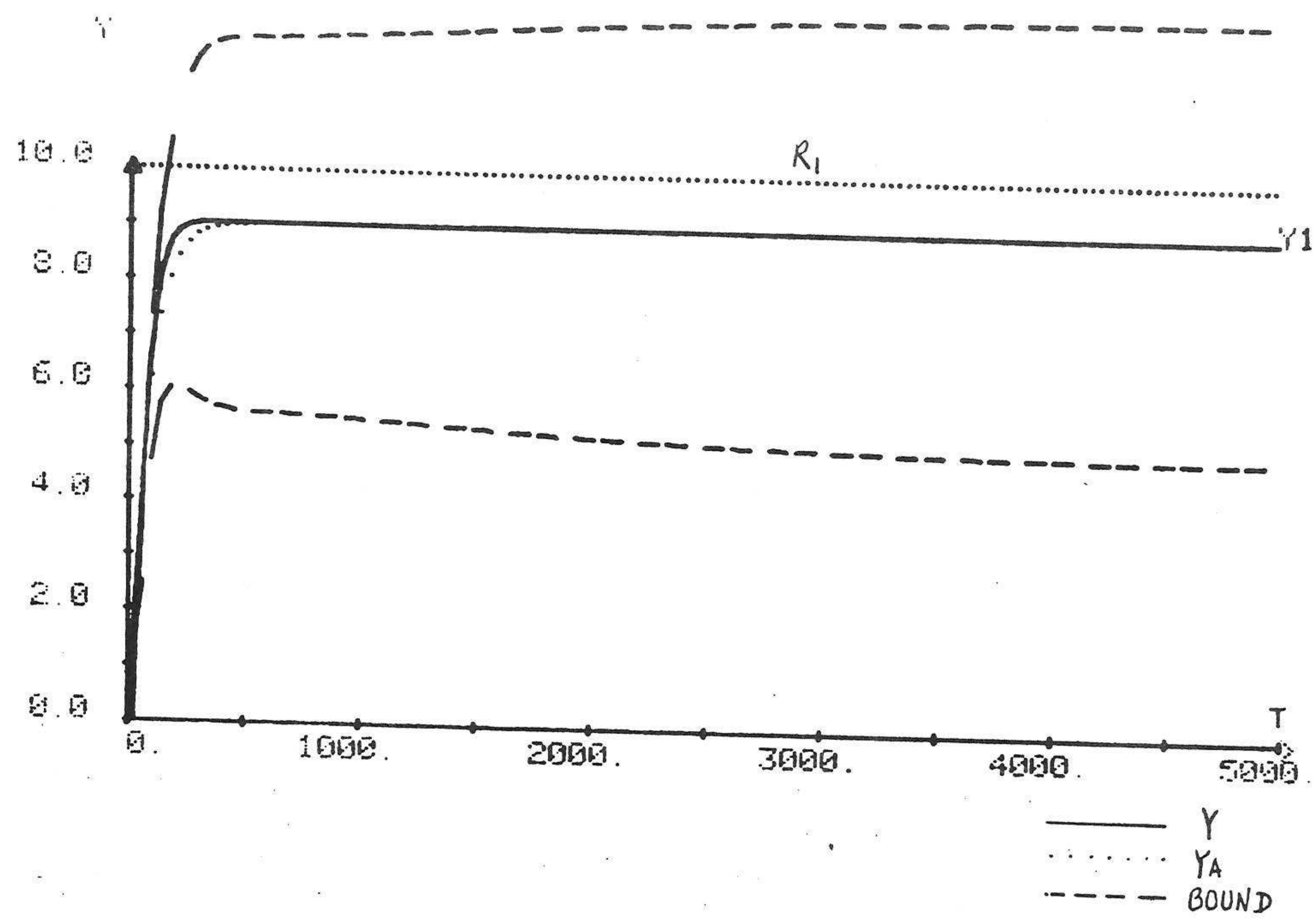


Fig 4. CLOSED LOOP RESPONSE

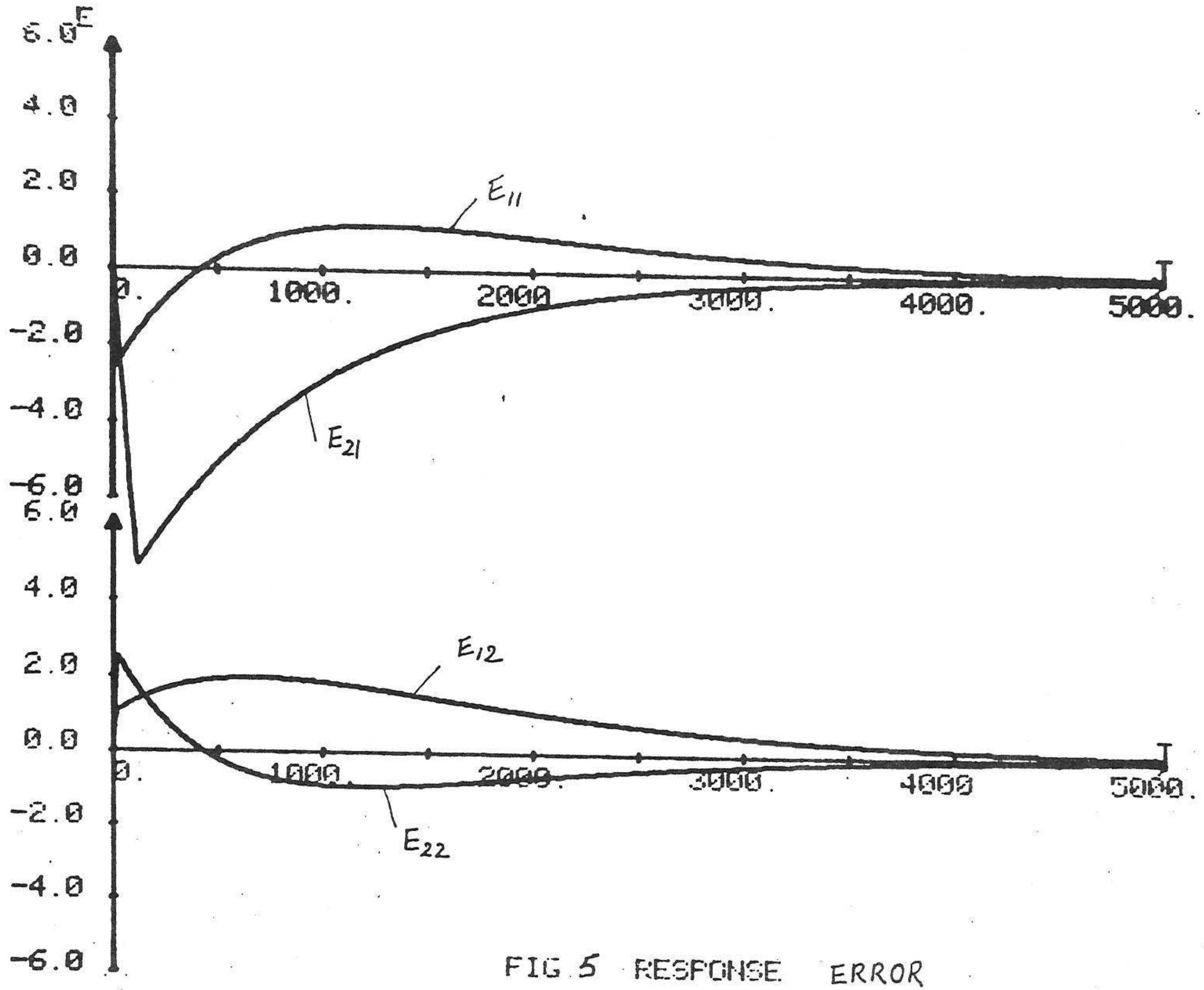


