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Controllability and Optimal Control of Partial Differential Equations on Compact Manifolds

by

S.P.Banks

Department of Control Engineering
University of Sheffield
Mappin Street
Sheffield S13JD

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Abstract

The controllability and optimal control problems for distributed systems defined on compact manifolds is considered and particular emphasis is placed on compact orientable surfaces .

(1) Introduction

The state-space theory of the control of distributed parameter systems defined in Euclidean spaces (for example , the heat flow problem on a one-dimensional bar , wave motion in three dimensional space etc.) is now well known and many papers and texts have appeared in the literature (see , for example , Lions (1971), Balakrishnan (1976) , Curtain and Pritchard (1978) and Banks (1983)). The freqency domain methods for such systems are also currently under investigation (Banks and Abbasi-Ghelmansarai (1983)) and we are soon likely to have a fairly complete (linear) theory for systems of this type .

Problems which do not seem to have been considered before are those relating to partial differential equations defined on compact manifolds (for example , heat flow on a spherical or toroidal surface , and , perhaps more fancifully , weather patterns on the surface of the Earth). We shall cosider , in this paper , the general control problem for such systems via the thoery of vector bundles on compact manifolds , and generalise controllability theory and the linear quadratic regulator problem with particular reference to compact orientable surfaces of genus g .

In sections 2 and 3 we shall present an introduction to vector bundle theory and partial differential operators on manifolds , for readers who may not be familiar with these concepts . A more extensive treatment may be found in Wells (1980). In section 4 we consider parabolic evolution equations on a compact manifold and in sections 5 and 6 the general controllability and optimal control problems are discussed . Finally in section 7 some aspects of the theory are exemplified for the sphere and torus .

(2) Vector Bundles on Compact Manifolds

In this section we shall briefly review the theory of vector bundles on a compact differentiable manifold X which will be required in the sequel . The differentiable functions on X will be denoted by $\mathcal{E}(X)$, and K will represent $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1 A differentiable K-vector bundle E of rank r over a differentiable manifold X is a differentiable manifold E together with a differentible map $\pi: E \to X$ such that

(i) $E_x \overset{\Delta}{=} \pi^{-1}(x)$, $x \in X$ is a K-vector space of dimension r

(ii) $\forall\, x \in X$, $\exists$ a neighbourhood U_x of x and a homeomorphism

$$h: \pi^{-1}(U_x) \to U_x x K^r$$

for which

$$h(E_x) \subseteq \{x\} x K^r ,$$

and h^x = $p o h: E_x \to K^r$ is a vector space isomorphism , where p is the projection on K^r. (h^x, U_x) is called a local trivialisation.

For two local trivialisations (h_α, U_α) , (h_β, U_β) the maps $g_{\alpha\beta}: U_\alpha \cap U_\beta \to GL(r,K)$ given by

$$g_{\alpha\beta}(x) = h_\alpha^x o (h_\beta^x)^{-1}: K^r \to K^r$$

are called the associated transition functions .

Conversely , if differentiable transition functions $g_{\alpha\beta}$ are given then we can construct a vector bundle $\pi: E \to X$ by defining

$$\tilde{E} = \bigcup_\alpha U_\alpha x K^r \qquad \text{(disjoint union)}$$

where (U_α) is an open covering of X , and then setting

$$E = \tilde{E}/\sim ,$$

where $\sim$ is the relation

$$(x,v) \sim (y,w)$$

iff

$$y=x \qquad \text{and} \qquad w=g_{\alpha\beta}(x)v \ .$$

We can use this construction to define the tensor product of vector bundles $\pi_X:E \to X$, $\pi_Y:F \to Y$ (which we shall need for the generalisation of Schwartz' kernel theorem later). In fact, if

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r,K) \ , \ h_{ab}:V_a \cap V_b \to GL(s,K)$$

are the associated transition functions then we define the functions

$$f_{\alpha\beta,ab}:(U_\alpha \cap U_\beta) x (V_a \cap V_b) \to GL(rs,K)$$

by

$$f_{\alpha\beta,ab} = g_{\alpha\beta} \otimes h_{ab} \ .$$

These can then be verified to be the transition functions for a vector bundle $\pi_X \otimes \pi_Y : E \otimes F \to XxY$ with fibre $E_x \otimes F_y$ over (x,y) .

The <u>tangent bundle</u> over a differentiable manifold X of dimension n is the bundle

$$\pi:T(X) = \bigcup_{x \varepsilon X} T_x(X) \to X$$

where $T_x(X)$ is the tangent space to X at x . In terms of local coordinates , $T_x(X)$ has basis $\{\partial/\partial x_1, \ldots\ldots, \partial/\partial x_n\}$. The bundle

$$\pi:T^*(X) = \bigcup_{x \varepsilon X} T^*_x(X) \to X \ ,$$

where $T^*_x(X)$ is the dual space of $T_x(X)$, is called the <u>cotangent bundle</u> and has local basis $\{dx_1, \ldots\ldots, dx_n\}$ dual to the basis of $T_x(X)$.

