# Realisation Theory and the Infinite Dimensional Root Locus

by

S.P.Banks and F.Abbasi-Ghelmansarai

Department of Control Engineering
University of Sheffield
Mappin Street
Sheffield S13JD

Research Report No. 210

February 1983

Abstract

The theory of realisation of linear input-output maps is applied to the study of root locus for distributed systems with bounded defining operators .

## (1) Introduction

In this paper we shall study the root locus of single input single output infinite dimensional systems which have a bounded realisation . The frequency domain approach to finite dimensional single input single output systems , which was developed by Bode, Nyquist and others gave way to the state space theory which is discussed in the book of Lee and Markus (1967) . The generalisation of the classical frequency domain methods was then given a new impetus with the theory of multivariable control developed largely by Rosenbrock (1974), Postlethwaite and MacFarlane (1979) , and Owens(1978) . On the other hand the control theory of distributed systems has been studied mainly from the state space viewpoint as discussed by Balakrishnan (1976) and Curtain and Pritchard (1978); the frequency domain methods have been neglected in infinite dimensional systems theory , apart from the realisation theory of Baras and Brockett (1975) and Fuhrmann and Brockett (1976) and work on systems with discrete spectrum (the heat equation , for example ) by Pohjolainen(1981,1982) .

In a recent paper , Banks and Abbasi-Ghelmansarai (1983) have studied the root locus of an approximation to a simple delay equation involving the left shift operator . In the present work we shall attempt to give a classification of the root loci for infinite dimensional systems with a bounded realisation , and we shall see some interesting phenomena related to the cuts in the plane which must be introduced to ensure the single-valuedness of the transfer functions . This will lead to a definition of 'generalised pole' which is , essentially , a connected component of the set of singularities of the transfer function , including the cuts . If a similar definition is given for 'generalised zeros'

then we can recover the classical criterion that if an open loop system has n generalised poles and m generalised zeros , then the root locus has n connected branches (including cuts) , m of which are attracted by the generalised zeros and n-m of which diverge to $\infty$ . Hence the correct objects to regard as poles are not the branch points , but the branch points which are 'glued together' by the connected cuts .

We shall start by introducing some notation and then recall the basic realisation theory of distributed systems ; the latter is then related to the classical 'canonical ' realisation of a finite dimensional system . The root locus for a system with a bounded realisation is then studied and we end with some simple examples .

(2) Notation and Terminology

In this paper we shall use the letters H,V and $\mathcal{H}$ to denote certain Hilbert spaces and , in particular , we shall use the right-shift operator $U_r$ with the matrix representation

$$U_r = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & 0 & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & \ddots \end{bmatrix}$$

in the standard basis of $\ell^2$ (the Hilbert space of square summable sequences ). If G is a function from $\mathbb{R}^+$ to $\mathbb{R}$ , then $\tilde{G}$ or $\mathcal{L}G$ will denote the Laplace transform of G . We shall also use some elementary complex function theory ( the complex plane being denoted , as usual , by $\mathbb{C}$ ). In particular we shall find it convenient to discuss the theory of analytic functions on the Riemann sphere $S_\infty$ , which is just the space $\mathbb{C} \cup \{\infty\}$ with the one point compactification topology . If V is a real Hilbert space ,

$V \oplus \mathbb{C}$ will denote the complexification of V and $\mathcal{L}(H_1, H_2)$ will denote the space of bounded operators from the Hilbert space $H_1$ to the Hilbert space $H_2$ .

Finally , we shall use the spectral theory of bounded operators , which can be found in Dunford and Schwartz (1959) . To summarise this theory , note that if A is a bounded operator on a Banach space X then $\sigma(A)$ (the spectrum of A) is compact and any set $\sigma$ which is open and closed in $\sigma(A)$ is called a spectral set . Assume that

$$\sigma(A) = \bigcup_{i=1}^{n} \sigma_i \quad ,$$

i.e. $\sigma(A)$ is the union of a finite number of spectral sets $\sigma_i$.

Then we define the operator

$$P_i = \frac{1}{2\pi i} \int_\Gamma e(\lambda)(\lambda I - A)^{-1} d\lambda$$

where $\Gamma$ is a Jordan curve containing $\sigma$ in its interior , and

$$e(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \sigma \\ 0 & \text{if } \lambda \in \sigma(A) \setminus \sigma \end{cases}$$

It is easy to see that $P_i$ is a projection operator . Let

$$X_{\sigma_i} = P_i X \qquad \text{and} \qquad A_{\sigma_i} = A\big|_{X_{\sigma_i}} \quad .$$

Then

$$X = \bigoplus_{i=1}^{n} X_{\sigma_i} \quad , \quad A X_{\sigma_i} \subseteq X_{\sigma_i} \quad ,$$

and

$$\sigma(A_{\sigma_i}) = \sigma_i \quad .$$

Hence A has a diagonal representation $A = \operatorname{diag}\left[A_{\sigma_1}, \ldots, A_{\sigma_n}\right]$ in X .

(3) Realisation Theory

In this section we shall recall the realisation theory of input output maps due to Baras and Brockett (1975) and Fuhrmann (1974) , and relate the bounded infinite dimensional realisations to limits of finite dimensional (i.e. rational ) transfer functions. Consider , then , the scalar input-output map $G:[0,\infty) \to \mathbb{R}$. The (bounded) realisation problem is to find a bounded operator A on some Hilbert space H and vectors $b,c \in H$ such that

$$G(t) = \langle c, e^{At} b \rangle_H \quad .$$

It is well known that this problem has a solution if and only if G (respectively $\tilde{G} \triangleq \mathcal{L}G$ ) is entire and of exponential order ( or G is analytic at infinity and vanishes there ) . In fact , G can be realised on $\ell^2$ in the following way :

If $\tilde{G}(s) = \Sigma_{i=0}^{\infty} a_i s^{-(i+1)}$ for $|s|>\rho$ (say) , then we can take

$$A = k \begin{bmatrix} 0 & & & \\ 1 & 0 & & \underline{0} \\ & 1 & 0 & \\ \underline{0} & & \ddots & \\ & & & \ddots \\ & & & \ddots \end{bmatrix} = kU_r \qquad (3.1)$$

$$b = \{1,0,0,\ldots.\} \quad ,$$

$$c = \{a_0, a_1/k, a_2/k^2, \ldots.\}$$

for any $k>\rho$ . Defining

$$\sigma(\tilde{G}) = \{ s \in \mathbb{C}: \tilde{G} \text{ is not analytic at } s \} \quad , \qquad (3.2)$$

we have

$$\sigma(\tilde{G}) \subseteq \sigma(A) = k\{ s \in \mathbb{C}: |s| \leq 1 \} \quad . \qquad (3.3)$$

and so k must be chosen large enough so that (3.3) holds .If $\sigma(\tilde{G}) \subseteq \{ s \in \mathbb{C}: |s-s_0| \leq 1 \}$ , then we may realise G with (A,b,c) where

$$A = \begin{bmatrix} s_0 & & \\ 1 & s_0 & \underline{0} \\ & 1 & s_0 \\ \underline{0} & & \ddots \end{bmatrix} = s_0 I + U_r \qquad (3.4)$$

$$b = \{1,0,0,\ldots\}$$

$$c = \{a_0, a_1, \ldots\}$$

where

$$\tilde{G}(s) = \sum_{i=0}^{\infty} a_i (s-s_0)^{-(i+1)} \quad , \quad |s-s_0| > 1.$$