FIG 5 RESPONSE ERROR

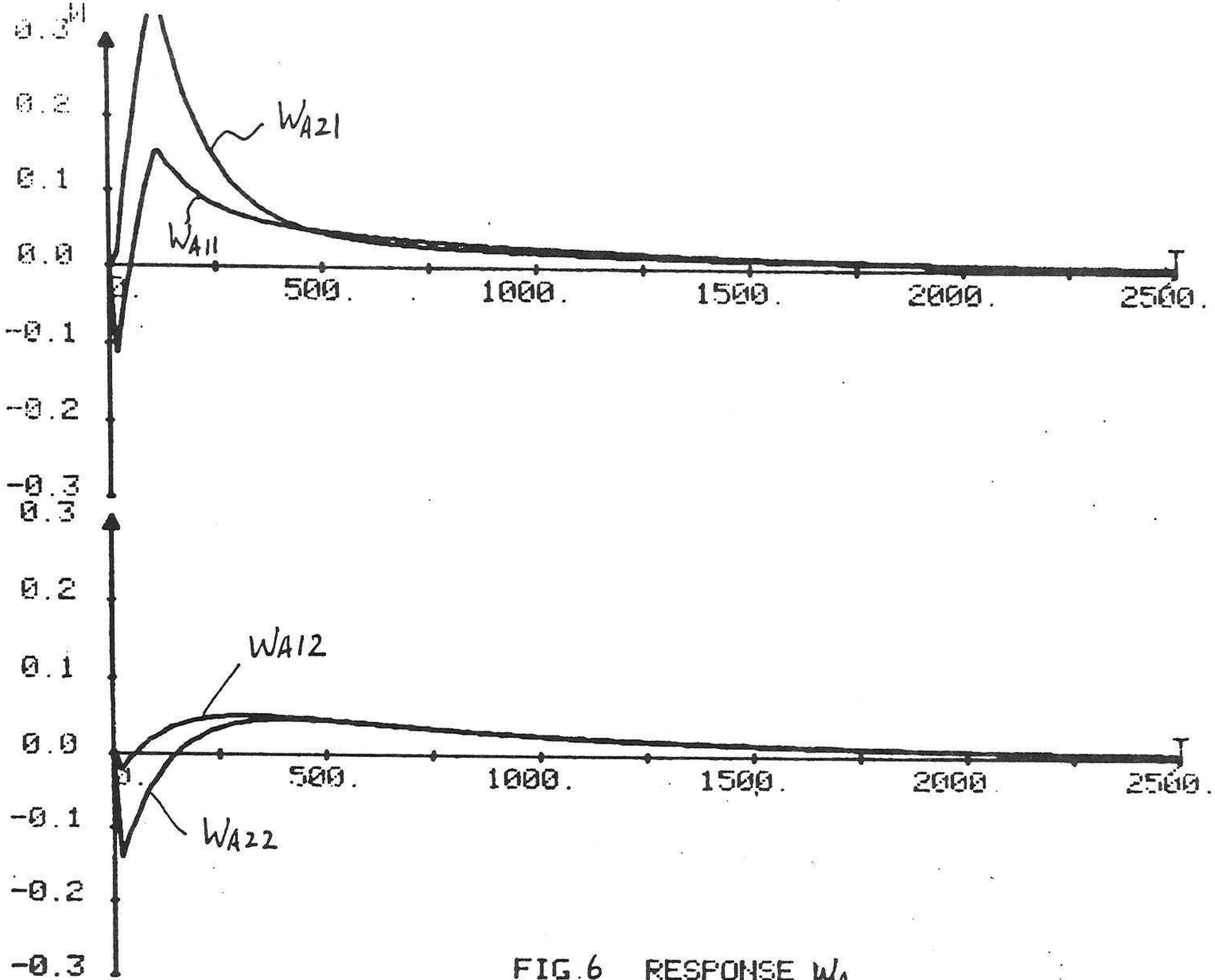


FIG.6 RESPONSE W_A