If $E_x \cong \mathbb{R}$ for each $x \varepsilon X$ then E is called a <u>real line bundle</u> . More generally , if

$$T_s^r(x) = \underbrace{T_x(X)\otimes\ldots\otimes T_x(X)}_{r \text{ terms}}\underbrace{\otimes T_x^*(X)\otimes\ldots\otimes T_x^*(X)}_{s \text{ terms}}$$

then the bundle with fibre $T_s^r(x)$ is the <u>bundle of tensors of type (r,s)</u> on X (cf. Kobayashi and Nomizu, 1963) . A <u>Riemannian metric</u> G on a compact manifold X is a tensor field (i.e. a section of the tensor bundle) of type (0,2) . Locally ,

$$G = \Sigma g_{ij} dx_i \otimes dx_j \ .$$

Similarly we define the <u>exterior product bundle</u> $\Lambda^p T^*(X)$ of order p with fibre

$$\Lambda^p T_x^*(X) \ .$$

<u>Definition 2.2</u> An <u>Hermitian vector bundle</u> E is a vector bundle with a C^∞ assignment of an Hermitian inner product $< \ , \ >_x$ to each fibre E_x .

<u>Definition 2.3</u> A section of a (differentiable) vector bundle $\pi: E \to X$ is a differentiable map $s: X \to E$ such that $\pi o s = 1_X$.

The space of sections of E is denoted by $\mathcal{E}(X,E)$, while the space of differentiable functions on the set X is denoted $\mathcal{E}(X)$. Note that if U is a coordinate neighbourhood we have , locally ,

$$\mathcal{E}(U,E) \cong [\mathcal{E}(U)]^p$$

where p is the rank of E .

If E is an Hermitian vector bundle with inner product $<.,.>_E$, then we define an inner product on (X,E) by

$$<\xi,\eta>_E = \int_X <\xi(x),\eta(x)>_E d\mu \quad , \qquad \xi,\eta \in \mathcal{E}(X,E)$$

where $d\mu$ is a volume measure on X given locally by

$$d\mu = \rho(x) dx_1 \ldots . dx_n \quad ,$$

where $\rho(x)$ may be taken as $|\det g_{ij}(x)|^{\frac{1}{2}}$ in the case of a

Riemannian manifold . If X is oriented and of dimension n then we can define a volume element by

$$v = *(1) \varepsilon\ \mathcal{E}(X, \Lambda^n T^*(X))$$

where $*$ is the usual Hodge star operator . (Note that

$$\mathcal{E}(X, \Lambda^0 T^*(X)) \cong \mathcal{E}(X, \mathbb{R}) \ ,$$

i.e. the real line bundle .)

If we consider the case of real bundles , then we can define a particular Hermitian inner product on $\Lambda^p T^*_x(X)$ by

$$\phi \wedge * \psi = \langle \phi, \psi \rangle_x v$$

and hence an inner product on $\mathcal{E}(X, \Lambda^p T^*(X))$ by

$$\langle \phi, \psi \rangle = \int_X \phi \wedge * \psi \tag{2.1}$$

Then if d is the exterior differential operator on $\mathcal{E}(X, \Lambda^p T^*_x(X))$ we define d^* to be the dual of d with respect to the inner product (2.1) and we let

$$\Delta = dd^* + d^* d$$

be the Laplacian on $\mathcal{E}(X, \Lambda^p T^*(X))$.

Note that

$$d^* = (-1)^{n+np+1} * d * \ ,$$

where n is the dimension of X .

We now finally introduce the Sobolev spaces of sections $\mathcal{E}(X,E)$ of a vector bundle E on a compact differentiable manifold X . Let $\{U_\ell, \phi_\ell\}_{1 \leq \ell \leq \alpha}$ be a finite open cover of X such that the maps ϕ_ℓ are local trivialisations of E . Then we have the diagram

$$\begin{array}{ccc} E|_{U_\ell} & \xrightarrow{\phi_\ell} & \tilde{U}_\ell \times C^k \\ \downarrow & & \downarrow \\ U_\ell & \xrightarrow{\tilde{\phi}_\ell} & \tilde{U}_\ell \end{array}$$

where $\tilde{\phi}_\ell$ is a local coordinate system on U . Hence we have the diagram of sections

$$\begin{array}{ccc} E|_{U_\ell} & \xrightarrow{\phi_\ell} & \widetilde{U}_\ell \times \mathbb{C}^k \\ s_\ell \uparrow & & \uparrow \widetilde{s}_\ell \\ U_\ell & \xrightarrow{\widetilde{\phi}_\ell} & \widetilde{U}_\ell \end{array}$$

and we write $\widetilde{s}_\ell = \phi_\ell^*(s_\ell)$. However , $\mathcal{E}(\widetilde{U}_\ell \times \mathbb{C}^k) \cong [\mathcal{E}(\widetilde{U}_\ell)]^k$ ($\mathbb{C}^k$-valued differential functions on $\widetilde{U}_\ell$) and so

$$\phi_\ell^* : \mathcal{E}(U_\ell, E) \longrightarrow [\mathcal{E}(\widetilde{U}_\ell)]^k \quad .$$

If $\{\rho_\ell\}_{1 \leq \ell \leq \alpha}$ is a partition of unity subordinate to $\{U_\ell\}$ we define

$$\| \xi \|_{s,E} = \sum_\ell \| \phi_\ell^* \rho_\ell \xi \|_{s,\mathbb{R}^n}$$

where $\| . \|_{s,\mathbb{R}^n}$ is the usual Sobolev norm for differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{C}^k$, i.e.

$$\| f \|^2_{s,\mathbb{R}^n} = \int |\hat{f}(y)|^2 (1+|y|^2)^s dy$$

where $\hat{\cdot}$ is the Fourier transform .

