Monograph:

Owens, D.H. and Chotai, A. (1983) On Eigenvalues, Eigenvectors and Singular Values in Robust Stability Analysis. Research Report. ACSE Report 216 . Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield

ON EIGENVALUES, EIGENVECTORS AND SINGULAR VALUES IN ROBUST STABILITY ANALYSIS

by

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Research Report No. 216

February 1983

This work is supported by SERC under grant GR/B/23250

Indexing terms: Multivariable systems; stability; robustness.

Abstract

Recent papers have examined the problem of robustness of the stability of multivariable feedback systems to perturbations ΔG in matrix form. Attention has been primarily focussed on the use of the maximal singular value $\bar{\sigma}(\Delta G)$. This paper considers how structured information on the uncertainty in each element $\Delta_{ij}(s)$ can be used in a similar way based on eigenvalue and singular value analysis.

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1. Introduction

The effects of uncertainty on the stability of multivariable feedback systems has recently been approached using singular values [1] (or principal gains [2]) and plant step data [3]. In particular, the problem of the stability of a unity negative feedback system with forward path controller K(s) and an m-input, m-output, invertible nominal plant G(s) subjected to additive perturbations

$$G \rightarrow G + \Delta_a G \quad , \qquad \ldots(1)$$

multiplicative input perturbations

$$G \rightarrow G(I_m + \Delta_i G) \quad , \qquad \ldots(2)$$

and multiplicative output perturbations

$$G \rightarrow (I_m + \Delta_o G)G \qquad \ldots(3)$$

have been considered with perturbations bounded in their maximal singular value

$$\bar{\sigma}(\Delta G(s)) \leq \ell(s) \qquad \ldots(4)$$

on the Nyquist 'infinite' D-contour. These ideas have been shown [1] to give powerful insight into the problem of design in the presence of uncertainty but they rely on the characterization of uncertainty in the singular value form of equation (4). In practice, however, uncertainty may be more structured and hence more easily characterized by bounds on the elements of the perturbation, $1 \leq i, j \leq m$,

$$|(\Delta G(s))_{ij}| \leq \ell_{ij}(s) \quad , \quad s \in D \qquad \ldots(5)$$

written in the form

$$\|\Delta G(s)\|_p \leq L(s) \quad , \quad s \in D \qquad \ldots(6)$$

where $L(s)$ is the matrix of elements $\ell_{ij}(s)$ and [3], for any $n_1 x n_2$ matrix M,

$$\|M\|_p \triangleq \begin{pmatrix} |M_{11}| \; \cdots\cdots & |M_{1n_2}| \\ \vdots & \\ |M_{n_1 1}| \cdots\cdots & |M_{n_1 n_2}| \end{pmatrix} \qquad \ldots(7)$$

denotes its matrix-valued 'absolute value'. The inequality in (6) is with respect to the partial ordering $A \leq B$ iff $A_{ij} \leq B_{ij}$, $1 \leq i, j \leq m$.

The purpose of this note is to present some results on robust stability paralleling those described in [1] in terms of the uncertainty described by (5) or (6). The work is strongly related to the recent independent results of Kantor and Andres [7] but we concentrate here on the use of both eigenvalue and singular value information which are described together with a discussion on problems in eigenvalue use at branch points. For simplicity, only the case of additive perturbations will be considered. The extension to multiplicative perturbations is achieved by similar analyses.

2. Robust Stability and Eigenvalues and Eigenvectors

Following Owens and Chotai [3], the fundamental result is stated as follows:

Theorem 1: If K stabilizes the nominal plant G, it will also stabilize the plant $G+\Delta G$ if

(a) ΔG is stable,

(b) $(G+\Delta G)K$ is both stabilizable and detectable, and

(c) $\sup_{s\in D} r((I_m+KG)^{-1}K\Delta G) < 1$ ...(8)

(Note: the spectral radius of the $\ell x \ell$ matrix M is $r(M) \triangleq \max_i |m_i|$ where $m_1, m_2, \ldots, m_\ell$ are the eigenvalues of M)

The problem with (8) is that ΔG is known only in terms of bounds (5) on its gains and hence it cannot be checked directly. It is convenient therefore to use the following results [3]:

Lemma 1: If M_1 and M_2 are complex matrices, then

$$||M_1M_2||_P \leq ||M_1||_P \, ||M_2||_P \qquad ...(9)$$

Lemma 2: If M is a square complex matrix, then

$$r(M) \leq r\,(\,||M||_P) \qquad ...(10)$$

Lemma 3: If A and B are real, square matrices satisfying $0\leq A\leq B$, then

$$r(A) \leq r(B) \qquad ...(11)$$

An immediate consequence of these results is that theorem 1 can be replaced by the result [3], [7]:

Theorem 2: The conclusions of theorem 1 remain valid if (c) is replaced by

$$\sup_{s\in D} r(\|(I_m+KG)^{-1}K\|_P L(s)) < 1 \qquad \ldots(12)$$

Proof: Simply note that

$$\|(I+KG)^{-1}K\Delta G\|_P \leq \|(I+KG)^{-1}K\|_P L \quad , \quad s \in D \qquad \ldots(13)$$

by lemma 1 and equation (8) and use lemma 2,3 to prove that equation (12) implies equation (8).

This result was used in [3] to obtain a stability criterion based on open-loop plant step data. In this section however we concentrate on results based on eigenvalue or characteristic locus analysis [4], [5]. More precisely, let GK (and hence KG) have eigenvalues $q_1(s)$, $q_2(s),\ldots,q_m(s)$ and suppose that there exists a similarity transformation T(s) defined almost everywhere on D such that

$$T^{-1}KGT = \text{diag}\,\{q_1,\ldots,q_m\} \qquad \ldots(14)$$

(Note: this assumption is generic and holds at all but branch points of KG on D).

As spectral radii are invariant under similarity transformation, condition (8) can be written in the form

$$\sup_{s\in D} \; r(\mathrm{diag}\{ \frac{q_j}{1+q_j} \}_{1\leq j\leq m} \; T^{-1}G^{-1}\Delta GT) < 1 \quad , \qquad \ldots(15)$$

Noting that

$$\| \mathrm{diag}\{ \frac{q_j}{1+q_j} \}_{1\leq j\leq m} \; T^{-1}G^{-1}\Delta GT \|_P$$

$$\leq (\max_{1\leq j\leq m} \left| \frac{q_j}{1+q_j} \right|) \, \|T^{-1}G^{-1}\|_P \, \|L\|_P \, \|T\|_P \qquad \ldots(16)$$

we deduce the following robustness result:

Theorem 3: The conclusions of theorem 1 remain valid if (c) is replaced by

$$\sup_{s\in D} \; \max_{1\leq j\leq m} \left| \frac{q_j}{1+q_j} \right| r(\|T\|_P \, \|T^{-1}G^{-1}\|_P \, \|L\|_P) < 1 \qquad \ldots(17)$$

Proof: Use the preceding discussion and lemmas 2,3 to note that (17) implies (8) as $r(AB) = r(BA)$ for any matrices A and B.

The result has the following graphical interpretation:

Corollary 3.1: Condition (17) is satisfied if the bounds generated by the inverse Nyquist plots $q_j^{-1}(s)$, $s\in D$, $1\leq j\leq m$ with circles superimposed at each frequency of radius

$$r(s) = r(\|T(s)\|_P \, \|T^{-1}(s)G^{-1}(s)\|_P \, \|L(s)\|_P) \qquad \ldots(18)$$

do not contain or touch the (-1,0) point of the complex plane.

<u>Proof</u>: Write (17) as

$$|1 + q_j^{-1}(s)| > r(s) \quad , \quad s \in D \quad , \quad 1 \leq j \leq m \qquad \ldots(19)$$

and interpret graphically.

The result provides a direct link between the element uncertainties (represented by L(s)), the characteristic loci $\{q_j(s)\}$ and the structure of the eigenframe T(s) and suggests that

(1) robustness is reduced if the characteristic loci move close to the (-1,0) point of the complex plane,

(2) robustness is reduced if the uncertainty increases as, if $L'(s) \geq L(s)$, $s \in D$, then

$$r(\|T(s)\|_P \|T^{-1}(s)G^{-1}(s)\|_P \|L'(s)\|_P)$$
$$\geq r (\|T(s)\|_P \|T^{-1}(s)G^{-1}(s)\|_P \|L(s)\|_P)$$
$$s \in D \qquad \ldots(20)$$

by lemma 3,

(3) robustness can be increased by replacing K by pK where p is an overall gain and letting p become small. Note that r(s) is independent of p and $q_j(s)$ is replaced by $pq_j(s)$.