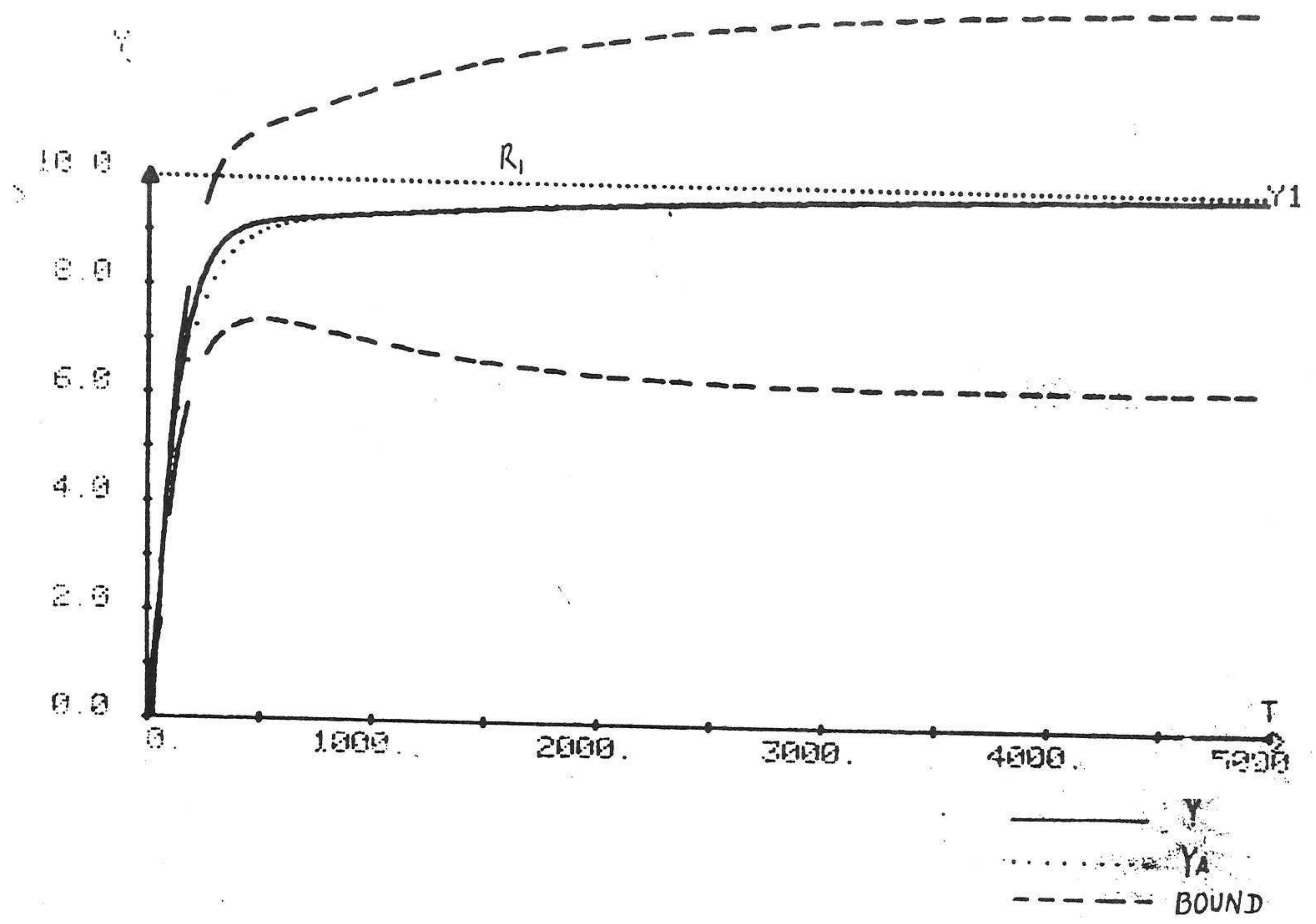


Fig 7. CLOSED LOOP RESPONSE

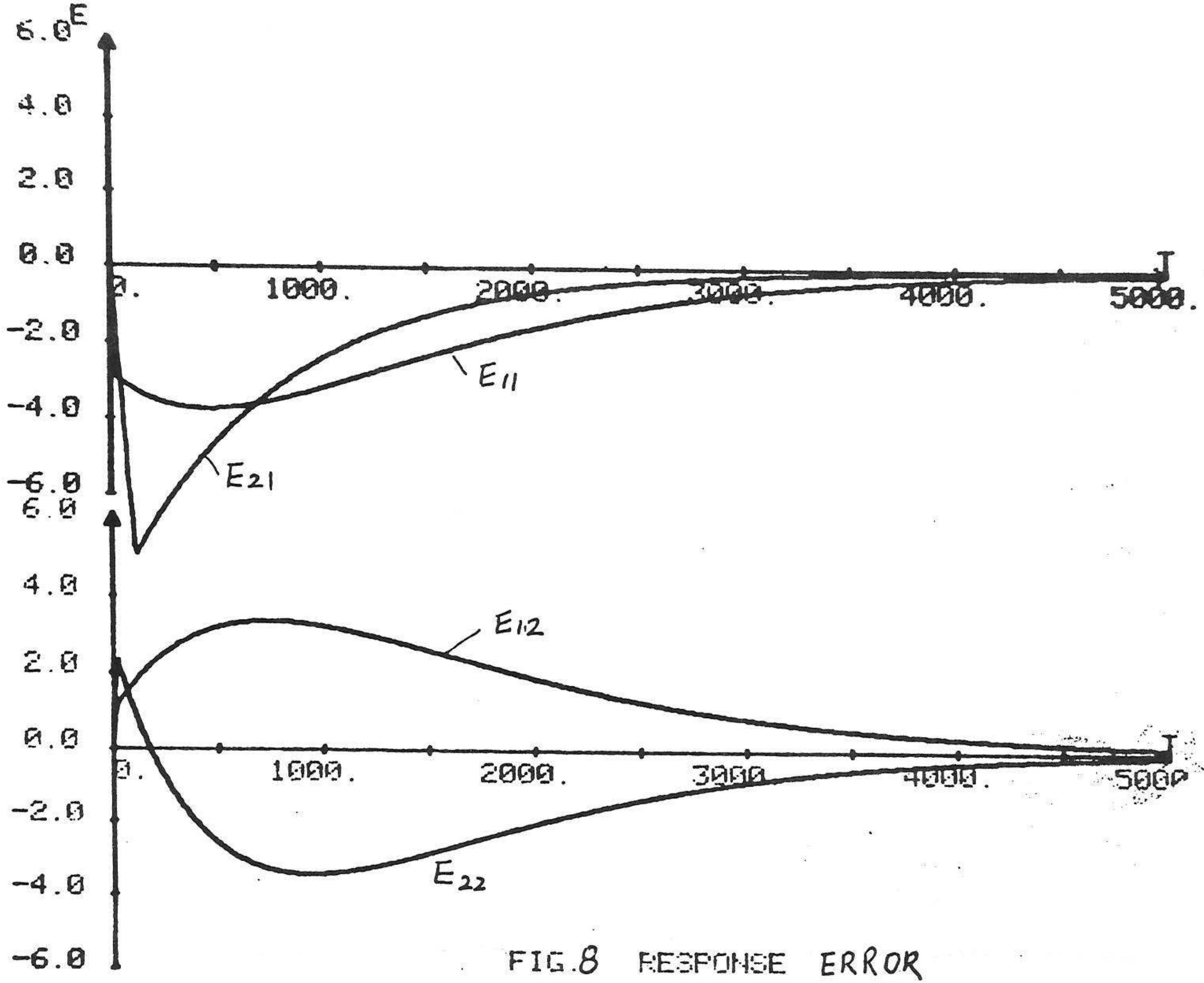