$H^s(X,E)$ will denote the completion of $\mathcal{E}(X,E)$ in the $\| . \|_{s,\mathbb{R}^n}$ norm .

(3) Partial Differential Operators and the Generation of Semigroups on Compact Manifolds

Let E and F be Hermitian vector bundles over a compact manifold X , and let the $\mathbb{C}$-linear map

$$L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,F)$$

be a <u>differential operator of order k</u> from the space of differentiable sections of E to that of F . Then in terms of local trivialisations there is a linear partial differential operator $\tilde{L}$ such that

$$\tilde{L}(f)_i = \sum_{\substack{j=1 \\ |\alpha| \leq k}}^{p} a_\alpha^{ij} D^\alpha f_j \ , \quad 1 \leq i \leq q \ ,$$

where $f = (f_1, \ldots, f_p) \in [\mathcal{E}(U)]^p$. This means that the diagram

$$\begin{array}{ccc} \mathcal{E}(X,E)|_U & \xrightarrow{L} & \mathcal{E}(X,F)|_U \\ \downarrow & & \downarrow \\ [\mathcal{E}(U)]^p & \xrightarrow{\tilde{L}} & [\mathcal{E}(U)]^q \end{array}$$

commutes . Note that if $L \in \mathrm{Diff}_k(E,F)$ (the set of all differential operators from E to F) , then L has a continuous extension

$$\bar{L} : H^s(E) \longrightarrow H^{s-k}(F) \quad .$$

If E = F we write $\mathrm{Diff}_k(E)$.

We now recall the definition of the <u>symbol</u> of a differential operator . If $T^*(X)$ is the real cotangent bundle of X , then we let $T'(X)$ denote $T^*(X)$ without the zero section . Then we define the pullbacks of E and F over $T'(X)$, denoted respectively by $\pi^* E$ and $\pi^* F$, to be the bundles over $T'(X)$ such that the following diagrams commute :

$$\begin{array}{ccc} \pi^* E & \longrightarrow & E \\ {\scriptstyle \pi^E}\downarrow & & \downarrow{\scriptstyle \pi_E} \\ T'(X) & \xrightarrow{\pi} & X \end{array} \qquad\qquad \begin{array}{ccc} \pi^* F & \longrightarrow & F \\ {\scriptstyle \pi^F}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ T'(X) & \xrightarrow{\pi} & X \end{array}$$

If $L \in \mathrm{Diff}_k(E,F)$, then $\sigma_k(L) \in \mathrm{Hom}(\pi^* E, \pi^* F)$ is defined by

$$\sigma_k(L)(x,v)e = L\left(\frac{i^k}{k!} (g-g(x))^k f \right)(x)$$

where $g \in \mathcal{E}(X)$ and $f \in \mathcal{E}(X,E)$ are chosen so that

$$dg_x = v \quad , \; f(x) = e \; ,$$

and $e \in E_x$, $(x,v) \in T'(X)$ (i.e. v is a cotangent vector at x) .

(Note that in this definition , an element $f \in \mathrm{Hom}(V,W)$ for two vector bundles V,W over X is an $\mathcal{E}$ -morphism $f:V \to W$ which preserves fibres and is linear on each fibre ; i.e. the diagram

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "\pi_V"'] & & W \arrow[dl, "\pi_W"] \\ & X & \end{tikzcd}$$

commutes . Hence , for each $x \in X$ we can associate a linear mapping

$$f_x : V_x \longrightarrow W_x \quad ,$$

i.e. we can regard f as a map from X to $\bigcup_{x \in X} \mathcal{L}(V_x, W_x)$. Now , the section of $\pi^* E \xrightarrow{\pi^E} T'(X)$ over (x,v) can be identified with E_x and so we may regard $\sigma_k(L)$ as a map

$$\sigma_k(L) \; : \; T'(X) \longrightarrow \bigcup_{x \in X} \mathcal{L}(E_x, F_x)$$

and this is precisely what has been done above .)

