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GEOMETRIC CONDITIONS FOR GENERIC STRUCTURE OF MULTIVARIABLE ROOT-LOCI

by

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Research Report No. 227

June 1983

This research is supported by SERC under grant GR/B/79455

Indexing Terms: Root-loci; asymptotic analysis; multivariable control systems; feedback systems.

ABSTRACT

Properties of the inverse $(A\text{-}pBC)^{-1}$ are used to characterise the parameters of the infinite zeros of the root-locus of an invertible, multivariable feedback system S(A,B,C) as the solutions of constrained eigenvalue problems and to generate generic subspace conditions that guarantee only integer order infinite zeros in terms of the subspaces in the {A,B}-invariant subspace algorithm.

1. Introduction

The root-locus of an m-input/m-output linear time-invariant system S(A,B,C) described by the model in R^n

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$y(t) = C\,x(t) \tag{1}$$

is the loci of the poles of the system generated by use of the input $u(t) = -\,p\,y(t) = -\,p\,C\,x(t)$ as the scalar gain parameter p varies from zero to infinity. At each gain p, the closed-loop poles are solutions of the characteristic equation $|s(p)\,I_n - A + p\,BC| = 0$ or the equivalent eigenvalue equation

$$(A - p\,BC)\,x(p) = s(p)\,x(p) \tag{2}$$

where the eigenvector $x(p) \neq 0$ and will be assumed to lie on the unit sphere in R^n. The behaviour of the finite and unbounded solutions s(p) as $p \to +\infty$ has now been well-studied using return-difference techniques combined with complex variable theory on Riemann surfaces [1], Markov parameter matrices [2] and transformation techniques [3]. The generic structure of the infinite zeros has been identified to be of the form [4]

$$s_{j\ell}(p) = p^{1/\nu_j}\,\eta_{j\ell} + \varepsilon_{j\ell}(p) \quad ,$$

$$\lim_{p\to\infty} p^{-1/\nu_j}\,\varepsilon_{j\ell}(p) = 0 \quad , \quad 1 \le \ell \le \nu_j \;,\; 1 \le j \le m \qquad (3)$$

where ν_j, $1 \le j \le m$, are the integer structural invariants of the Morse group [5] and $\eta_{j\ell}$, $1 \le \ell \le \nu_j$, are the distinct ν_j^{th} roots of a non-zero complex number.

The nature of the genericity of (3) has recently been identified using [6] input/output Markov parameter methods. More precisely, it has been seen that the characterization (3) holds for systems $S(A + BF + KC , BM , NC)$ with (F,K,M,N) in an open, dense subset of $L^{mxn} \times L^{nxm} \times L^{mxm} \times L^{mxm}$ and, in particular, in $\{0\} \times \{0\} \times \{I_m\} \times \psi$ with ψ open and dense in L^{mxm}. The details can be found in [6]. The problem with Markov analysis is that it provides no direct insight into the state space geometry of the root-locus. In particular, it provides no simple computational framework in the state-space and sheds little light on the relative subspace structure of the triple (A,B,C) that guarantees the validity of the generic structure (3). A preliminary attempt at this problem was provided [7] by the author by examination of the set of eigenvalue equations generated by (2)

$$(A - pBC)^k\, x(p) = (s(p))^k\, x(p) \quad , \quad k \ge 1 \qquad (4)$$

Although successful, the conditions generated are highly complex and have no clear system theoretical connections or interpretation. The purpose of this paper is to provide a simple and thorough treatment of the problem by considering the related eigenvalue problems

$$(A - pBC)^{-k} (s(p))^k\, x(p) = x(p) \quad , \quad k \ge 1 \qquad (5)$$

derived from (4) and the properties of the polynomial matrix inverse

$(A - pBC)^{-1}$ relative to the $\{A,B\}$ - invariant subspace algorithm [8],

$$V_o = R^n$$
$$V_{j+1} = N(C) \cap A^{-1}(R(B) + V_j) \quad , \quad j \geq o \qquad (6)$$

which generates a nested sequence $V_o \supset V_1 \supset V_2 \supset \ldots$. converging to the maximal $\{A,B\}$ - invariant subspace V^* in the kernel of C and the dual sequence [5] $\tau_o \subset \tau_1 \subset \tau_2 \subset \ldots$. generated by

$$\tau_o = \{o\}$$
$$\tau_{j+1} = R(B) + A(\tau_j \cap N(C)) \quad , \quad j \geq o \qquad (7)$$

and converging to τ^*.