FIG.8 RESPONSE ERROR

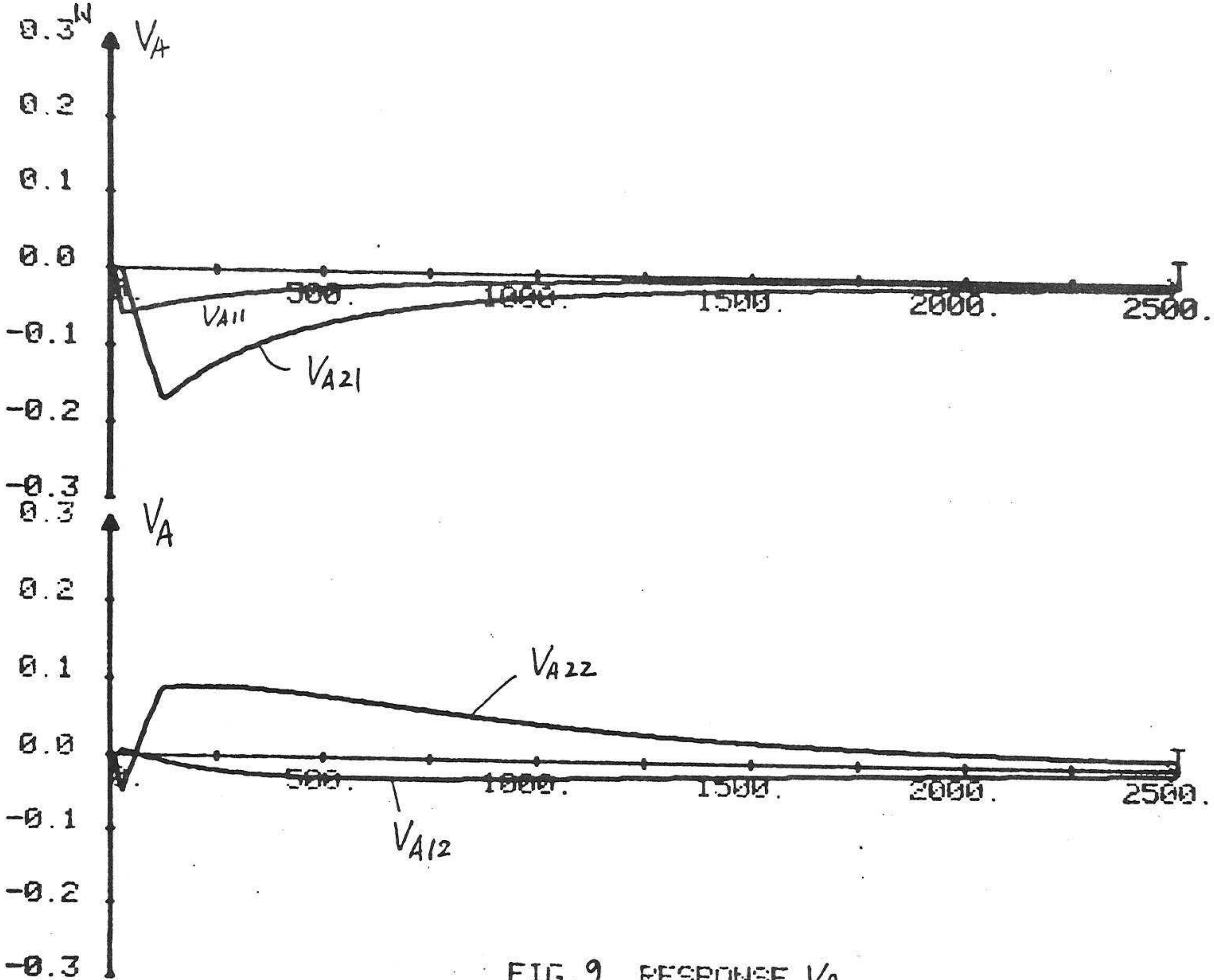


FIG. 9 RESPONSE V_A