It follows from the above definition that $\sigma_k(L)$ is homogeneous of degree k , i.e.

$$\sigma_k(L)(x,\rho v) = \rho^k \sigma_k(L)(x,v) \; , \qquad \forall \, (x,v) \in T'(X) \; , \; \rho > 0 \; .$$

Also , if we have

$$L = \sum_{|\nu| \leq k} A_\nu D^\nu \qquad , \; A_\nu \in \mathcal{E}(U, \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q))$$

locally on $U \subseteq X$, then it is easy to see that

$$\sigma_k(L)(x,v) = \sum_{|\nu| = k} A_\nu(x) \xi^\nu$$

where $v = \xi_1 dx_1 + \ldots + \xi_n dx_n \in T'(X)$. Hence , the symbol of a partial differential operator is just the principal polynomial form familiar in classical elliptic operator theory . As a simple generalisation of the well known criterion for ellipticity we introduce

Definition 3.1 If $L \in \mathrm{Diff}_k(E,F)$ we say that L is *elliptic* if the linear map

$$\sigma_k(L)(x,v) : E_x \longrightarrow F_x$$

is an isomorphism for each $(x,v) \in T'(X)$. (This implies that $p=q$ above .)

Recalling now the definition

$$\langle \xi,\eta\rangle_E = \int_X \langle \xi(x),\eta(x)\rangle_E \, d\mu \quad , \qquad \xi,\eta \in \mathcal{E}(X,E)$$

of the inner product on $H^0(X,E) = L^2(E)$, we define the dual L^* of $L \in \mathrm{Diff}_k(E,F)$ by

$$\langle L\xi,\eta\rangle_F = \langle \xi, L^*\eta\rangle_E \quad ,$$

and let

$$\mathcal{H}_L = \{\xi \in \mathcal{E}(X,E) : L\xi = 0\} \quad .$$

As usual , $\mathcal{H}_L^\perp$ denotes the orthogonal complement of $\mathcal{H}_L$ in $L^2(E)$. The following result can then be proved (cf. Wells,1980) :

Theorem 3.2 Let $L \in \mathrm{Diff}_k(E,F)$ be an elliptic operator and suppose that $\tau \in \mathcal{H}_{L^*}^\perp \cap \mathcal{E}(X,F)$. Then there exists a unique $\xi \in \mathcal{E}(X,E)$ such that $\xi \in \mathcal{H}_L^\perp$ and $L\xi = \tau$. Moreover , if $L \in \mathrm{Diff}_k(E)$ is self-adjoint $(E=F)$, we can associate with L an operator

$$L_k : H^k(E) \cap \mathcal{H}_L^\perp \longrightarrow H^0(E) \cap \mathcal{H}_L^\perp$$

which is bijective . □

Note that $\mathcal{H}_L$ is finite dimensional . In the case of the classical theory of elliptic equations , of course , we usually have an elliptic differential operator L specified in an open submanifold Ω of R^n together with boundary conditions defined on $\partial\Omega$. Under certain conditions on these boundary values and on the nature of the (n-1) dimensional manifold $\partial\Omega$, one can then prove the existence of a unique solution of the equation

$$L\phi = \psi ,$$

for $\psi \varepsilon L^2(\Omega)$, say. Then, the subspace $\mathcal{H}_L$ introduced above is trivial. The reason that $\mathcal{H}_L \neq \{0\}$ on a compact manifold is precisely because we have no boundary on which to apply boundary conditions. Hence we have to factor out $\mathcal{H}_L$ to obtain uniqueness of solutions.

We now consider real vector bundles, so that k is even for an elliptic operator. In order to prove that L_k defines a semi-group, suppose that $\tau \varepsilon \mathcal{H}_L^\perp \cap \mathcal{E}(X,E)$ and let $\xi \varepsilon \mathcal{E}(X,E)$ be the unique solution which is shown to exist in theorem 3.2. Then let $\{U_\ell\}_{1 \leq \ell \leq \alpha}$ be an open covering of X and $\overline{\phi}_\ell : U_\ell \longrightarrow V_\ell \subseteq \mathbb{R}^n$ be local coordinates on U_ℓ. (This is possible for finite α since the manifold X is compact.) Moreover we can clearly choose each U_ℓ so that ∂U_ℓ is as smooth as desired. Then restricting ξ and τ to U_ℓ, we have

$$\tilde{L}\xi_\ell = \tau_\ell$$

where $\xi_\ell, \tau_\ell \varepsilon [\mathcal{E}(U_\ell)]^p \cong [\mathcal{E}(V_\ell)]^p$. Now let

$$g_j^\ell = \frac{\partial^j}{\partial\nu^j}\xi_\ell \;\varepsilon\; [\mathcal{E}(\partial V_\ell)]^p ,$$

where $\partial/\partial\nu$ is the normal derivative to ∂V_ℓ. Then ξ_ℓ is the unique solution to the Dirichlet problem

$$\begin{aligned} \tilde{L}\phi &= \tau_\ell \\ \frac{\partial^j \phi}{\partial\nu^j} &= g_j^\ell \end{aligned} \qquad (3.1)$$

for $\phi \varepsilon [\mathcal{E}(V_\ell)]^p$. However , we know from classical elliptic theory (Friedman , 1969) that this is equivalent to solving

$$\tilde{L}\phi' = \tau'_\ell \qquad\qquad (3.2)$$
$$\frac{\partial^j \phi}{\partial \nu^j} = 0$$