(Remark: Throughout the paper the notation R(L) and N(L) will be used to represent the range and kernel of the linear map L respectively, the symbol $L|V$ will denote the restriction of L to a linear subspace V and $\bar{V}$ will denote a matrix whose columns are a basis for V).

2. On the Structure of $(A - pBC)^{-1}$

Equation (4) indicates that the polynomial matrix $(A-pBC)^{-1}$ has a direct connection with root-locus structure. We first indicate that the inverse is well-defined:

Proposition 1: If S(A,B,C) has no zero at the origin of the complex plane, then (A-pBC) exists at all but a finite number of complex gains p. Furthermore, under these conditions

$$(A-pBC)^{-1} = A_o + p^{-1}A_1 + p^{-2}A_2 + \ldots \qquad (8)$$

the series converging absolutely for all large enough gains p.

Proof: For the first part, it is sufficient to prove that $|A - pBC| \not\equiv 0$. Suppose the converse and hence that, for each p, there exists a non-zero $\psi(p) \in R^n$ such that $(A-pBC)\,\psi(p) = 0$. Without loss of generality, take $\psi(p)$ on the unit sphere and let ψ_∞ be an arbitrary cluster point of $\{\psi(p)\}_{p \geq o}$ as $p \to \infty$. Clearly $\psi_\infty \in N(C) - \{o\}$ and $A\,\psi_\infty \in R(B)$ indicating that S has a zero at the origin contrary to assumption. Next, to prove the validity of (8), note that $(A - pBC)^{-1}$ is a matrix of rational functions of p and hence can be expanded in the form

$$(A - pBC)^{-1} = \sum_{j=-k}^{\infty} A_j \, p^{-j} \tag{9}$$

for some <u>finite</u> integer $k \geq 0$ and for all large enough gains p. Suppose that $k \geq 1$ and $A_{-k} \neq 0$. Substitution of (9) into the obvious identity $(A - pBC)(A - pBC)^{-1} = I_n$ and equating coefficients leads in particular, to the equations

$$BCA_{-k} = 0 \quad , \quad AA_{-k} - BC\,A_{-k+1} = 0 \tag{10}$$

and hence the inclusion $R(A_{-k}) \subset N(C) \cap A^{-1}R(B)$. The assumption that S has no zero at $s = 0$ ensures that the right-hand-side is $\{0\}$ and hence that $A_{-k} = 0$ contrary to assumption. The result is hence proven.

The assumption that $S(A,B,C)$ has no zero at $s=0$ is fundamental to the following development due to its simplifying effect on $(A - pBC)^{-1}$ as expressed by (8). Although on apparent restriction, we note that it will almost always be satisfied in practice and can be arranged by the following easily proved construction:

<u>Proposition 2</u>: Suppose that $S(A,B,C)$ is invertible. Then $S(A - \alpha I_n, B, C)$ has no pole or zero at the origin of the complex plane for all but a finite number of shifts α.

Throughout the remainder of this paper we will therefore assume that $S(A,B,C)$ is invertible with no pole or zero at the origin of the complex plane. The invertibility assumption cannot be removed but if $S(A,B,C)$

has a pole or zero at $s = o$, it will be assumed that a suitable 'shift' has been implemented to remove it. The effect of the shift on the root-locus is easily estimated by noting that the map $A \rightarrow A - \alpha I$ induces the eigenvalue map $s \rightarrow s - \alpha$ i.e. the root-locus is shifted by α!

The coefficients A_j , $j \geq o$, will play an important role in the paper. The following resultis easily obtained by using (8) in (A-pBC) $(A-pBC)^{-1} = I_n$ and $(A - pBC)^{-1}(A-pBC) = I_n$ and equating coefficients.