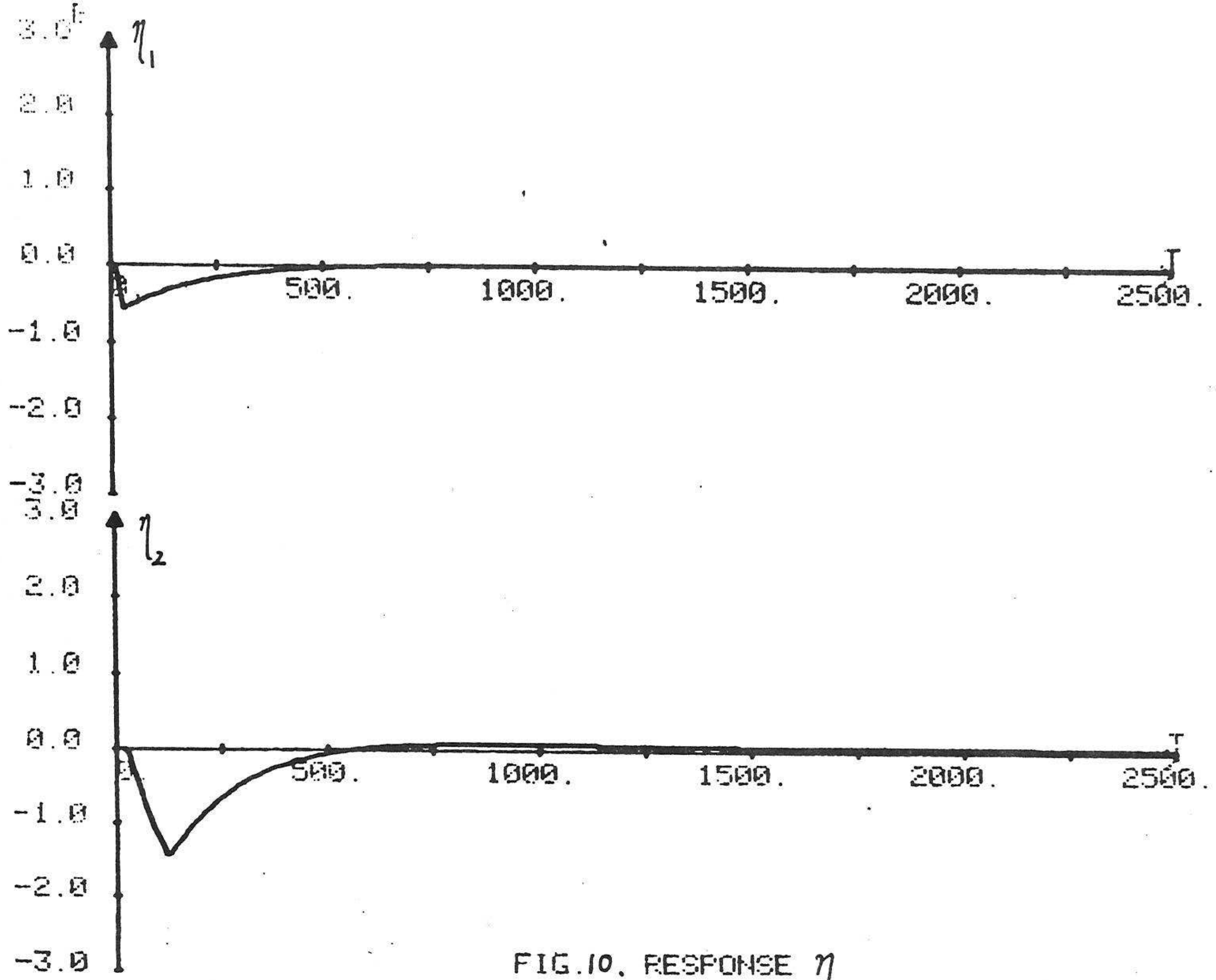
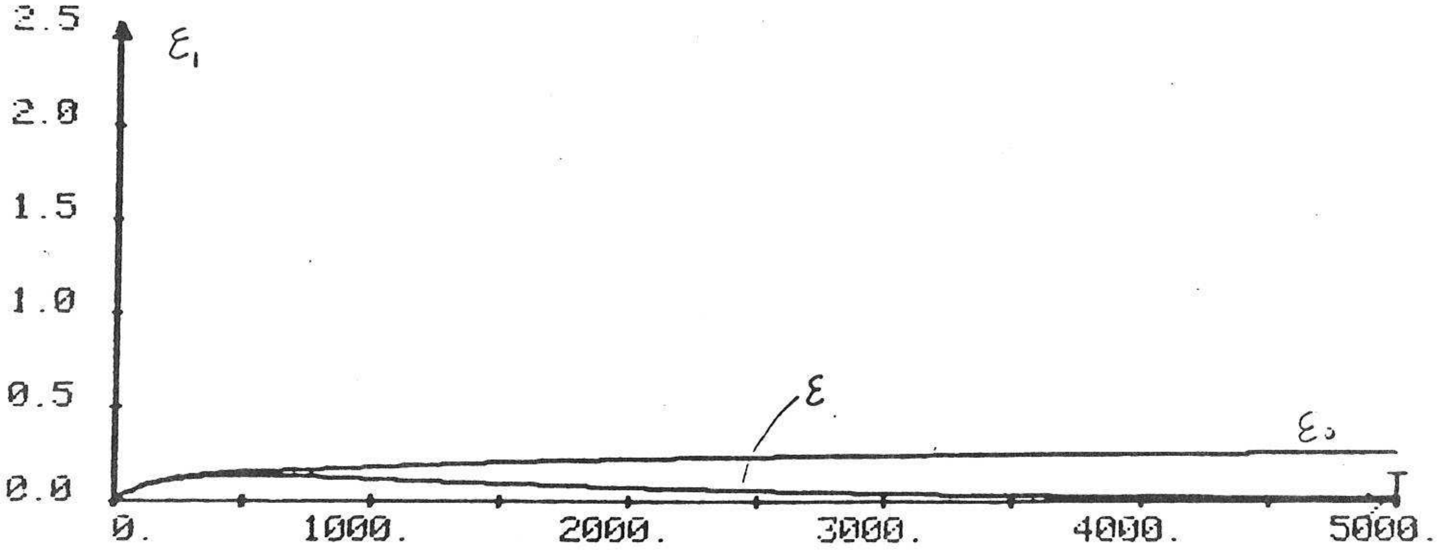
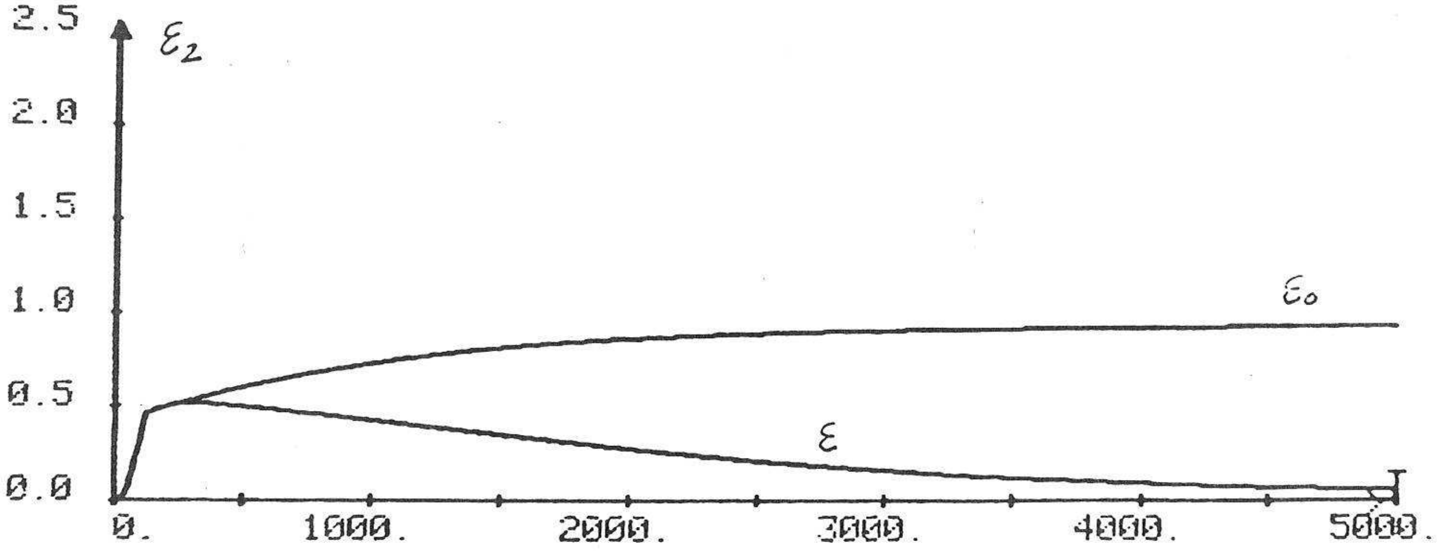