Also , if we define the operator A'_ℓ with domain $H^k(V_\ell) \cap H_0^{k/2}(V_\ell)$ by

$$A'_\ell \phi' = \tilde{L}\phi' ,$$

then we have that A'_ℓ is closed and the resolvent $(\lambda I - A'_\ell)^{-1} : L^2(V_\ell) \longrightarrow L^2(V_\ell)$ exists for all $\lambda \varepsilon \mathbb{C}$ in a sector

$$S_\ell = \{\lambda : \tfrac{1}{2}\pi < \arg(\lambda + \beta_\ell) < (3/2)\pi \quad , \text{ some real } \beta_\ell\}$$

and

$$\|(\lambda I - A'_\ell)^{-1}\|_{\mathcal{L}(L^2(V_\ell))} \leqslant \frac{C_\ell}{|\lambda| + 1} \qquad\qquad (3.3)$$

for some constant C_ℓ . Hence , a similar inequality holds for the operator A_ℓ defined by (3.1) with boundary conditions g_j^ℓ , and so we may regard (3.3) to hold also for A_ℓ . Note that the intersection of the sectors S_ℓ , $\cap_{\ell=1}^{\alpha} S$, contains a sector of the same type which we denote by S . (This is , of course , dependent on the compactness of X . It may be false for non-compact manifolds. It now follows easily that if A denotes the operator with domain $H^k(E) \cap \mathcal{H}_L^\perp$ defined by

$$A\xi = L_k \xi , \qquad \xi \varepsilon H^k(E) \cap \mathcal{H}_L^\perp$$

(there are no boundary conditions , of course) , then A is a sectorial operator with sector S . Moreover , if we let

$$^\perp H^k(E) \triangleq H^k(E) \cap \mathcal{H}_L^\perp$$

and in particular ,

$$^{\perp}L^2(E) \overset{\Delta}{=} H^0(E) \cap \mathcal{H}_L^{\perp}$$

then we have , for $\lambda \varepsilon S$,

$$\|(\lambda I - A)^{-1}\xi\|_{{}^{\perp}L^2(E)} = \sum_{\alpha=1}^{\ell} \|\phi_\alpha^* \rho_\alpha (\lambda I - A_\ell)^{-1}\xi_\ell\|_{L^2(U_\ell)}$$

$$= \sum_{\alpha=1}^{\ell} \|(\lambda I - A'_\ell)^{-1}\xi_\ell\|_{L^2(V_\ell)}$$

$$\leq \sum_{\alpha=1}^{\ell} \frac{C_\ell}{1+|\lambda|} \|\xi_\ell\|_{L^2(V_\ell)}$$

$$\leq \frac{C}{1+|\lambda|} \sum_{\alpha=1}^{\ell} \|\phi_\alpha^* \rho_\alpha \xi_\ell\|_{L^2(U_\ell)}$$

$$= \frac{C}{1+|\lambda|} \|\xi\|_{{}^{\perp}L^2(E)}$$

where $C = \max_\ell C_\ell$. Hence ,

$$\|(\lambda I - A)^{-1}\|_{{}^{\perp}L^2(E)} \leq \frac{C}{1+|\lambda|}$$

for $\lambda \varepsilon S$.

It follows , therefore , by classical semigroup theory that A generates an analytic semigroup on the Hilbert space ${}^{\perp}L^2(E)$. We shall denote this semigroup by $T(t;E)$ or simply by $T(t)$ if the vector bundle E from which T is defined is clear .

(4) Evolution Equations on Compact Manifolds

In this section we let

$$L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,E)$$

be an $\mathbb{R}$-linear (elliptic) differential operator of order 2k (which must be even) , and let

$$A : {}^{\perp}L^2(E) \longrightarrow {}^{\perp}L^2(E) \ , \ \mathcal{D}(A) = {}^{\perp}H^{2k}(E) \ .$$

be the sectorial operator associated with it in section 3 . Then an equation of the form

$$\frac{\partial \xi(t)}{\partial t} = A\xi(t) \ , \quad \xi(t)\epsilon{}^{\perp}L^2(E) \ , \ t\epsilon[0,T] \ , \quad \xi(0) \text{ given}$$

is called an evolution equation on ${}^{\perp}L^2(E)$. Since A generates a semigroup T(t) , classical semigroup theory implies that this equation has a unique solution

$$\xi(t) = T(t)\xi(0) \ , \ t > 0 \ , \quad \xi(0)\epsilon{}^{\perp}L^2(E) \ .$$

Consider now the inhomogeneous equation

$$\frac{\partial \xi(t)}{\partial t} = A\xi(t) + f(t) \ , \qquad \xi(0)\epsilon{}^{\perp}L^2(E) \tag{4.1}$$

where f is locally Holder continuous , i.e.

$$\|f(t) - f(s)\|_{{}^{\perp}L^2(E)} < h|t-s|^{\theta}$$

for some constants $h,\theta > 0$.