Proposition 3: (a)

$$BCA_o = o$$

$$AA_o - BC A_1 = I_n$$

$$AA_j - BCA_{j+1} = 0 \quad , \quad j > o \tag{11}$$

(b) $A_oBC = o$

$$A_oA - A_1 BC = I_n$$

$$A_jA - A_{j+1}BC = 0 \quad , \quad j > o \tag{12}$$

The properties of A_o and A_1 will prove to be particularly important. Some basic properties are described below:

Proposition 4:

$$N(A_o) = R(B) \tag{13}$$

$$R(A_o) = N(C) \tag{14}$$

Proof: Using (11), (12) $A_oBC = 0$ indicates that $R(B) \subset N(A_o)$, but if $x \in N(A_o)$ then $x = (AA_o - B C A_1)x = B C A_1 x \in R(B)$ proving (13). To prove (14) note that $B C A_o = 0$ implies $R(A_o) \subset N(C)$. But (13) indicates tha rank $A_o = n-m$ and hence equality must hold as $\dim N(C) = n-m$.

Proposition 5: (a) $R(A_o) \cap R(A_1) = \{o\}$ (15)

(b) $R^n = R(A_o) \oplus R(A_1)$ (16)

(c) $R(A_1) = R(A_1B) = N(A_oA) = A^{-1}R(B)$ (17)

(d) $B C A_1|B = - I$ (i.e. $C A_1 B = - I_m$) (18)

Proof: By assumption $|A| \neq 0$ and hence rank AA_o = rank A_oA = rank A_o = n-m by proposition 4. Clearly rank $A_1BC \leq m$ with equality holding as the identity $A_oA - A_1BC = I_m$ (equation (12)) also indicates that rank $A_1BC \geq n - \text{rank } A_oA = m$. Conditions (a) and (b) are now trivially proved. To prove (d) use (11) and (13) to verify that $B = (AA_o - BCA_1)B = -BCA_1B$. Finally, to prove (c) note that (d) indicates that dim $A_1R(B)$ = dim $R(B)$ = m. In fact, dim $A_1R(B)$ = dim $R(A_1)$ = m as dim $R(A_1) \geq$ dim $A_1R(B)$ = m whilst (equation (11)) $AA_1 - BCA_2 = 0$ indicates that dim $R(A_1) \leq$ dim $R(B)$ = m. A dimension argument now indicates that $A_1R(B) = R(A_1)$. Next note that $x \in N(A_oA)$ together with (12) indicates that $x = (A_oA - A_1BC)x = -A_1BCx \in R(A_1B)$ whilst $x \in R(A_1B)$ yields $A_oAx = x + A_1BCx \in R(A_1B)$. Clearly $A_oAx = o$ from (15) and the observation that $A_oAx \in R(A_o) \cap R(A_1)$. This proves that $R(A_1B) = N(A_oA)$. Finally, $N(A_oA) = A^{-1}N(A_o) = A^{-1}R(B)$ by (13).

Proposition 6: (a) $R(AA_o) \cap N(A_o) = \{o\}$ (19)

(b) $R^n = R(AA_o) \oplus N(A_o)$ (20)

(c) $N(A_1) = N(BCA_1) = R(AA_o) = A\,N(C)$ (21)

(d) $A_oA \mid R(A_o) = I$ (22)

Proof: The proof is similar in structure to that of Proposition 5 and is outlined below. Note that rank AA_o = rank A_o = n-m and also that rank BCA_1 = m as rank $BCA_1 \leq$ rank B = m and the equation $AA_o - BCA_1 = I_n$ indicates that rank $BCA_1 \geq n - \text{rank } AA_o = m$. Conclusions (a) and (b) follows directly from the identity $AA_o - BCA_1 = I_n$ by a dimension argument. To prove (d), simply note that $x \in R(A_o) = N(C)$ together with $A_oA - A_1BC = I_n$ indicates that $x = A_oAx$. To prove (c), note that $R(AA_o) = AN(C)$ by (14). Next note that $N(BCA_1) = R(AA_o)$ as $x \in N(BCA_1)$ together with (11) indicates that $x = AA_ox \in R(AA_o)$ whilst $x \in R(AA_o)$ plus (11) indicates that $x - AA_ox = BCA_1x \in R(AA_o) \cap N(A_o)$ by (13). Part (a) yields immediately that $BCA_1x = 0$ i.e. $x \in N(BCA_1)$. Finally, $N(A_1)$