FIG.10. RESPONSE η





$\mathcal{E}_0$ ORIGINAL BOUND

$\mathcal{E}$ IMPROVED BOUND

Fig 11. RESPONSE ERROR

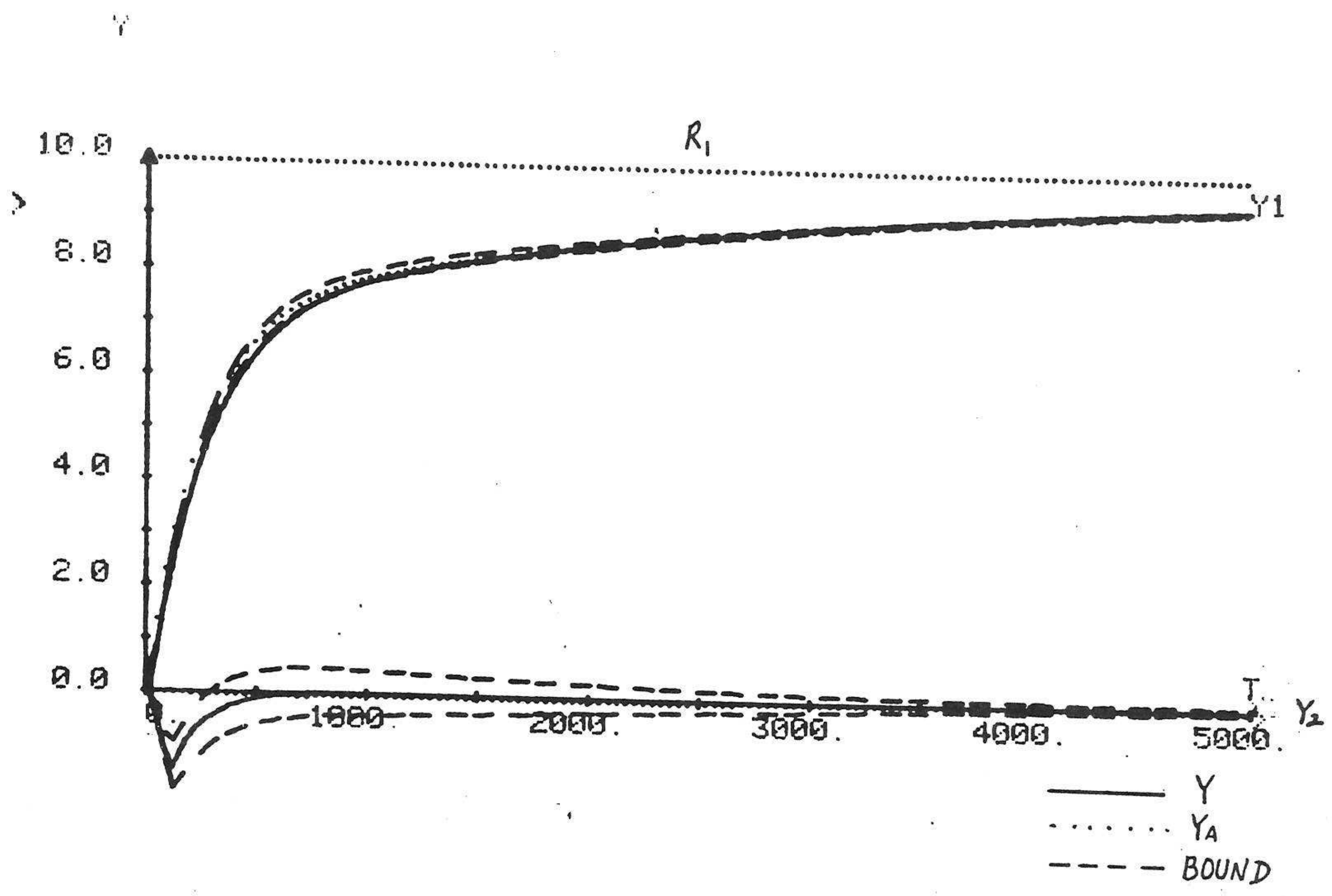


Fig 12. · CLOSED LOOP RESPONSE

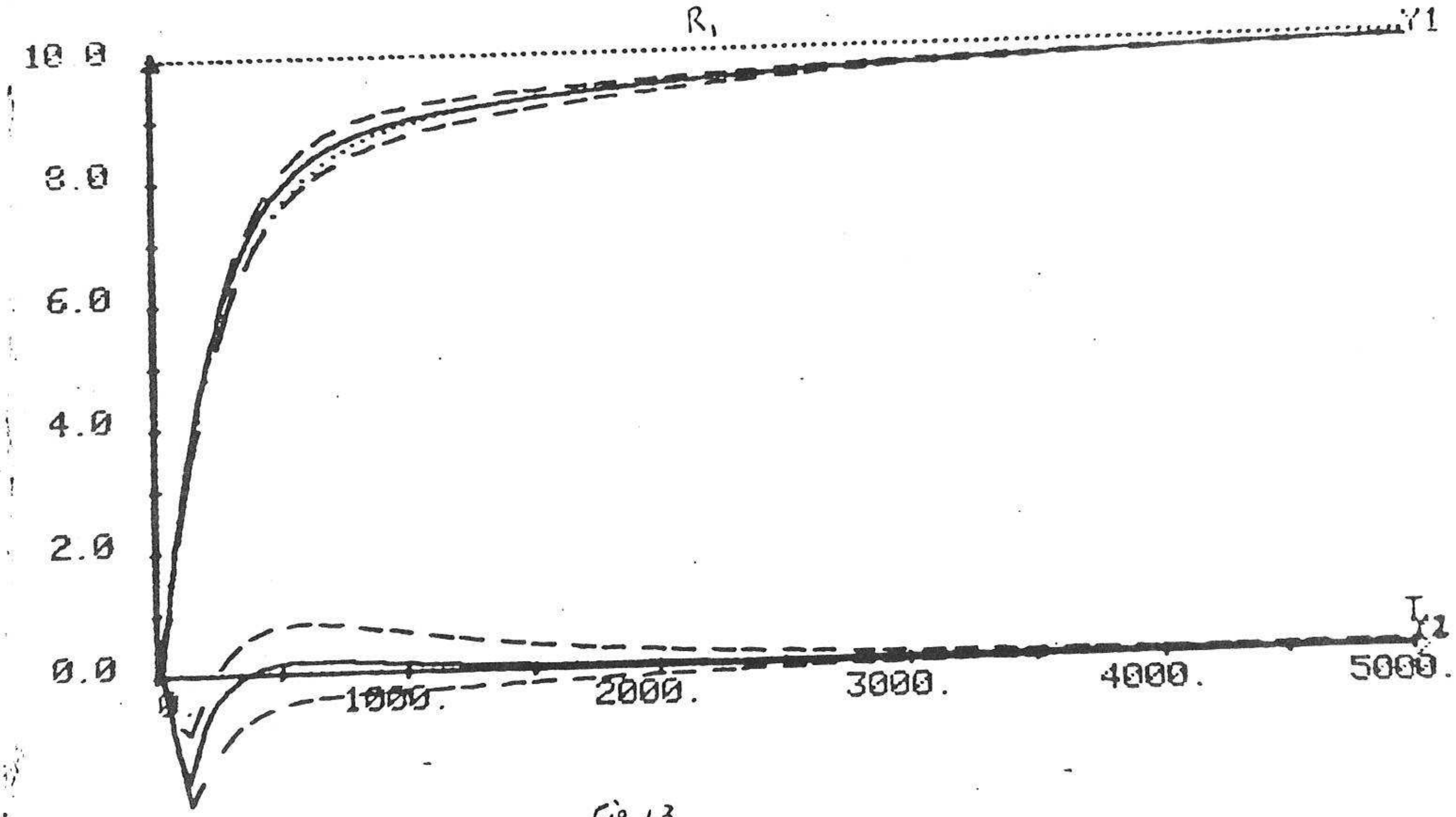


Fig 13.