Then we have the following well known result (Henry,1981):

<u>Theorem 4.1</u> If A is a sectorial operator in ${}^{\perp}L^2(E)$ which generates the semigroup T(t) , $f:(0,T)\longrightarrow {}^{\perp}L^2(E)$ is locally Holder continuous and $\int_0^{\rho}\|f(t)\|dt < \infty$ for some $\rho > 0$, then there exists a unique (strong) solution ξ of the equation (4.1) given by the variation of constants formula

$$\xi(t) = T(t)\xi(0) + \int_0^t T(t-s)f(s)ds \quad . \ \square$$

<u>Corollary 4.2</u> If $f:(0,T) \longrightarrow {}^{\perp}L^2(E)$ satisfies

$$\| f_{\ell}(t) - f_{\ell}(s)\|_{L^2(V_{\ell})} < h_{\ell}| t-s|^{\theta_{\ell}}$$

for each $\ell\epsilon\{1,\ldots.,\alpha\}$, where $f_{\ell} = f\big|_{L^2(V_{\ell})}$ and V_{ℓ} is a local coordinate neighbourhood image in $\mathbb{R}^n$, then the conclusion of theorem 4.1 holds . $\square$

If $f:(0,T)\longrightarrow L^2(E)$ then we can write $f = (f_1,f_2)$, where

$$f_1:(0,T)\longrightarrow {}^{\perp}L^2(E)$$

$$f_2:(0,T)\longrightarrow L^2(E)\ominus{}^{\perp}L^2(E)\triangleq{}_{\perp}L^2(E)$$

and equation (4.1) can be extended to $L^2(E)$ by writing

$$\frac{\partial}{\partial t}\begin{pmatrix}\xi_1(t)\\ \xi_2(t)\end{pmatrix} = \begin{pmatrix}A & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1(t)\\ \xi_2(t)\end{pmatrix} + \begin{pmatrix}f_1(t)\\ f_2(t)\end{pmatrix}$$

where $(\xi_1,\xi_2) \in L^2(E) = {}_{\perp}L^2(E) + {}^{\perp}L^2(E)$. Hence , theorem 4.1 is again valid if $f:(0,T) \longrightarrow L^2(E)$ provided we extend the semigroup $T(t)$ from ${}^{\perp}L^2(E)$ to $L^2(E)$ by

$$T(t)\xi = (T(t)\xi_1, \xi_2)$$

where $\xi=(\xi_1,\xi_2)\varepsilon L^2(E)$. We shall continue to denote the extension of the semigroup $T(t)$ by the same symbol , since no confusion is likely . The trivial extension $\begin{pmatrix}A & 0\\ 0 & 0\end{pmatrix}$ of A to $L^2(E)$ will also be denoted by A .

In order to consider control which is applied on zero- or one-dimensional submanifolds of X , it is necessary to consider delta functions defined on X . For this reason we define the following distributions on X (note that these are section-valued distributions and $\mathcal{D}(X,E)$ denotes the sections with compact support):

(a) If $x \varepsilon X$, we define , in the usual way ,

$$<\delta_{x_0}, \phi> \;=\; \phi(x_0)$$

where $\phi\varepsilon\, \mathcal{D}(X,E)$

(b) If Y is a smooth submanifold of X of dimension less than that of X , then we define

$$<\delta_Y, \phi> \;=\; \phi|_Y \quad , \qquad \phi\varepsilon\mathcal{D}(X,E)$$

where the right hand side denotes the section ϕ restricted to the submanifold Y . Note that $\delta_Y \varepsilon\, \mathcal{L}(\mathcal{D}(X,E),\mathcal{D}(Y,E))$.

Then we have the following generalisation of the familiar result which states that the δ -function belongs to $H^{-s}(\mathbb{R}^n)$ for some s .

Lemma 4.3 If Y is a submanifold of dimension m of the n dimensional manifold X , then

$$\delta_Y \in H^{((m-n)/2)-\varepsilon}(E) \quad ,$$

for any $\varepsilon>0$.

Proof If $y \in Y$ then we can choose local coordinates $x_1,\ldots,x_n$ in a neighbourhood U of y so that Y is given by the equations

$$x_{m+1} = \ldots = x_n = 0 \ .$$

Define the Fourier transform $\hat{f}(\xi) = (2\pi)^{-n}\int e^{-i<x,\xi>}f(x)dx$, for $x,\xi \in R^n$, in the usual way . Then

$$\|\delta_{Y\cap U}\|^2_{\frac{m-n}{2}-\varepsilon,R^n} = (2\pi)^{-n}\int\left(\int \exp(-i\sum_{j=1}^{m} x_j\xi_j)dx_1\ldots dx_m\right)^2 .$$

$$(1+|\xi|^2)^{((m-n)/2)-\varepsilon}d\xi$$

$$= (2\pi)^{-n}\int(1+\xi_{m+1}^2+\ldots+\xi_n^2)^{((m-n)/2)-\varepsilon} \cdot \prod_{j=m+1}^{n} d\xi_j$$

$$< (2\pi)^{-n}\prod_{j=m+1}^{n}\int_{-\infty}^{\infty}\frac{1}{(1+\xi_j^2)^{\frac{1}{2}+\varepsilon/(n-m)}}d\xi_j$$

$$< \infty \quad ,$$

since $\int \exp(-i\sum_{j=1}^{m} x_j\xi_j)dx_1\ldots dx_m = \delta_{\xi_1,\ldots,\xi_m}$. The result now follows by using a partition of unity . □