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$= N(BCA_1)$ as $A_1\ x = o$ implies $BCA_1\ x = o$ whilst $BCA_1\ x = o$ indicates that $A_1\ x \in N(C) = R(A_o)$ i.e. $A_1\ x \in R(A_o) \cap R(A_1) = \{o\}$ by (15) and hence $x \in N(A_1)$.

These results are sufficient to provide a simple characterization of A_o and A_1 that is suitable for computation:

Proposition 7: Let N and M be, respectively, $(n-m) \times n$ and $n \times (n-m)$ full rank matrices satisfying the annihilator equations

$$NB = O \quad , \quad CM = O \tag{23}$$

Then A_o and A_1 can be computed from the formulae

$$A_o = M(NAM)^{-1}N \tag{24}$$

$$A_1 = -A^{-1}B(CA^{-1}B)^{-2}C\ A^{-1} \tag{25}$$

Proof: Proposition 4 indicates that $A_o = MQN$ for some nonsingular $(n-m) \times (n-m)$ matrix Q whilst (22) indicates that $M = A_o\ A\ M = M\ Q\ N\ A\ M$ and hence that $Q\ NAM = I_{n-m}$ proving (24). In a similar manner (17) and (21) indicate that $A_1 = A^{-1}\ B\ P\ C\ A^{-1}$ for some nonsingular $m \times m$ matrix P. Condition (18) then yields $-\ I_m = C\ A_1\ B = (CA^{-1}B)P(CA^{-1}B)$ i.e. $P = -(CA^{-1}B)^{-2}$ proving (25).

It is of interest to note the presence of the term NAM in A_o. This term also appears in the theory of zero computation [9,10] using annihilators One could conjecture at this stage that A_o has a strong connection with the zero structure of the system. This will be made more precise below by relating A_o to the subspace sequence $\{V_j\}_{j \geq o}$ defined by (6).

Proposition 8: $$V_j = R(A_o^j) \quad , \quad j \geq o \tag{26}$$

Moreover $A_o V^* = V^*$ and $A_o \bar{V}^* K = \bar{V}^*$ for some nonsingular matrix K whose eigenvalues are the zeros of $S(A,B,C)$.

Proof: We prove (26) by induction. It is trivially true for j=o so suppose that $V_j = R(A_o^j)$. From the definition it is clear that $A V_{j+1} \subset V_j + R(B)$ and hence that $A\overline{V}_{j+1} = \overline{V}_j L_{j+1} + B M_{j+1}$ for some L_{j+1} and M_{j+1}. Now (12) indicates that $(A_oA - A_1 B C)\overline{V}_{j+1} = \overline{V}_{j+1} = A_oA \overline{V}_{j+1}$ as $V_{j+1} \subset N(C)$. Clearly $\overline{V}_{j+1} = A_oA \overline{V}_{j+1} = A_o(\overline{V}_j L_{j+1} + B M_{j+1}) = A_o\overline{V}_jL_{j+1}$ by (13), and hence $V_{j+1} \subset A_oV_j = R(A_o^{j+1})$. To prove the reverse inclusion (and hence equality) use (11) in the form $(AA_o - BCA_1)A_o^j = A_o^j$ to deduce that $A\, R(A_o^{j+1}) \subset R(A_o^j) + R(B) = V_j + R(B)$. Using (14), it follows that $R(A_o^{j+1}) \subset N(C) \cap A^{-1}(V_j + R(B)) = V_{j+1}$ as required. Finally, the statement $A_o V^* = V^*$ follows from (26) and the fact that $V_j \to V^*$. $A_o|V^*$ is clearly a bijection so the existence of nonsingular K is trivially proved. Using (11) to write $(A A_o - B C A_1)\overline{V}^* K = \overline{V}^* K$ we deduce that $A \overline{V}^* = \overline{V}^* K + B C A_1 \overline{V}^* K$ and hence that the eigenvalues of K are the zeros of S(A,B,C).