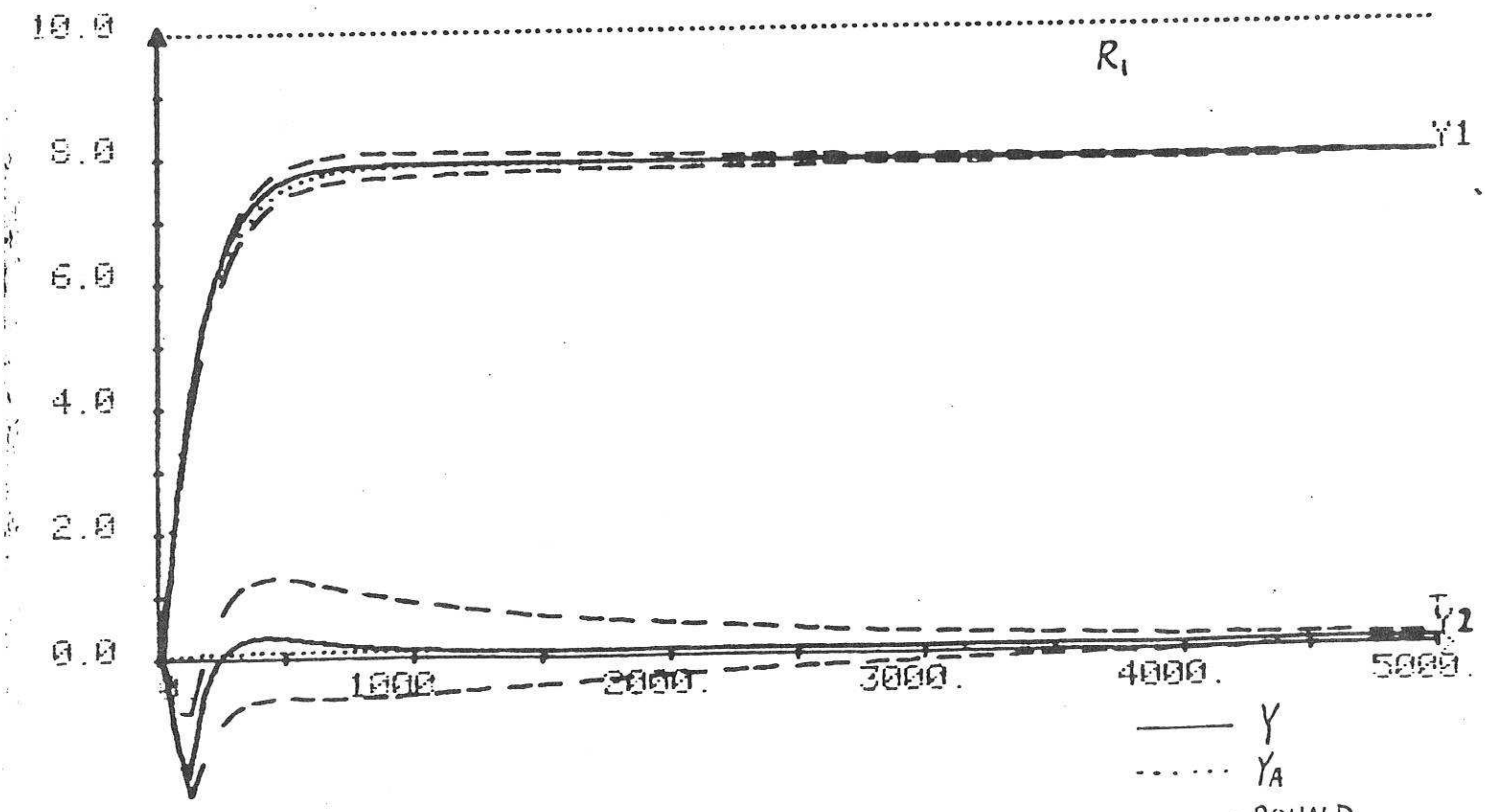


Fig 14. CLOSED LOOP RESPONSE

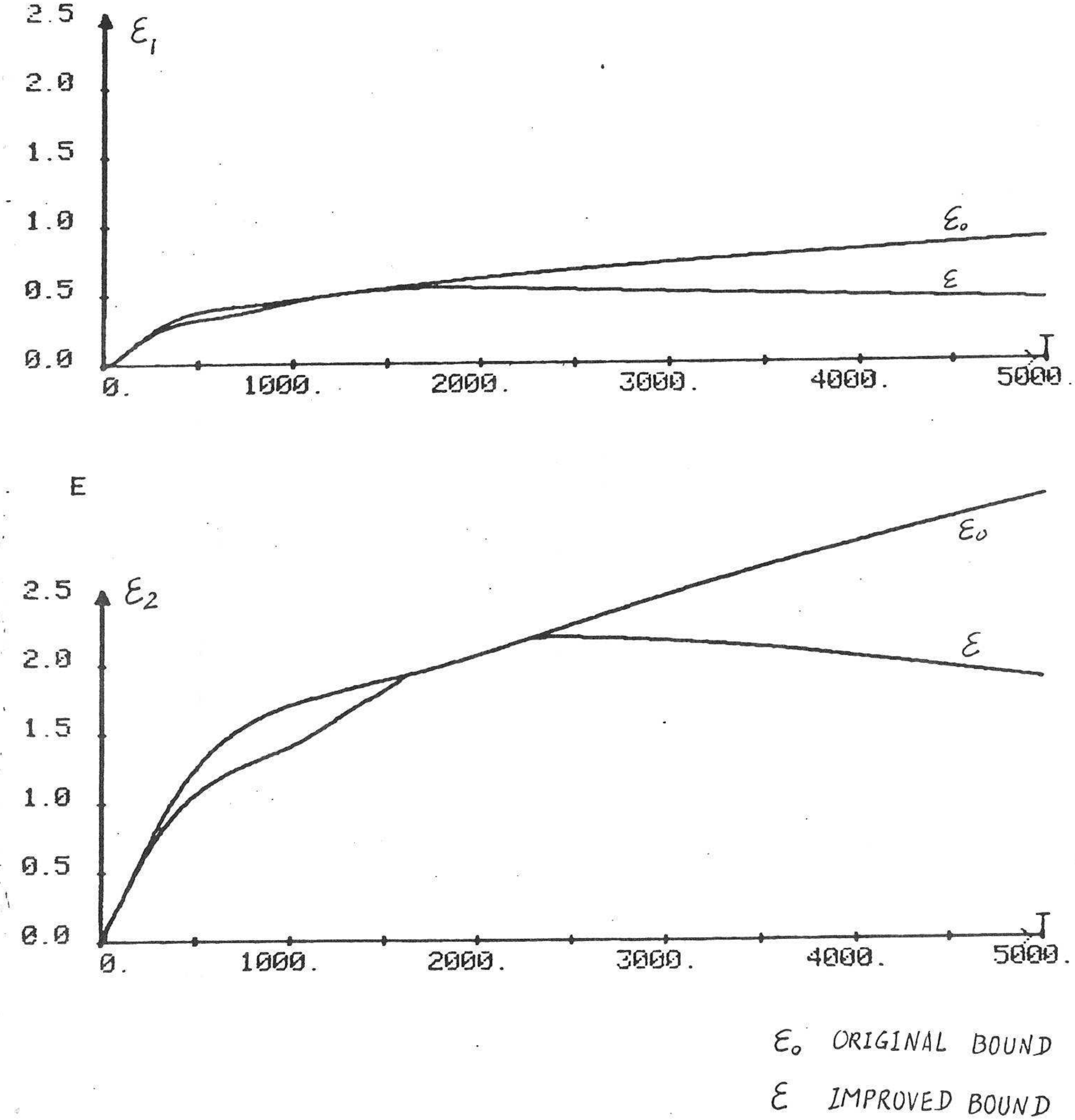