We can easily generalise the above result to the case where $G: [0,\infty) \to \mathcal{L}(V,V)$ for some Hilbert space V ; we have

Theorem 3.1 The weighting function G has a bounded realisation iff $\tilde{G}(s)$ is analytic at infinity and vanishes there .( $\tilde{G}$ is an analytic function with values in $\mathcal{L}(V\oplus\mathbb{C}, V\oplus\mathbb{C})$ . )

Proof The necessity is obvious . For sufficiency , write

$$\tilde{G}(s) = \sum_{i=0}^{\infty} \mathcal{A}_i s^{-(i+1)} \quad , \quad |s| > \rho$$

Then we define $\mathcal{H} = \oplus_{i=0}^{\infty} V$ and the operator A on $\mathcal{H}$ by

$$A = k \begin{bmatrix} 0_{\mathcal{H}} & & & \\ I_{\mathcal{H}} & 0_{\mathcal{H}} & & \underline{0}_{\mathcal{H}} \\ & I_{\mathcal{H}} & 0_{\mathcal{H}} & \\ \underline{0}_{\mathcal{H}} & & \ddots & \ddots \end{bmatrix} \quad , \quad k > \rho \qquad (3.5)$$

$$B = \{I_{\mathcal{H}}, 0_{\mathcal{H}}, \ldots\}$$

$$C = \{\mathcal{A}_0, \mathcal{A}_1/k, \mathcal{A}_2/k^2, \ldots\} \quad ,$$

where $0_{\mathcal{H}}$ , $I_{\mathcal{H}}$ are respectively the zero and identity operators on $\mathcal{H}$ . We then have

$$G(t) = Ce^{At}B \ .$$

A is again a unilateral shift ( of multiplicity $\tau$ , where $\tau$ is the dimension of V ). In the case where $V = \mathbb{R}^n$ (n inputs,n outputs)

we have

$$A = k\begin{bmatrix} 0_n & & \underline{0} \\ I_n & 0_n & \\ \underline{0} & \cdots\cdots & \\ & \cdots\cdots & \end{bmatrix} .$$

Returning to the single input-single output case recall that if the system has finite dimensional realisation , then the classical 'canonical' realisation is of the form $c^T e^{At} b$ , where

$$A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \underline{0} \\ & & \cdots & & \\ & & & \cdots & 0 \quad 1 \\ b_0 & -b_1 & \cdots\cdots & b_{n-2} & b_{n-1} \end{bmatrix} \qquad (3.6)$$

$$b = \{0,0,\ldots,1\}$$

$$c = \{a_0,a_1,\ldots,a_m,0,\ldots,0\} .$$

It will be important for us to relate this representation to that given by (3.1) . Suppose that the transfer function of a finite dimensional system is given by

$$\tilde{G}(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n-1} b_i s^i + s^n} \quad , \; m < n \qquad (3.7)$$

$$= \sum_{i=0}^{\infty} c_i s^{(m-n)-i} \quad ,$$

where we find inductively ,

$$c_0 = a_m$$

$$c_k = a_{m-k} - \left( \sum_{\substack{0 \le i < n \\ n-k-i \le 0}} b_i c_{-n+k+i} \right) \quad , \; k \ge 1 \qquad (3.8)$$

Hence , by (3.1) , we have the realisation defined by

$$A = kU_r$$

$$b = \{1,0,0,\ldots.\} \qquad (3.9)$$

$$c = \{\underbrace{0,0,\ldots.,0}_{n-m-1}, c_0/k^{n-m-1}, c_1/k^{n-m}, \ldots\}$$

where $k > \max_{s\in\sigma(\tilde{G})} |s|$ . Of course , in this case , $\sigma(\tilde{G})$ consists of the poles of $\tilde{G}$ . The important point to notice , however , is that whereas the representation (3.6) is s-minimal (i.e. $\sigma(A) = \sigma(\tilde{G})$ ) that given by (3.9) is not . Nevertheless , it is precisely the form (3.9) which provides the connection with infinite dimensional realisation theory . This connection is brought out more clearly if we use the following well-known theorem of complex function theory (cf. for example , Rudin 1966):

Theorem 3.2 (Runge's Theorem) Let $\Omega$ be open in $\mathbb{C}$ , and let A be a set which has one point in each component of $S_\infty - \Omega$ , and assume that f is analytic in $\Omega$ . Then there exists a sequence $\{R_n(s)\}$ of rational functions , with poles in A , such that $R_n \to f$ uniformly on compact subsets of $\Omega$ . □

In this theorem , $S_\infty = \mathbb{C}\cup\{\infty\}$ refers to the Riemann sphere (i.e. the compactified complex plane ) . Before applying the theorem , we rewrite (3.9) in a different form assuming this time that $\tilde{G}$ is given in terms of poles and zeros ;i.e.

$$\tilde{G}(s) = \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} \quad . \tag{3.10}$$

In this case ,

$$a_m = b_n = 1$$

and

$$a_i = (-1)^{m-i}\sigma_{m-i}(z_1,\ldots,z_m) \tag{3.11}$$

$$b_i = (-1)^{n-i}\sigma_{n-i}(p_1,\ldots,p_n)$$

where $\sigma_i$ is the $i^{th}$ elementary symmetric function . Hence , the realisation is again given by (3.9) with

$$c_0 = 1$$

$$c_k = (-1)^{m-k}\sigma_{m-k}(z_1,\ldots,z_m) - \sum_{\substack{0\le i<n \\ n-k-i\le 0}} \{(-1)^{n-i}\sigma_{n-i}(p_1,\ldots,p_n)c_{-n+k+i}\} \tag{3.12}$$

where $k > \max_{1\leq i\leq n} \{|p_i|\}$ .

Consider again a single input single output system with transfer function $\tilde{G}(s)$ and suppose this has a bounded realisation. Then $\sigma(\tilde{G}(s))$ contains poles and branch cuts , the latter necessarily being in the finite plane since $\tilde{G}(s)$ is analytic at $s = \infty$ . $\sigma(\tilde{G})$ is closed and so we may apply Runge's theorem with $A = \sigma(\tilde{G}) \cup \{\infty\}$. However , since $|\tilde{G}(s)| \to 0$ as $|s| \to \infty$, we may obviously assume that the rational approximations just have poles in $\sigma(\tilde{G})$ . Therefore , taking $A = \sigma(\tilde{G})$ , theorem 3.2 implies that there exists a sequence of rational functions $R_n$ such that $R_n \to \tilde{G}(s)$ uniformly on compact sets . Suppose that

$$R_n = \frac{\prod_{i=1}^{m_n} (s-z_i^n)}{\prod_{i=1}^{n} (s-p_i^n)} \quad ,$$

where we have assumed that $R_n$ has n poles (it is easy to see that there is no loss of generality in this assumption ) which are at the points $p_i^n$ $(1 \leq i \leq n)$ and $m_n$ $(< n)$ zeros at $z_i^n$ $(1 \leq i \leq m_n)$. Then we see that the function $\tilde{G}(s)$ has a realisation of the form (3.1) which can be obtained as a limit of a sequence of realisations of the form (3.9) , where

$$c_0^n = 1$$

$$c_k^n = (-1)^{m_n-k} \sigma_{m_n-k}(z_1^n, \ldots, z_{m_n}^n) - \sum_{\substack{0\leq i<n \\ n-k-i\leq 0}} \left\{(-1)^{n-i} \sigma_{n-i}(p_1^n, \ldots, p_n^n) c_{-n+k+i}^n\right\}$$

for $n = 1,2,\ldots$ (cf (3.12)).