The following result also relates A_o to the subspace sequence $\{\tau_j\}_{j\geq o}$ defined by (7).

Proposition 9: $\tau_j = N(A_o^j) \quad , \quad j \geq o$ (27)

Proof: Use induction noting that (28) is trivially true for j = o and supposing that $\tau_j = N(A_o^j)$. Take $x \in \tau_{j+1}$ in the form $x = b + A w$ with $b \in R(B)$ and $w \in \tau_j \cap N(C)$ and subsequently examine $A_o^{j+1} x = A_o^{j+1}(b+Aw) = A_o^{j+1}Aw$ (by (13)). Using (12) $A_oA w = w$ as $w \in N(C)$ and hence $A_o^{j+1} x = A_o^j(A_oA w) = A_o^j w = o$ as $w \in \tau_j = N(A_o^j)$. This proves that $\tau_{j+1} \subset N(A_o^{j+1})$. To prove the reverse inclusion, take $x \in N(A_o^{j+1})$ and write $A_o^{j+1} x = o$ in the form $A_o x \in N(A_o^j) \cap N(C) = \tau_j \cap N(C)$. Using (11) in the form $x = A A_o x - B C A_1 x \in A(\tau_j \cap N(C)) + R(B) = \tau_{j+1}$ we conclude that $N(A_o^{j+1}) \subset \tau_{j+1}$ as required.

Propositions 8 and 9 essentially relate $\{V_j\}$ and $\{\tau_j\}$ to the Jordan structure of A_o. To connect these ideas to the root-locus structure in the next section we will also need the following results.

Proposition 10: For all $k \geq 0$,

$$\{x : A_o^k A_1 x \in R(A_o^{k+1})\} = A_1^{-1}(N(C) + \tau_k) \tag{28}$$

Proof: If $A_o^k A_1 x \in R(A_o^{k+1})$ then $A_1^k(A_1 x - A_o z) = 0$ for some $z \in R^n$ i.e. $A_1 x - A_o z \in N(A_o^k) = \tau_k$ by (27) and hence using (14), $x \in A_1^{-1}(\tau_k + N(C))$. Conversely, if $x \in A_1^{-1}(N(C) + \tau_k)$ then $A_1 x = A_o z + V$ for some $z \in R^n$ and V such that $A_o^k V = 0$ i.e. $A_o^k A_1 x = A_o^{k+1} z \in R(A_o^{k+1})$ and the proof is completed.

Proposition 11: $R(B) \cap A_1^{-1}(N(C) + \tau_k) = B\,C\,\tau_k \;,\; k \geq 0$ (29)

Proof: Suppose that $x \in R(B)$ and $A_1 x \in N(C) + \tau_k$. Using (14), write $A_1 x = A_o z + b$ with $b \in \tau_k$ and $z \in R^n$ and note from (11) that $(AA_o - BCA_1)x = x = - B\,C\,A_1\,x$ as $A_o x = 0$ from (13). Eliminating $A_1 x$ from this expression yields $x = - BC(A_o z + b) = - B\,C\,b$ (from (14)) and hence $x \in B\,C\,\tau_k$. Conversely, if $x \in B\,C\,\tau_k$ then $x \in R(B)$ is clear and $x = B\,C\,V$ for some $V \in \tau_k$. Again invoking (11), write $(AA_o - B\,C\,A_1)x = x = - B\,C\,A_1\,x$ as $A_o x = 0$ by (13). A simple calculation yields $B\,C\,(A_1 x + V) = 0$ and hence $A_1 x \in \tau_k + N(C)$ as required.