Fig 15 RESPONSE ERROR

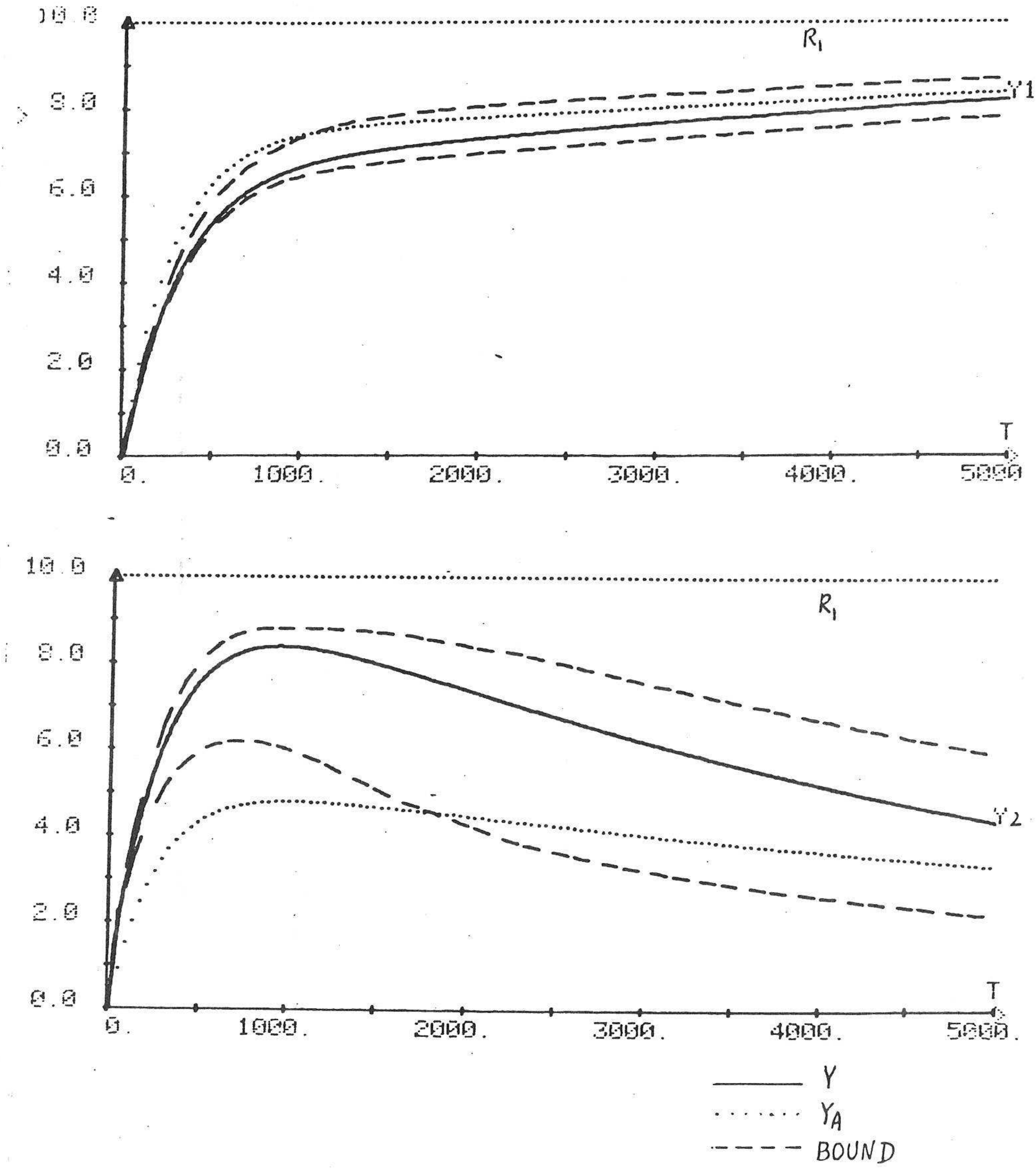


Fig 16. CLOSED LOOP RESPONSE