(4) robustness in a frequency range is reduced if the eigenvectors of KG are skew in the sense that T is almost singular. This <u>always</u> occurs in the vicinity of branch points as these correspond to frequencies where KG has only a <u>non-diagonal</u> Jordan form. In such situations T^{-1} is unbounded even if T is bounded and hence r(s) can take arbitrarily large values.

Observation (4) is related to the problem of characteristic locus methods noted in [1] albeit using a different framework and uncertainty characterization. Consider the example in [1],

$$G(s) = \frac{1}{(s+1)(s+2)} \begin{pmatrix} -47s+2 & 56s \\ -42s & 50s+2 \end{pmatrix} \qquad \ldots(21)$$

with $K = I_2$ generating the characteristic transfer functions

$$q_1(s) = \frac{1}{s+1} \quad , \quad q_2(s) = \frac{2}{s+2} \qquad \ldots(22)$$

and frequency independent (skew) eigenvector matrix

$$T(s) = \begin{pmatrix} 7 & 8 \\ 6 & 7 \end{pmatrix} \qquad \ldots(23)$$

This yields

$$\|T\|_P \, \|T^{-1}\|_P = \begin{pmatrix} 97 & 112 \\ 84 & 97 \end{pmatrix} \qquad \ldots(24)$$

with spectral radius equal to 194.0. Taking $s = 0$, then $G(o) = I_2$ and if the uncertainty is represented by $L(s) = \varepsilon(s)I_2$ corresponding to uncertainty in the diagonal terms of G only then $r(o) = 194.0\varepsilon(o)$ and the conditions of corollary 3.1 require that $r(o) < 2$. This means that the permissible uncertainty in steady state characteristics must be less than 1%. This result should be compared with the system

$$G(s) = \frac{1}{(s+1)(s+2)} \begin{pmatrix} s+1.5 & 0.5 \\ 0.5 & s+1.5 \end{pmatrix} \qquad \ldots(25)$$

generating characteristic transfer functions given in (22) with orthogonal eigenvector matrix

$$T(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \ldots(26)$$

Simple calculations then yield $r(o) = 2\varepsilon(o)$ and hence that we require $\varepsilon(o) < 1.0$ ie almost 100% uncertainties in diagonal steady state characteristics is permissible!

The unboundedness of $r(s)$ in the vicinity of branch points is not present if branch points are not present as proved by the following result:

Proposition 1: If $T(s)$ is uniformly bounded on D and $\inf|\det T(s)| \geq \varepsilon > 0$ for some scalar ε, then $r(s)$ is uniformly bounded on D.

Proof: Let $T(s)$ have characteristic polynomial $t(\lambda) = \sum_{i=o}^{m} t_i(s)\lambda^i$, then the $\{t_i(s)\}$ are uniformly bounded on D and $|t_o(s)| \geq \varepsilon$ for all $s \in D$. Write

$$T^{-1}(s) = \frac{(-1)}{t_o(s)} \sum_{i=1}^{m} t_i(s) T^{i-1}(s) \qquad \ldots(27)$$

by the Cayley-Hamilton theorem and take norms.

Although uniformly bounded, $r(s)$ could still be large if ε is small. This can be avoided if we use unitary transformations. The following boundedness proposition is easily proved.

<u>Proposition 2</u>: If U is mxm and unitary then $||U^*||_P = ||U||_P^T$ and both $||U||_P$ and $||U^*||_P$ have column and row sums of squares of moduli of elements equal to unity. The spectral radius of $||U||_P ||U^*||_P$ is less than or equal to m.

<u>Proof</u>: The result follows from the definition of conjugate transpose, the orthogonality of the rows and columns of U and the Cauchy-Schwarz inequality which yields $||U||_P ||U^*||_P \leq M$ where M is an mxm matrix of unity elements and hence of spectral radius $\leq m$.

Consider now the use of eigenvalue information as revealed by unitary transformations. More precisely, there exists ([6], p144) a unitary transformation U(s) such that

$$U^*(s)(I_m+K(s)G(s)^{-1}K(s)G(s)U(s) = D_1(s) \qquad \ldots(28)$$

where $D_1(s)$ is upper triangular with diagonal elements $q_j/1+q_j$, $1\leq j\leq m$. Writing

$$D_1(s) = \text{diag}\{\frac{q_j}{1+q_j}\}_{1\leq j\leq m} D_2(s) \qquad \ldots(29)$$

where $D_2(s)$ is upper triangular with unit diagonal elements then the following results are proved in a similar manner to theorem 3 and corollary 3.1:

<u>Theorem 4</u>: The conclusions of theorem 1 remain valid if (c) is replaced by

$$\sup_{s \in D} \max_{1 \leq j \leq m} \left| \frac{q_j}{1+q_j} \right| r(\|U(s)\|_P \|D_2(s)U^*(s)G^{-1}(s)\|_P \|L(s)\|_P)$$

$$< 1 \qquad \ldots(30)$$

Corollary 4.1: The results of corollary 3.1 hold with the replacement

$$r(s) = r(\|U(s)\|_P \|D_2(s)U^*(s)G^{-1}(s)\|_P \|L(s)\|_P) \qquad \ldots(31)$$

Consider the application of the result to the example of equation (21) with

$$U(s) = \frac{1}{\sqrt{85}} \begin{pmatrix} 7 & -6 \\ 6 & 7 \end{pmatrix} \qquad \ldots(32)$$

At $s = 0$, it is easily seen that $G(o) = I_2$, $D_2(o) = I_2$ and, with $L(s) = \varepsilon(s)I_2$, we obtain $r(o) = 1.99\varepsilon(o)$. Corollary 3.1 and 4.1 then require $r(o) < 2$ which is equivalent to $\varepsilon(o) < 1.0$. The improvement in the prediction of permissible steady state errors over those due to corollary 3.1 are self-evident.

In conclusion, the use of unitary transformations can enable the use of element information in robust stability analysis based on characteristic locus behaviour without the singularity problems due to branch points. Note however that the matrix $D_2(s)$ depends on the detailed structure of both K and G and hence must be recomputed if K is changed. This is true even on the ray $\{pK\}_{p \geq o}$. There is therefore an incentive to try to make the eigenvector matrix T of KG unitary when we can choose $T = U$ and $D_2 = I_m$.

3. Robust Stability and Singular Values

The above ideas can also be used in the incorporation of element uncertainty in robust stability analysis based on singular values [1]. More precisely, write

$$(I+KG)^{-1}KG \quad = \quad HV \qquad \ldots(33)$$

where H is hermitean, positive-definite and V is unitary. The eigenvalues $0<\sigma_1<\sigma_2<\ldots<\sigma_m$ of H are the singular values of $(I+KG)^{-1}KG$. Suppose that H is diagonalized by the unitary transformation U_o, then the following result follows in a similar manner to theorem 3:

Theorem 5: The conclusions of theorem 1 hold if (c) is replaced by

$$\underline{\sigma}(I+(KG)^{-1}) > r(s) \quad , \quad s\in D \qquad \ldots(34)$$

where

$$r(s) \triangleq r(\, \|U_o(s)\|_p \, \|U_o^*(s)V(s)G^{-1}(s)\|_p \, \|L(s)\|_p) \qquad \ldots(35)$$

Proof: It is easily shown that (c) can be replaced by

$$\sup_{s\in D} \bar{\sigma}((I+KG)^{-1}KG)r(\, \|U_o(s)\|_p \, \|U_o^*(s)V(s)G^{-1}(s)\|_p \, \|L(s)\|_p) < 1 \qquad \ldots(36)$$

which is equivalent to (34) as [1]

$$\bar{\sigma}((I+KG)^{-1}KG) \quad = \quad (\underline{\sigma}(I+(KG)^{-1}))^{-1} \qquad \ldots(37)$$

This result should be compared with the equivalent result in [1] where the robust stability criterion reduces to

$$\underline{\sigma}(I+(KG)^{-1}) > \bar{\sigma}(G^{-1}\Delta G) \quad , \quad s\in D \qquad \ldots(38)$$

Note however that theorem 5 provides a direct link between element uncertainty and robust stability, hence removing the need to represent structured uncertainties by unstructured bounds on singular values.

4. Conclusions

The paper has illustrated how structured uncertainty in the elements of a multivariable plant G can be incorporated directly into robust stability analyses. The use of characteristic locus structure of the nominal plant has been demonstrated and it has been seen that skewness of the eigenvector matrix has a significant effect on robustness by reducing stability margins. In the worst cases, the techniques are highly sensitive to uncertainty in the vicinity of branch points. The use of unitary transformations removes the problem both in the use of eigenvalues and singular values but at the expense of increased complexity of the result as the transformations interact with the uncertainty to effect stability margins. This is to be expected however as eigenvalue and/or singular value information on nominal plant dynamics is not defined in the natural input-output basis.

Acknowledgements

This work is supported by the UK Science and Engineering Research Council under grant GR/B/23250.

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