Combining Propositions 4,8 - 11 immediately yields

Proposition 12: For all $k \geq 0$,

$$R(A_o^k) \cap N(A_o) \cap \{x : A_o^k A_1 x \in R(A_o^{k+1})\}$$

$$= V_k \cap R(B) \cap A_1^{-1}(N(C) + \tau_k)$$

$$= V_k \cap B\,C\,\tau_k \tag{30}$$

The subspaces $V_k \cap B C \tau_k$ will play an important role in the root-locus theory of the following section. Noting that V_k and τ_k are independent of input mappings $B \rightarrow BK$ if $|K| \neq 0$, the following proposition highlights a generic property of the family generated by taking $K \in G\ell(m)$.

<u>Proposition 13</u>: $V_k \cap B K C \tau_k = \{o\} \quad , k \geq 0$ (31)

for all K in an open-dense subset of $G\ell(m)$

(Remark (i) the open-dense subset is in fact the complement of a proper algebraic variety in $G\ell(m)$,

(ii) the result can be strengthened to include the validity of (31) in the presence of arbitrary state feedback, output injection and change of basis in the input and output spaces but these results are not needed here.)

<u>Proof</u>: Bearing in mind (13), (14), (26) and (27), consideration of the Jordan structures of A_o leads to the identity,

$$\dim V_k \cap R(B) + \dim C \tau_k = m, \quad k \geq o \tag{32}$$

The result now follows from the observation that, for each $k \geq o$, $B K C \tau_k$ is a subspace of $R(B)$ of dimension $m - \dim V_k \cap R(B)$ and that to each subspace $W_k \subset R(B)$ of that dimension there exists a K such that $W_k = BKC \tau_k$. Using a dimension arguement based on (32) it is clear that $W_k \cap V_k = \{o\}$ for all K except those on a proper algebraic variety A_k of $G\ell(m)$ and the result is proven as there are only a <u>finite</u> number of distinct entries in both $\{V_j\}$ and $\{\tau_j\}$ and hence $G\ell(m) - \bigcup_{k \geq o} A_k$ is open and dense.

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3. Root-locus Structure Theory

In this section we apply the results of section 2 in the construction of an eigenvalue-like characterization of the asymptotic directions of the infinite zeros of the root-locus and the identification of conditions that guarantee the generic structure of equation (3). To begin the analysis, examine the eigenvalue equation (4) in the form of equation (5) with k=1 and use (8) to write it in the form

$$(A_o + p^{-1}A_1 + p^{-2} A_2 + \ldots.) \, s(p) \, x(p) = x(p) \tag{33}$$

Suppose that $p \to \infty$ and that attention is focussed on a branch of the root-locus corresponding to a cluster point x_∞ of $\{x(p)\}$ on the unit sphere in R^n. If $\{s(p)\}$ has a finite cluster point s_∞ on this branch then (33) indicates that $A_o \, s_\infty \, x_\infty = x_\infty$. Proposition 8 indicates that $x_\infty \in V^*$ and recovers the well-known result [1-3] that all finite cluster points of the root locus are zeros of S(A,B,C). We therefore concentrate on the case when $|s(p)| \to \infty$ and $p \to \infty$.

First we note the following result:

Proposition 14: If $|s(p)| \to \infty$ as $p \to \infty$ then $x_\infty \in N(A_o)$ and $p^{-1}s(p)$ remains bounded as $p \to \infty$.

Proof: Dividing (33) by s(p) and letting $p \to \infty$ reduces to $A_o x_\infty = 0$. Next note that if $p^{-1}s(p)$ is unbounded then division of (33) by $p^{-1}s$ yields $A_1 \, x_\infty + \lim_{p\to\infty} \, p \, A_o \, x(p) = 0$ and hence, using (28) and (29) with $k = o$, we obtain $x_\infty \in B \, C \, \tau_o = \{o\}$ contradicting the assumption that $x_\infty \neq 0$. The result is therefore proved.