If we consider the example given by Baras and Brockett (1975), defined by the transfer function

$$\tilde{G}(s) = \frac{1}{\sqrt{(s^2 + 1)}} \quad ,$$

then $\sigma(\tilde{G}) = \{s \in \mathbb{C}: \operatorname{Re} s = 0, |\operatorname{Im} s| \leq 1\} \triangleq L$ , i.e. $\sigma(\tilde{G})$ consists of the points $\pm i$ together with the cut along the imaginary axis joining these points . Then by the above theory , we find a sequence $\tilde{G}_n(s)$ of rational functions with n poles on L such that

$$\tilde{G}_n \to \tilde{G} \text{ , uniformly on compacta .}$$

( It should be noted that the authors cited above give an S-minimal realisation using the Laurent operator ; the same theory as that above could be given using this operator in place of $U_r$.)

Note finally that we have expressed the realisations above in terms of the right shift operator $U_r$ ; we could have used instead the left shift $U_\ell = U_r^*$ , since

$$\langle c, e^{U_r t} b\rangle = \langle e^{U_\ell t} c, b\rangle \quad .$$

(4) Root Locus Theory

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + kbu \\ y &= cx \end{aligned} \tag{4.1}$$

where A,b,c are bounded operators belonging , respectively , to the spaces $\mathcal{L}(H,H)$ , $\mathcal{L}(R,H)$ , $\mathcal{L}(H,R)$ (we shall consider the single-input single output case for simplicity ). Suppose that

$$\sigma(A) = \bigcup_{i=1}^{n} \sigma_i \tag{4.2}$$

where each $\sigma_i$ is a spectral set of A . Then we can write

$$H = \bigoplus_{i=1}^{n} H_{\sigma_i} \tag{4.3a}$$

$$A_{\sigma_i} = A\big|_{H_{\sigma_i}} \tag{4.3b}$$

and

$$bu = (b_1 u, \ldots, b_n u) \text{ , } cx = c_1 x_1 + \ldots + c_n x_n \text{ ,}$$

where

$$x_i \in H_{\sigma_i} \quad , \quad b_i u \in H_{\sigma_i} \quad \text{and} \quad c_i = c\big|_{H_{\sigma_i}} \quad .$$

Hence ,

$$\begin{aligned} \dot{x}_i &= A_i x_i + k b_i u \\ y &= \Sigma c_i x_i \end{aligned} \tag{4.4}$$

where we have abbreviated $A_{\sigma_i}$ to $A_i$ .

Taking Laplace transforms in the usual way , we obtain

$$sX_i(s) = A_i X_i(s) + k b_i U(s)$$

$$Y(s) = \Sigma c_i X_i(s) \quad ,$$

where

$$X_i(s) = \mathcal{L}(x_i(t)), U(s) = \mathcal{L}(u(t)) \ , \ Y(s) = \mathcal{L}(y(t)) \ .$$

If the control u is just the error from feedback of y , then for any input v to the overall system we have

$$(1+k \sum_{i=1}^{n} c_i R(s;A_i) b_i) Y(s) = k( \sum_{i=1}^{n} c_i R(s;A_i) b_i) V(s) \ , \tag{4.5}$$

provided $s \notin \sigma(A_i) = \sigma_i$ , where $R(s;A_i) \triangleq (sI-A_i)^{-1}$ is the resolvent of $A_i$ . Hence , if

$$s \in \{ \lambda \in \mathbb{C} : 1+k \sum_{i=1}^{n} c_i R(\lambda;A_i) b_i = 0 \}$$

then it follows that

$$\begin{aligned} Y(s) &= \left\{ \frac{k( \sum_{i=1}^{n} c_i R(s;A_i) b_i)}{1 + k \sum_{i=1}^{n} c_i R(s;A_i) b_i} \right\} V(s) \\ &= \tilde{G}(s) V(s) \end{aligned}$$

where

$$\tilde{G}(s) \triangleq \frac{k( \sum_{i=1}^{n} c_i R(s;A_i) b_i}{1 + k \sum_{i=1}^{n} c_i R(s;A_i) b_i} \tag{4.6}$$

is called the closed-loop transfer function . If we wish to

emphasize the dependence of $\tilde{G}$ on k we shall write $\tilde{G}(s;k)$ . The root locus L of the system is defined by

$$L = \bigcup_{k \geq 0} \{s \in \mathbb{C}: \tilde{G}(s;k) \text{ is not analytic at } s\} \qquad (4.7)$$

Remark 4.1 In the case of finite dimensional realisations , the representation (4.1-4.3) of a system corresponds to the Jordan decomposition of A . In fact , the spectral sets of the matrix A are just the (isolated) eigenvalues and the subspaces $H_{\sigma_i}$ correspond to the generalised eigenspaces . The decomposition of the open loop transfer function

$$\tilde{G}_0(s) = \sum_{i=1}^{n} c_i R(s;A_i) b_i$$

is then just the partial fraction expansion .

In order to demonstrate that the classical properties of the root locus can be generalised to the present situation , it is important to define transmission zeros for our systems . Pohjolainen (1981) has defined such zeros,in the case where A has compact resolvent (and so has isolated eigenvalues ) , as the finite eigenvalues of A+kBC as $k \to \infty$. It is then easy to show that this is equivalent to the condition

$$\lambda \notin \sigma(A) \text{ and } \det[C(A-\lambda I)^{-1}B] = 0 \qquad (4.8)$$

However , these systems are spectral minimal , as shown by Brockett and Fuhrmann (1976) and the definition of transmission zeros in terms of $\sigma(A+kBC)$ is then reasonable .

In the case where the original system (A,b,c) and the transfer function are not spectral minimal , it is necessary to define the transmission zeros directly by (4.8) ; ie. for single input single output systems by

$$\lambda \notin \sigma(A) \quad \text{and} \quad c(A-\lambda I)^{-1}b = 0 \qquad (4.9)$$

It then follows directly from (4.6) that the finite singularities of $\tilde{G}(s;k)$ tend to the transmission zeros as $k \to \infty$ . However ,even

the definition (4.9) is not quite correct since we must not only include the zeros of $c(A- I)^{-1}b$ but also any 'cut' in the plane which is necessary to make $c(A- I)^{-1}b$ single valued . Hence , by comparison with (3.2) we introduce

Definition 4.2 The transmission zero set $\zeta$ for the system $(A,b,c)$ is defined by

$$\zeta(\tilde{G}_0) = \{ s \in \mathbb{C} : s \notin \sigma(A) \ , \ \tilde{G}_0^{-1} \text{ is not analytic at } s \} \tag{4.10}$$

where $\tilde{G}_0(s) = c(A-sI)^{-1}b$ , the open loop transfer function .

Just as in (3.2) , this does not uniquely specify $\sigma(\tilde{G}_0)$ . However , it is to be understood in the same sense as (3.2) , i.e. $\sigma(\tilde{G}_0)$ consists of the solutions of (4.9) together with any cuts which are necessary for the single-valuedness of $\tilde{G}_0$ .(Note that this ambiguity can be removed by interpreting the root locus on an appropriate Riemann surface.)

For reasons which will become clear shortly , we also introduce

Definition 4.3 A connected component of $\sigma(\tilde{G}_0)$ is called a generalised open-loop pole . Similarly , a component of $\sigma(\tilde{G}(s;k))$ (where G is defined by (4.6)) is called a generalised closed-loop pole . A connected component of $\zeta(\tilde{G}_0)$ is called a generalised open-loop zero .