We can now consider equations of the form

$$\frac{\partial\xi(t)}{\partial t} = A\xi(t) + Bu \ , \qquad (4.2)$$

where A is a sectorial operator with domain $\mathcal{D}(A) \subseteq L^2(E)$ and

$$B: U \longrightarrow W$$

for some Hilbert spaces U,W such that

(i) $W \supseteq \mathcal{R}(B) \supsetneq L^2(E)$

(ii) $B \in \mathcal{L}(U,W)$ (4.3a)

Theorem 5.1 The system (5.1) is approximately controllable on $[0,t_1]$ if and only if

$$B^*T^*(t)\xi = 0 \text{ for } t \in [0,t_1] \ , \xi \in L^2(E) \ .$$

implies $\xi=0$. (We have identified $L^2(E)$ with its dual.)

Since A has compact resolvent on $L^2(E)$ there are a countable number of eigenfunctions (or , more precisely , eigensections) $\phi_i \in L^2(E)$ of A corresponding to eigenvalues λ_i , each with finite multiplicity , and we may write

$$A\xi = \sum_{i=1}^{\infty} \lambda_i \sum_{j=1}^{n_i} <\xi, \phi_{ij}>\phi_{ij} \ .$$

Moreover , the semigroup generated by A is given by

$$T(t)\xi = \sum_{i=1}^{\infty} e^{\lambda_i t} \sum_{j=1}^{n_i} <\xi, \phi_{ij}>\phi_{ij} \ ,$$

as shown in Dunford and Schwartz (1963) .

Consider now the system

$$\frac{d\,\xi(t)}{dt} = A\,\xi(t) + \delta_Y u(t) \ , \qquad (5.2)$$

where A is a self-adjoint operator defined in $L^2(E)$, $Y \subseteq X$ is a differentiable submanifold of X of dimension m and $u(t) \in \mathbb{R}$. This system is of the form (4.2) where

$$B : R \longrightarrow H^s(E) \ , \text{ where } s = \frac{m-n}{2} - \varepsilon \ .$$

To find the dual of B note that

$$<\eta, \delta_Y u>_{H^{-s}, H^s} = \sum_{\alpha} \int (\eta|_Y . u)\phi_\alpha|_Y . \rho_\alpha|_Y d\phi^\alpha_{m+1} \ldots d\phi^\alpha_n \ ,$$

where we have chosen local coordinates ϕ^α in each neighbourhood U_α of a covering of Y together with a partition of unity $\{\rho_\alpha\}$, so that locally Y is given by $\phi^\alpha_1 = \phi^\alpha_2 = \ldots\ldots = \phi^\alpha_m = 0$ in each neighbourhood . Hence ,

$$<\eta, \delta_Y u>_{H^{-s}, H^s} = u \int \eta|_Y d\mu_Y$$

where $d\mu_Y$ is a measure on Y . (Of course , if Y is zero dimensional, i.e. a collection of points , then δ_Y^* is just the evaluation map at these points .)

Hence , if A is self-adjoint , then

$$B^*T(t)\xi = \sum_{i=1}^{\infty} e^{\lambda_i t} \sum_{j=1}^{n_i} <\xi,\phi_{ij}>\int \phi_{ij}|_Y d\mu_Y .$$

Similarly , if we extend the system (5.2) to the case of m controllers on the submanifolds $Y_1,\dots\dots,Y_m$; i.e. the system

$$\frac{d\xi(t)}{dt} = A\xi(t) + \sum_{i=1}^{m} \delta_{Y_i} u_i(t) \qquad (5.3)$$

then it is easy to see that

$$B^*T(t) = \left\{ \sum_{i=1}^{\infty} e^{\lambda_i t} \sum_{j=1}^{n_i} <\xi, \phi_{ij}>\int \phi_{ij}|_{Y_k} d\mu_{Y_k} \right\}_{1 \leqslant k \leqslant m} .$$

The next result follows easily from theorem 5.1.

<u>Theorem 5.2</u> If the submanifolds $(Y_k)_{1 \leqslant k \leqslant m}$ are points (i.e. 0-dimensional submanifolds) P_k , $1 \leqslant k \leqslant m$, then the system (5.3) is approximately controllable if and only if the matrix

$$B_i = \begin{pmatrix} \phi_{i1}(P_1) & \dots & \phi_{i1}(P_k) \\ \dots & \dots & \dots \\ \phi_{in_i}(P_1) & \dots\dots & \phi_{in_i}(P_k) \end{pmatrix} \qquad (5.4)$$

has rank n_i for each $i \geqslant 1$. In particular , if each $n_i = 1$, then (5.3) is approximately controllable with a single controller at the point P if and only if $\phi_i(P) \neq 0$ for each $i \geqslant 1$, where we have written $\phi_i \overset{\Delta}{=} \phi_{i1}$. $\square$

We shall now consider the case of controllability of (5.3) on a compact orientable manifold of dimension 2 , i.e. a (Riemann) surface . Let us first recall that such a manifold is topologically a sphere with g handles attached (Stillwell , 1980) , and can be obtained from a plane polygon and its interior by identifying edges as shown in fig.5.1. (This is the 'normal form' of a compact surface of genus $g > 0$.)