Suppose that attention is restricted to unbounded branches of the root-locus such that $p^{-1}s(p)$ has a cluster point $\lambda^{(1)}$. From (33) it is clear that $\lambda^{(1)}$ and x_∞ satisfy the constrained eigenvalue problem

$$(I_n - \lambda^{(1)} A_1) x_\infty \in R(A_o) \quad , \quad x_\infty \in N(A_o) - \{o\} \tag{34}$$

which can be written in the form

$$(I_n - \lambda^{(1)} A_1)x_\infty \in V_1 \quad , \quad x_\infty \in (R(B) \cap V_o) - \{o\} \tag{35}$$

This relation can, in principle, be used to calculate the asymptotic directions of the first-order infinite zeros of S(A,B,C) by searching for non-zero solutions $\lambda^{(1)}$. There may however be zero solutions corresponding to the existence of higher order infinite zeros. If $\lambda^{(1)} = o$ then it is clear that $\lim_{p\to\infty} p^{-1} s(p) = o$ and that $\lim_{p\to\infty} A_o\, s(p)x(p) = x_\infty$. In order to produce an inductive procedure for characterization of the higher order root-locus, it is convenient to introduce the following property in a similar manner to the development of ref [7]:

<u>Definition</u>: The property GA(k) holds true if

$$\lim_{p\to\infty} p^{-1}(s(p))^k = o, \lim_{p\to\infty} A_o^k\,(s(p))^k\, x(p) = x_\infty \tag{36}$$

(Remark: Note that GA(o) trivially holds true).

The main result of the paper can now be stated:

<u>Theorem 1</u>: If GA(k) is valid and the condition

$$V_k \cap B\, C\, \tau_k = \{o\} \tag{37}$$

holds true, then $p^{-1}(s(p))^{k+1}$ only has finite cluster-points $\lambda^{(k+1)}$ that can be obtained from the 'constrained' eigenvalue problem

$$(I_n - \lambda^{(k+1)} A_o^k\, A_1)\, x_\infty \in V_{k+1}$$

$$x_\infty \in (R(B) \cap V_k) - \{o\} \tag{38}$$

Moreover, if $\lambda^{(k+1)} = o$, GA(k+1) holds true.

Before proving this result, it is vital to recognize its inductive nature. More precisely <u>provided that the subspace condition (37) is</u>

satisfied the result can be applied recursively for k=0,1,2,... to obtain the asymptotic structure of the infinite zeros. It has already been noted that (37) holds in the generic sense defined by proposition 13 and hence we must conclude that it is exactly the state-space geometric counterpart to the generic conditions derived in [6] based upon algebraic properties of the Markov parameter sequence!

Proof of Theorem 1: Using (8) in (5) with k replaced by k+1 yields

$$\{A_o^{k+1} + p^{-1}(F_k A_o + A_o^k A_1) + O(p^{-2})\}(s(p))^{k+1} x(p) = x(p) \qquad (39)$$

for some nxn matrix F_k. Suppose that $p^{-1} s^{k+1}$ has an unbounded branch and hence, dividing (39) by $p^{-1} s^{k+1}$ and using Proposition (14), that

$$(F_k A_o + A_o^k A_1)x_\infty = A_o^k A_1 x_\infty \in R(A_o^{k+1}) \qquad (40)$$

where, using Proposition (14) and GA(k), $x_\infty \in N(A_o) \cap R(A_o^k)$. Proposition 12 then indicates that $x_\infty \in V_k \cap BC\tau_k$ or $x_\infty = o$ by (37). This possibility has been excluded so we must conclude that $p^{-1}s^{k+1}$ remains bounded. Equation (38) follows by letting $p \to \infty$ in (39) to yield

$$(I_n - (F A_o + A_o^k A_1)\lambda^{(k+1)})x_\infty = \lim_{p\to\infty} A_o^{k+1}(s(p))^{k+1}x(p) \qquad (41)$$

and noting that $A_o x_\infty = o$ by Proposition 14 and, together with GA(k) that $x_\infty \in N(A_o) \cap R(A_o^k) - \{o\} = R(B) \cap V_k - \{o\}$. Finally, if $\lambda^{(k+1)} = o$ then $\lim_{p\to\infty} p^{-1}s^{k+1} = o$ by definition and (41) indicates that $x_\infty = \lim_{p\to\infty} A_o^{k+1}(s(p))^{k+1}$ x(p) proving the validity of GA(k+1).