The transfer function $\tilde{G}(s)$ of a finite dimensional system is , of course , rational and can be factorised in the form

$$\tilde{G}(s) = \frac{\prod_{i=1}^{m} (s-z_i)}{\prod_{i=1}^{n} (s-p_i)}$$

where each zero $z_i$ and pole $p_i$ represents a single connected component of $\sigma(\tilde{G})$ . We would like to obtain a similar factorisation

when $\tilde{G}(s)$ is irrational , of the form

$$\tilde{G}(s) = \frac{\prod_{i=1}^{m} \Psi_i(s)}{\prod_{i=1}^{n} \Phi_i(s)} \qquad (4.11)$$

where $\Psi_i$ ( $\Phi_i$ ) corresponds to a generalised zero (pole ) . We shall prove this in the case when $\tilde{G}$ has a finite number of algebraic singular points . Let us assume for simplicity that in the spectral factorisation of (4.2)-(4.4) each operator $A_{\sigma_i}$ is such that $\sigma_i$ ($= \sigma(A_{\sigma_i})$ ) contains a single connected component of $\sigma(\tilde{G}(s))$ . Then such a spectral representation of the system (4.1) will be called <u>simple</u> .

<u>Theorem 4.4</u> Suppose that all the branch points for the system (4.1) are algebraic . Then we may write the open loop transfer function $\tilde{G}_0(s)$ in the form

$$\tilde{G}_0(s) = \left( \prod_{i=1}^{m_1} \Psi_i(s) \Big/ \prod_{i=1}^{n_1} \Phi_i(s) \right) R(s) \qquad (4.12a)$$

where R is a rational function with $m_2$ zeros and $n_2$ poles , $m_1+m_2 < n_1+n_2$ and $\Psi_i$ ($\Phi_i$) correspond to generalised zeros (poles). Moreover , we have

$$\Psi_i(s)-s \ , \ \frac{s}{\Phi_i(s)} - 1 \quad \text{are analytic at } \infty \text{ and vanish there .} \qquad (4.12b)$$

<u>Remark</u> The term R(s) corresponds to those operators $A_{\sigma_i}$ for which $\sigma_i$ contains a single isolated pole and consequently which have a finite dimensional realisation . We shall therefore ignore the term R (ie. set R=1) since this result is well known in this case . Hence we shall prove the result with R=1, $m=m_1, n=n_1$ .

<u>Proof of theorem (4.4)</u> Since we are considering only components

of $\sigma(\tilde{G}_0(s))$ which are nontrivial (ie. which are not just isolated poles ) we can suppose that the singularities (which are assumed to be algebraic ) occur at isolated branch points . Now at such a point b it is well known that $\tilde{G}_0(s)$ must have branches of the form

$$F(\sqrt[n]{s-b}) \qquad (4.13)$$

for some n , where F is holomorphic in a neighbourhood of 0 except possibly at 0 where F can have at most a pole . (See , for example , aks and Zygmund ,1971) . We can assume without loss of generality that if 0 is a pole then it is simple . Consider , then , two cases :

Case 1 0 is a pole of F .

If $n > 1$ , then in order that $F(\sqrt[n]{s-b})$ be well-defined we must cut the plane from b to $\infty$ . Hence , in this case , since we know that $\tilde{G}_0$ is analytic at $\infty$ we must be able to find another term $F_1$ in $\tilde{G}_0$ so that

$$FF_1 = F_2(\sqrt[n]{s^n+a_{n-1}s^{n-1}+....+a_0})$$

for some new function $F_2$ , holomorphic except at 0 . This term will then correspond to the generalised pole of $\tilde{G}_0(s)$ containing b. Also , if $\tilde{G}_0$ contains a factor of the form $1/(\sqrt{s^4-a^4})$ , for example , then this will not represent a single component of $\sigma(\tilde{G}_0(s))$ since it can be factorised as $1/(\sqrt{s^2+a^2}\sqrt{s^2-a^2})$ . Hence , if we collect together the representations (4.13) for each branch point we must have a function of the form

$$P(s) = \Phi_1(s)....\Phi_n(s)$$

which satisfies (4.12b). Now write $Z(s) = \tilde{G}_0(s)P(s)$ . Then it is clear that Z(s) has branch points which satisfy

Case 2 0 is not a pole of F . The same argument as in case 1 now completes the proof . □

Corollary 4.5 Under the conditions of theorem 4.4 , if the system (4.1) has a simple spectral representation in the form (4.4) , we may write

$$\tilde{G}_0(s) = \sum_{i=1}^{n} \frac{h_i(s)}{\Phi_i(s)} \qquad (4.14)$$

where each $h_i(s)$ is holomorphic in $\mathbb{C}$ .

The expression (4.14) is a generalised partial fraction expansion . We can state an obvious conjecture :

Conjecture 4.6 A decomposition of the form (4.12a) exists even for transcendental singularities . □

Returning to the case of algebraic singularities , in view of corollary 4.5 and the importance of the partial fraction expansion of a rational function , it is of interest to determine when we can obtain a simple spectral representation of a given system . It turns out that if the system has a bounded realisation and only algebraic singularities , we can always do this . In order to prove this statement , we must first recall the theory of G-C Rota (1960) which says , essentially , that the left-shift operator (on a certain Hilbert space ) is a 'universal model ' for any bounded operator whose spectrum is contained in the unit disc . In fact , if A is a bounded operator with $\sup |\sigma(A)| \leq 1$ defined on a Hilbert space H , then we define the new Hilbert space $H^\infty \triangleq \Theta_{k=1}^{\infty} H$ with the inner product

$$\langle \underline{x}, \underline{y} \rangle = \sum_{k=1}^{\infty} \langle x_k, y_k \rangle_H \quad , \quad x_k, y_k \in H \qquad (4.15)$$

for all $\underline{x}, \underline{y} \in H^\infty$ for which the sums $\sum_{k=1}^{\infty} x_k^2$ , $\sum_{k=1}^{\infty} y_k^2$ exist .Here,

$$\underline{x} = (x_1, x_2, \ldots) \ , \ \underline{y} = (y_1, y_2, \ldots) \ .$$

Now define the left-shift $U_\ell$ on $H^\infty$ by

$$U_\ell(x_1,x_2,x_3,....) = (x_2,x_3,....) ,$$

and the operator $I_A : H \to H^\infty$ given by

$$I_A x = (x,Ax,A^2x,....) .$$

Then it follows that $I_A$ is invertible and

$$I_A^{-1} U_\ell I_A x = Ax ,$$

and so A is similar to $U_\ell$ .

Now consider again the general spectral decomposition of a system as in (4.1-4.3) , and suppose that a particular projected subsystem defined by $A_i, b_i, c_i$ as in (4.4) has a transfer function $\tilde{G}_i(s) = c_i R(s;A_i) b_i$ which has several singularities ( which are generalised poles ) contained in $\sigma(A_i)$ . (cf. fig. 4.1) . Assume without loss of generality that $\|A\| \leq 1$.