Finally, the condition GA(k) can be used to provide information on the eigenvector x(p):

Theorem 2: If GA(j) holds true for $1 \leq j \leq k$ then

$$x(p) = x_\infty + (s(p))^{-1}x_1 + \ldots + (s(p))^{-k}x_k + \varepsilon_k(p) \tag{42}$$

where $x_j \in \tau_{j+1}$, $1 \leq j \leq k$, are any solutions of the algebraic equations

$$x_\infty = A_o^j x_j \quad , \quad 1 \leq j \leq k \tag{43}$$

and $\varepsilon_k(p)$ has the properties that

$$\lim_{p\to\infty} (s(p))^j \varepsilon_k(p) = o \pmod{\tau_j}$$

$$o \leq j \leq k \tag{44}$$

Proof: Consider first the case of k=1 when GA(1) indicates that $x_\infty = \lim_{p\to\infty} A_o s(p)\, x(p)$. Let x_1 be any solution of $x_\infty = A_o x_1$ and note that $o = A_o x_\infty = A_o^2 x_1$ so that $x_1 \in \tau_2$ by (27). Now write $x(p) = x_\infty + (s(p))^{-1} x_1 + \varepsilon_1(p)$ where $\lim_{p\to\infty} \varepsilon_1(p) = o$ and $\lim_{p\to\infty} A_o s(p)\, \varepsilon_1(p) = O$. Clearly $\lim_{p\to\infty} (s(p))^j \varepsilon_1(p) = O \pmod{N(A_o^j)}$ for $j = 0,1$ which, together with (27), verifies (44) in the case of k=1. Proceeding by induction we suppose that the result is true for k replaced by $\ell < k$ and proceed to prove that the validity of GA(ℓ+1) ensures that the result is true with k replaced by ℓ+1. To do this note that GA(ℓ+1) means that $x_\infty = \lim_{p\to\infty} A_o^{\ell+1}(s(p))^{\ell+1} x(p) = \lim_{p\to\infty} A_o^{\ell+1} (s(p))^{\ell+1} \varepsilon_\ell(p)$ as $x_j \in \tau_{j+1} \subset \tau_{\ell+1} = N(A_o^{\ell+1})$, $1 \leq j \leq \ell$. Write $\varepsilon_\ell(p) = (s(p))^{-(\ell+1)} x_{\ell+1} + \varepsilon_{\ell+1}(p)$ where $x_\infty = A_o^{\ell+1} x_{\ell+1}$ and hence $x_{\ell+1} \in \tau_{\ell+2}$ as $A_o^{\ell+2} x_{\ell+1} = A_o x_\infty = O$. The function $\varepsilon_{\ell+1}(p)$ has the property that $\lim_{p\to\infty} s^j \varepsilon_{\ell+1}(p) = \lim_{p\to\infty} s^j \varepsilon_\ell(p) = o \pmod{\tau_j}$ for $1 \leq j \leq \ell$ and $\lim_{p\to\infty} s^{\ell+1} A_o^{\ell+1} \varepsilon_{\ell+1}(p) = O$ from the definition of $x_{\ell+1}$. The validity of the inductive assumption is evident by writing this expression in the form $\lim_{p\to\infty} s^{\ell+1} \varepsilon_{\ell+1} = O \pmod{N(A_o^{\ell+1})} = O \pmod{\tau_{\ell+1}}$, and the theorem is proved.

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4. Conclusions

Using the properties of the inverse $(A-pBC)^{-1}$ the asymptotic directions of the infinite zeros of the multivariable root-locus have been characterised by constrained eigenvalue problems. Generic conditions for the existence of only integer order infinite zeros have been identified in terms (equation (37)) of the subspaces $\{V_j\}$ and $\{\tau_j\}$ in the $\{A,B\}$ invariance algorithm and its dual. Major assumptions in the analysis are that the system is invertible with no pole or zeros at the origin of the complex plane. Origin shift techniques (proposition 2) indicate that there is no real loss of generality here but it would be nice if future proofs do not need this simplification.

5. References

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