Consider the term $\Phi_1(s)$ (say) in (4.11) due to the generalised pole $p_1$ . Then we can draw a Jordan curve $\gamma$ in $\sigma(A_i)$ containing $p_1$ in the interior and with $p_2, p_3$ in the exterior of $\gamma$ , as in fig . 4.2 . Since $\gamma$ is a Jordan curve , the Riemann mapping theorem (Rudin , 19 ) states that there exists a conformal mapping f which takes $\gamma$ in the s-plane into the unit circle in the f(s)-plane. However , $f^{-1}$ is analytic and so we can define $A_{i,1} = f^{-1}(U_\ell)$ , where $U_\ell$ is the left shift operator in H . Then $A_{i,1}$ has spectrum which consists of $\gamma$ and its interior (by the spectral mapping theorem ) . However , $A_{i,1}$ has the model $U_\ell$ on $H^\infty$ , i.e.

$$A_{i,1} = I_{A_{i,1}}^{-1} U_\ell I_{A_{i,1}}$$

and we can realise $(\Phi_1(s))^{-1}$ on $H^\infty$ by the results of §3 in terms of the triple $(U_\ell, b, c)$ ,say; i.e. by the system

$$\dot{\xi} = U_\ell \xi + bv$$
$$y = c\xi$$

If we change variables to $\Xi = I_{A_{i,1}}^{-1} \xi$ , we obtain

$$\dot{\Xi} = I_{A_{i,1}}^{-1} U_\ell I_{A_{i,1}} \Xi + I_{A_{i,1}}^{-1} bv$$
$$y = cI_{A_{i,1}} \Xi$$

or

$$\dot{\Xi} = A_{i,1} \Xi + b'v \qquad (4.16)$$
$$y = c'\Xi$$

where

$$b' = I_{A_{i,1}}^{-1} b \quad , \quad c' = cI_{A_{i,1}} .$$

(4.16) is a system defined on H with transfer function $(\Phi_1(s))^{-1}$ whose system operator $A_{i,1}$ has spectrum $\gamma$ and its interior . The stated result now follows by induction .

We now return to the general structure of the root locus for a system with bounded realisation having algebraic singularities . It follows by an elementary spectral continuity argument that generalised poles remain generalised poles as k increases . (Of course , at some values of k a generalised pole consisting of , say , a pair of singularities joined by a cut may coalesce into a single ordinary pole .)

<u>Proposition 4.7</u> Suppose that the open loop transfer function $\tilde{G}_0(s)$ of the system (4.1) is written in the form (4.12a) with $m = m_1 + m_2$ and $n = n_1 + n_2$ , then we have

$$\tilde{G}_0(s) = k\left( \frac{a_{n-m}}{s^{n-m}} + \frac{a_{n-m+1}}{s^{n-m+1}} + \cdots \right)$$

where $a_{n-m} \neq 0$ , in a neighbourhood of $s = \infty$ .

<u>Proof</u> This follows easily from (4.12a) since each term $\Psi_i(s)$ , $\Phi_i(s)$ has an expansion $\sum_{i=1}^{\infty} \alpha_i s^i$ with $\alpha_1 \neq 0$ , by (4.12b). □

Corollary 4.8 If the feedback system (4.5) has m generalised zeros and n generalised poles , then m of the poles converge , as $k \to \infty$ to the generalised zeros and the remaining n-m generalised poles tend to $\infty$ with asymptotic directions given by the angles

$$\theta = \ell\pi/(n-m)$$

and which intersect the x-axis at $\sigma_0 = \left(\frac{a_{n-m+1}}{a_{n-m}}\right) \Big/ (n-m)$ . □

It is clear from the above results , therefore , that an infinite dimensional system which has a bounded realisation with only algebraic singularities has a root locus in which the generalised poles and zeros behave in essentially the same way as the corresponding root locus of a finite dimensional system . In the next section we shall give some examples to clarify these remarks . However , before presenting these examples , we shall first mention briefly the generalised pole assignment problem for our systems .

Consider again a system og the form (4.1) which is canonical (i.e. controllable and observable ) and suppose that the open loop transfer function $\tilde{G}_0(s)$ is of the form (4.12a) with only algebraic singularities . The generalised pole assignment problem for this system is to move the poles ( generalised or ordinary) with state feedback to any other finite set of generalised poles . The next result shows that we can solve this problem arbitrarily closely .

Theorem 4.9 The generalised pole assignment problem is approximately soluble for the system (4.1) in the sense that there exists a sequence of finite dimensional systems $S_{0i} = (A_{0i}, b_{0i}, c_{0i})$ approximating $\tilde{G}_0(s)$ , with transfer functions $\tilde{G}_{0i}(s)$ , $1 \leq i < \infty$ such that , if $\tilde{G}_d(s)$ is the desired closed loop transfer function of (4.1) , then we can assign the poles of $S_{0i}$ with state feedback

so that

$$\overline{\bigcup_{i=1}^{\infty} \sigma(\tilde{G}_i)} = \sigma(\tilde{G}_d) \quad .$$

i.e. the union of all the poles of the systems $S_i$ is dense in $\sigma(\tilde{G}_d)$ where $S_i$ is the closed loop system of $S_{0i}$ and $\tilde{G}_i$ is its transfer function .

Proof The proof of this result follows easily from the above remarks and Runge's theorem . Indeed , using Runge's theorem we can find a sequence of transfer functions $\tilde{G}_{0i}(s)$ of finite dimensional systems such that $\tilde{G}_{0i}(s) \to \tilde{G}_0(s)$ uniformly on compact subsets of $\mathbb{C} \setminus \sigma(\tilde{G}_0)$ . Similarly , we can find a sequence $\tilde{G}_i(s)$ such that $\tilde{G}_i(s) \to \tilde{G}_d(s)$ uniformly on compact subsets of $\mathbb{C} \setminus \sigma(\tilde{G}_d)$ . Without loss of generality we may assume that $\tilde{G}_{0i}$ and $\tilde{G}_i$ each have i poles (counted with multiplicity ) . However , the classical result now implies that the poles of $S_{0i}$ can be arbitrarily assigned . If we assign them precisely to $\sigma(\tilde{G}_i)$ then the result follows . □

(5) Examples

Example 5.1 Consider the transfer function

$$\tilde{G}_0(s) = \frac{s+1}{\sqrt{(s^2+1)}} \quad ,$$

with one zero at s=-1 and a generalised pole consisting of the branch points $s=\pm i$ joined by a cut . (This transfer function does not have a bounded (A,b,c) realisation , since $\tilde{G}_0(s) \to 1$ as $|s| \to \infty$. However , it is the simplest nontrivial case and will show very well some of the peculiarities of irrational transfer functions .)

The root locus is given by

$$1+k\frac{s+1}{\sqrt{(s^2+1)}} = 0$$

i.e.

$$s^2+1 = k^2(s+1)^2 \quad .$$

(Note that in squaring we have effectively lost the cut between +i and -i , but we must bear in mind that this cut is still present .) Hence ,

$$s^2 - \frac{2k^2}{1-k^2} s + 1 = 0$$

and so the root locus consists of the loci of the poles

$$s = \frac{k^2 \pm \sqrt{(2k^2-1)}}{1 - k^2} \quad .$$

Of course , when k=0 the locus starts at $s=\pm i$ and has complex roots until $k=1/\sqrt{2}$ , when the roots become real at the point s=1 . It is easy to see that for $k\in[0,1/\sqrt{2}]$ we obtain the locus of a semicircle . Now consider $k> 1/\sqrt{2}$ . If $k\in[1/\sqrt{2},1)$ then we obtain two branches ; one tends to s=0 as $k\to 1$ and one tends to $\infty$ as $k\to 1$ . When $k\in(1,\infty)$ we obtain branches which tend to s=-1 as $k\to\infty$ . The root locus is therefore as shown in fig.5.1(a),(b),(c). The poles $s=\pm i$ are shown at various positions on the root locus as $s_1, s_2$ connected by a cut (shown by a dotted line ). Note that the cut disappears when $k=1/\sqrt{2}$ and that when k=1 one pole ($s_2$) is at $\infty$ . The behaviour of the generalised pole as $k\to\infty$ with respect to the zero at s=-1 can be shown more easily on the Riemann sphere as in fig.5.2(a),(b) where the branch points and cuts are shown with a continuous line for various values of k . The point at infinity is marked $\infty$ .

Example 5.2 Consider the system with open loop transfer function

$$\tilde{G}_0(s) = \frac{(s+i)}{s\sqrt{(s^2 - 1)}}$$

i.e. $\tilde{G}_0$ has a zero at -i , a pole at 0 and a generalised pole ($\pm 1$). The root locus is shown in fig.5.3.

Note that the apparent splitting of the zero at s=0 is caused by squaring $\tilde{G}_0(s)$ to find the root locus . Alternatively , we can regard the pole at s=0 as a 'degenerate' generalised pole which

splits into two singularities joined by a cut for $k > 0$ and coalesces at $s=-i$ when $k = \infty$ .

Example 5.3 Consider finally the system

$$\tilde{G}_0(s) = \frac{\sqrt{\{(s+1)^2 + \frac{1}{4}\}}}{\sqrt{(s^2+4)}\sqrt{(s^2+1)}}$$

which has two generalised poles $(\pm i), (\pm 2i)$ and a generalised zero $(-1 \pm i/2)$ . The root locus is shown in fig.5.4.

As expected , one generalised pole diverges while the other $(\pm i)$ is attracted by the generalised zero .

(6) Conclusions

In this paper we have discussed the application of realisation theory to the root locus of infinite dimensional systems . In particular we have generalised the classical theory to the case of systems with bounded realisations whose transfer functions have a finite number of isolated algebraic singularities . A complete theory for such systems has been developed in terms of generalised poles and zeros which must be regarded as the appropriate counterparts of the classical poles and zeros . Indeed , we have seen that the latter behave in many ways as degenerate generalised singularities .

We leave two problems for further research : firstly , of course , we could use the theory developed here to generalise the classical compensator theory and secondly it is important to attempt to extend the theory to unbounded system operators A . The latter problem is complicated by the fact that the spectrum of A may now accumulate at infinity ; i.e if we compactify $\mathbb{C}$ to the Riemann sphere $S_\infty$ then any neighbourhood of $\infty$ contains infinitely many singularities . This suggests that we should study such systems on $S_\infty$ and regard $\infty$ as just another point ; we must then extend the theory of this paper to the case of transfer functions $\tilde{G}_0$ with accumulation points in $\sigma(\tilde{G}_0)$ .

(7) References

Balakrishnan,A.V. 1976 , Applied Functional Analysis (New York: Springer )

Banks,S.P. and Abbasi-Ghelmansarai,F.,1983, Int. J. Control, 37 , 235

Baras,J,S. and Brockett,R.W., 1975, Siam J. Control ,13,221

Curtain,R.F. and Pritchard,A.J. , 1978 , Infinite Dimensional Linear Systems Theory (Berlin:Springer)

Dunford,N. and Schwartz,J.T., 1959, Linear Operators (New York: Interscience)

Fuhrmann,P.A.,1974, Math. Systems Theory,8 , 132

Fuhrmann,P.A. and Brockett,1976, Siam J. Control,14 ,107

Lee,E.B. and Markus,L. 1967 , Foundations of Optimal Control Theory , (New York:Wiley)

Owens,D.H.,1978 , Feedback and Multivariable Systems , (Stevenage: Peter Perigrinus)

Pohljanen,S. 1981, Int. J. Control ,33,199;1982,IEEE Trans, Aut. Contr.,AC-27,17

Postlethwaite,I. and MacFarlane,A.G.J.,1979, A Complex Variable Approach to the Analysis of Linear Multivariable Feedback Systems (Berlin:Springer)

Rosenbrock,H.H, 1974, Computer-Aided Control System Design , (London:Academic Press)

Rota,G-C, 1960, Comm. on Pure and Applied Maths.,13,469

Rudin,W. 1966 , Real and Complex Analysis (New York:McGraw-Hill)

Saks,S. and Zygmund,A. 1971, Analytic Functions (Warsaw:Elsevier)

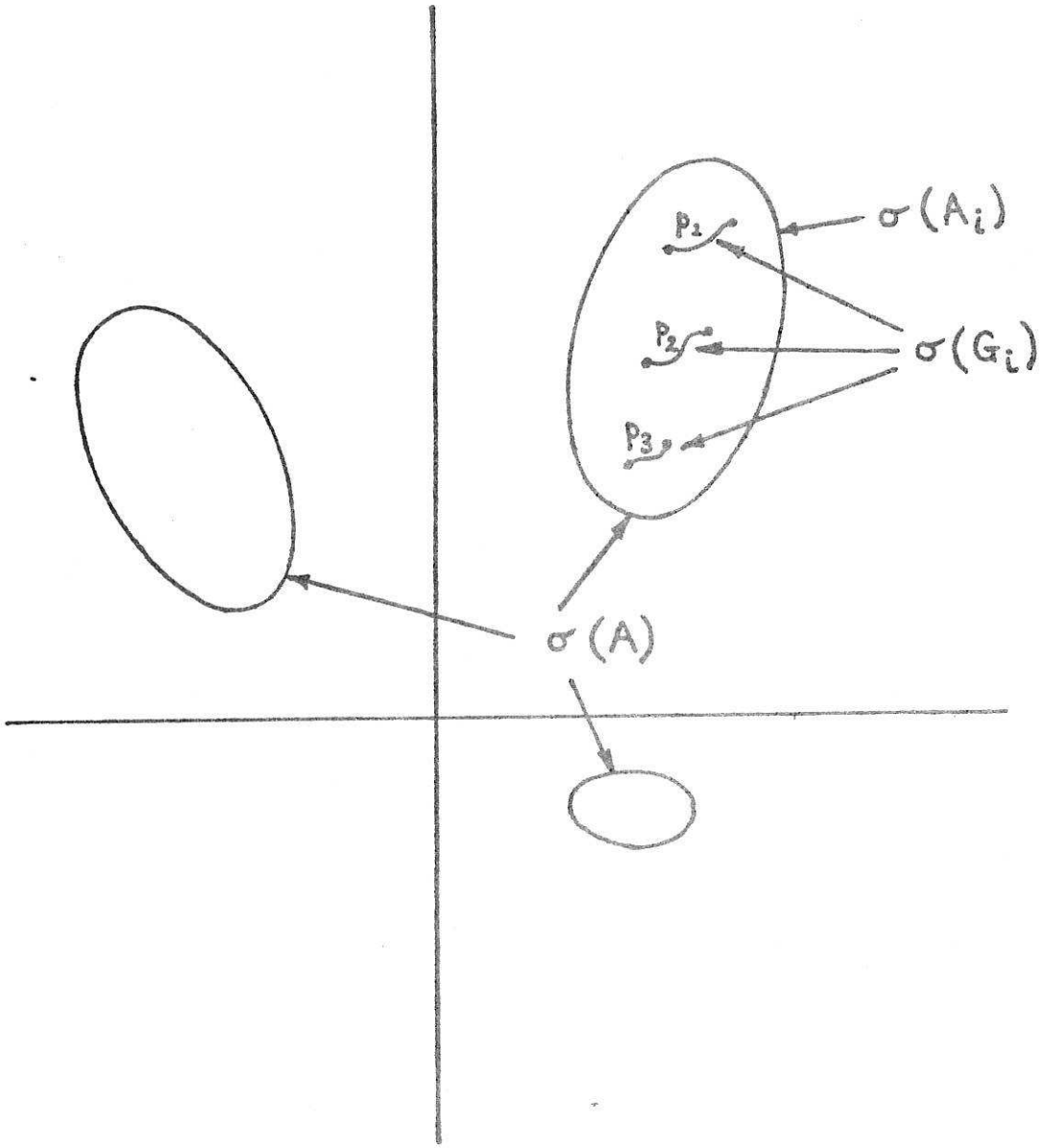



fig. 4.1.

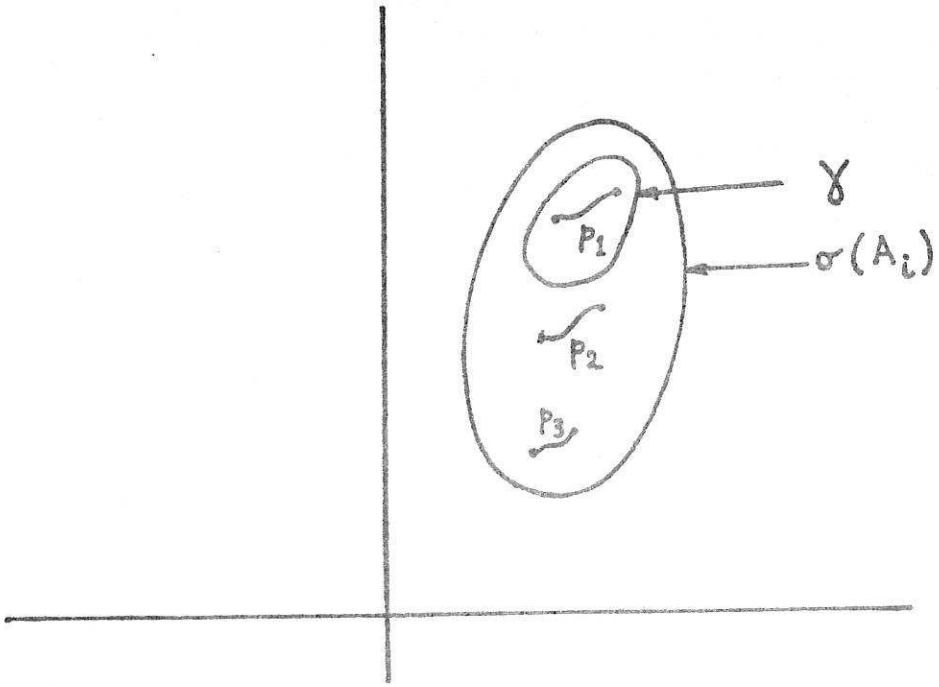
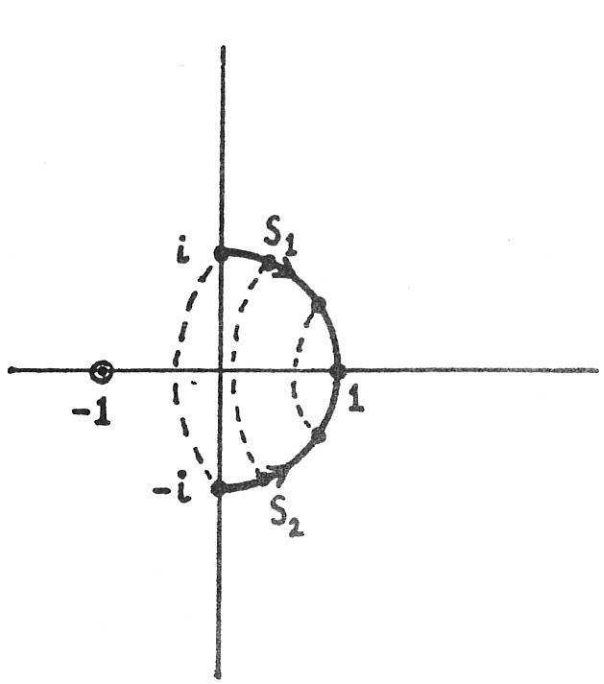
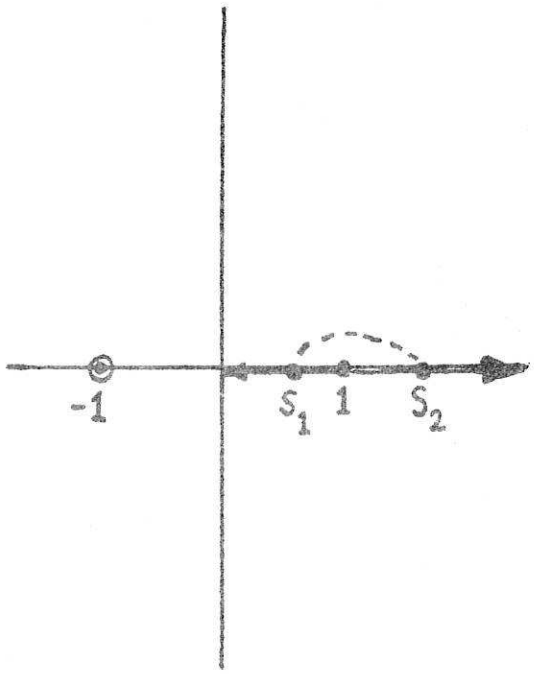


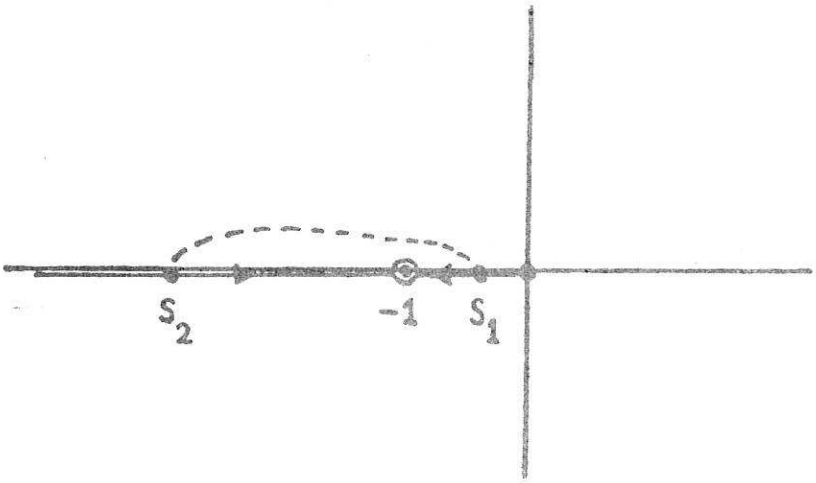

fig. 4.2.




(a). $0 < k < 1/\sqrt{2}$

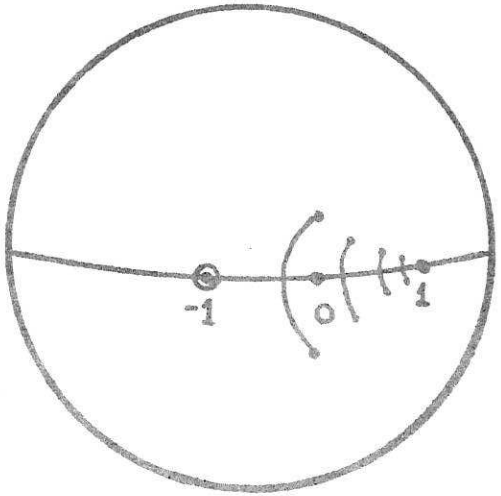



(b). $1/\sqrt{2} \leq k < 1$

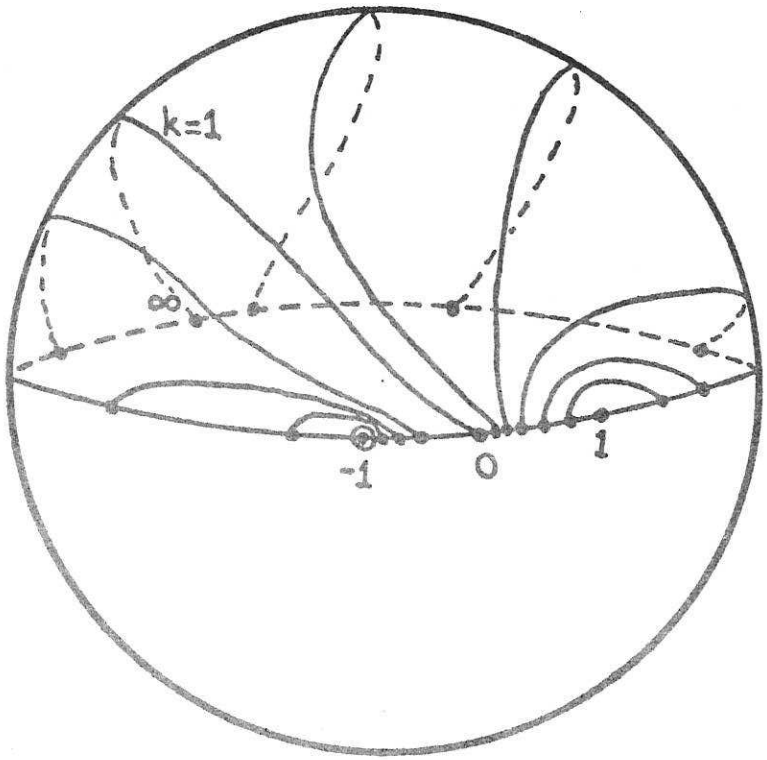



(c). $k > 1$

fig 5.1




(a). $k < 1/\sqrt{2}$




(b). $k \geq 1/\sqrt{2}$

fig 5.2

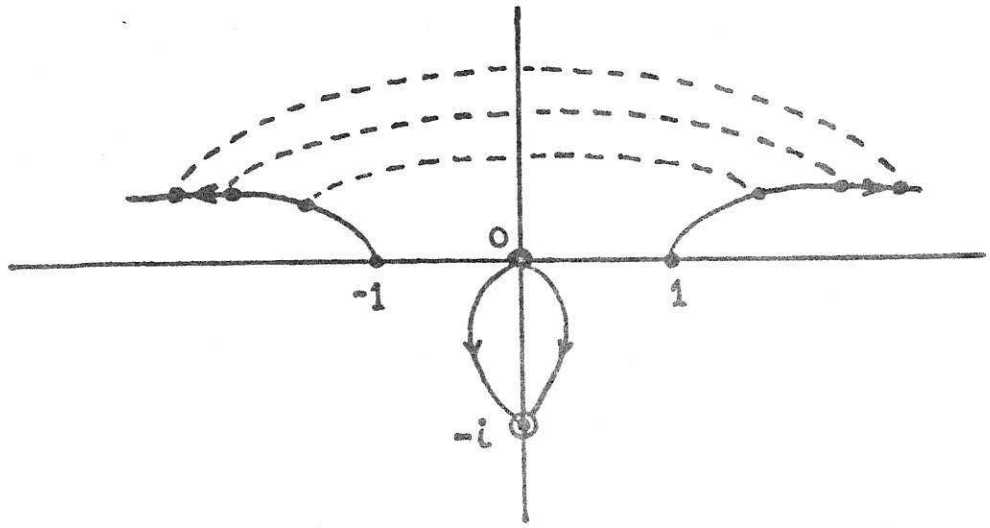



fig. 5.3

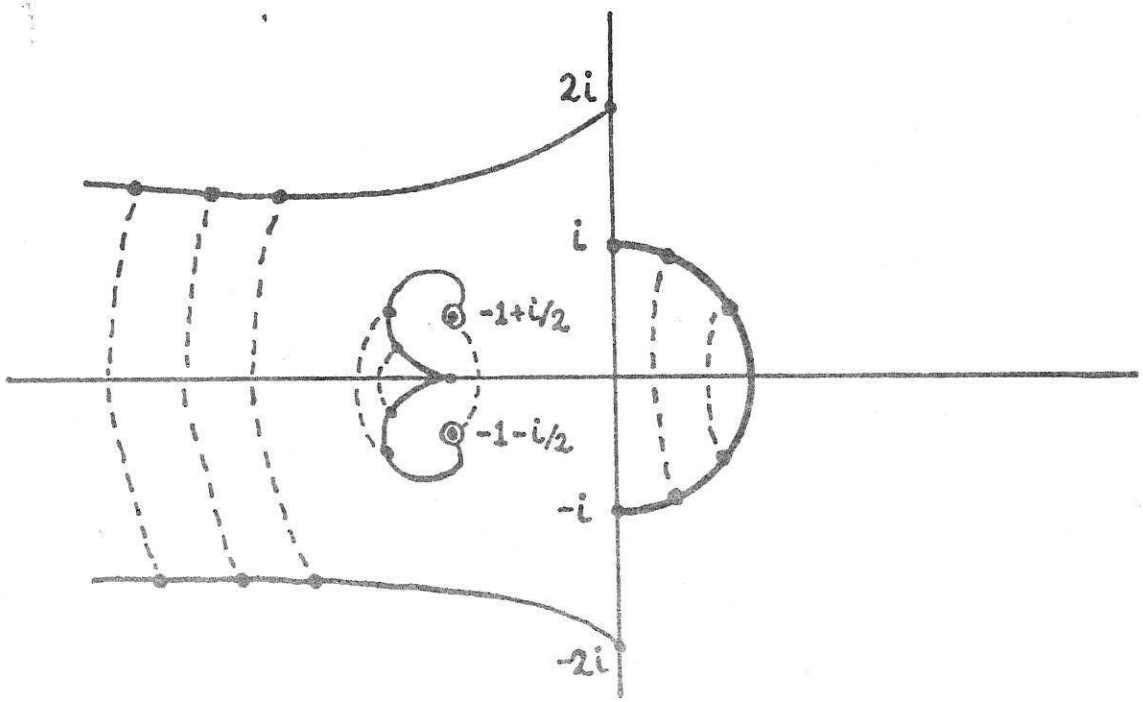



fig. 5.4

Legends for the figures

fig.4.1 A typical spectral decomposition into spectral sets

fig.4.2 Surrounding a generalised pole by a Jordan curve.

fig.5.1 Root locus for example 5.1

fig.5.2 Root locus for example 5.1 shown on $S_\infty$ .

fig.5.3 Root locus for example 5.2.

fig.5.4 Root locus for example 5.3.