If we regard the surface as a sphere with g handles , then the edges a_i, b_i of the polygon appear as latitude and meridian circles on the surface as in fig. 5.2.

Lemma 5.3 If X is an orientable surface S of genus g and

$$A: L^2(E) \longrightarrow L^2(E)$$

is a differential operator on X with analytic eigenfunctions ϕ_i $(1 \leq i < \infty)$ then for any system of defining cycles $a_1, b_1, \ldots, a_g, b_g$ of S there exist an infinite number of points P_j arbitrarily close to $C \overset{\Delta}{=} a_1 \cup b_1 \ldots . a_g \cup b_g$ such that $\phi_i(P_j) \neq 0$ for each i,j . Similarly , there exist an infinite number of one-dimensional submanifolds (with boundary) Y_j such that

$$\int_{Y_j} \phi_i d\mu_{Y_j} \neq 0 .$$

Proof Let N be a neighbourhood of C in S and let $P \in N$. Then for any neighbourhood M of P each ϕ_i can have at most a finite number of zeros in $M \cap N$. Since there are only countably many ϕ_i's , the first part follows directly . A similar argument may be used to prove the second part . □

Then we have

Theorem 5.4 If $a_1, b_1, \ldots, a_g, b_g$ is any set of generating cycles of an orientable surface of genus g then there exist an infinite of points P_i or submanifolds Y_i arbitrarily close to $a_1 \cup b_1 \ldots a_g \cup b_g$

such that the systems

$$\frac{d\xi(t)}{dt} = A\xi(t) + \sum_{i=1}^{\infty} \delta_{P_i} u_i(t)$$

and

$$\frac{d\,\xi(t)}{dt} = A\xi(t) + \sum_{i=1}^{\infty} \delta_{Y_i} u_i(t)$$

are approximately controllable . □

(We just note that invertibility is generic and so the points P_i can be chosen so that the matrix (5.4) is invertible , for $k = n_i$.)

Of course , if the multiplicities n_i of the eigenvalues λ_i of A are bounded then we would require only a finite number of controllers .

(6) Optimal Control of Systems on Compact Manifolds

In this section we shall consider the linear-quadratic problem for the system

$$\xi(t) = T(t)\xi(0) + \int_0^t T(t-s)Bu(s)ds \qquad (6.1)$$

with the cost functional

$$J(u) = \langle\xi(T),G\xi(T)\rangle_{L^2(E)}$$

$$+ \int_0^T \{\langle\xi(t),M\xi(t)\rangle_{L^2(E)} + \langle u(t),Ru(t)\rangle_U\}dt \qquad (6.2)$$

where M and $G\varepsilon\mathcal{L}(L^2(E))$ are non-negative self-adjoint operators and $R\varepsilon\mathcal{L}(U)$ is a positive definite self-adjoint operator . It will be assumed that the conditions (4.3) hold for some space W . Then the optimal control for the system (6.1) with the cost functional (6.2) (defined on the Hilbert space $L^2(E)$) is well known (Curtain and Pritchard ,1978) and is given by

$$u^* = -R^{-1}B^*Q(t)\xi(t)$$

where Q satisfies the inner product Riccati equation

$$\frac{d}{dt} \langle Q(t)h,k\rangle + \langle Q(t)h,Ak\rangle + \langle Ah,Q(t)k\rangle$$
$$= \langle Q(t)BR^{-1}B^*Q(t)h,k\rangle - \langle Mh,k\rangle \qquad (6.3)$$

where $Q(T) = G$ and $h,k \in \mathcal{D}(A)$. Note that $Q(t) \in \mathcal{L}(W,L^2(E))$. We shall need the following generalisation of Schwartz' Kernel Theorem (cf. Treves , 1967):

<u>Lemma 6.1</u> If X,Y are compact manifolds and E,F are vector bundles over X,Y respectively , then we have

$$\mathcal{D}'(X\text{x}Y,E\otimes F;Z) \cong \mathcal{D}'(\mathcal{D}(Y,F);\mathcal{D}'(X,E;Z))$$

for any topological vector space Z . Here ,

$$\mathcal{D}'(X,E;Z) = \mathcal{L}(\mathcal{D}(X,E);Z) \quad .$$

<u>Proof</u> This result follows easily from the classical Kernel Theorem by localisation and by application of the well known vector space isomorphism

$$\mathrm{Hom}_K(\mathrm{Hom}_K(M,Z),N) \cong \mathrm{Hom}_K(M\otimes N,Z)$$

for vector spaces M,N over K. □

Hence , with any continuous linear map $L:\mathcal{D}(Y,F) \longrightarrow \mathcal{D}'(X,E;Z)$ we associate a distribution $K \in \mathcal{D}'(X\text{x}Y,E\otimes F;Z)$ which satisfies

$$\langle K,\phi\otimes\psi\rangle = \langle L\psi,\phi\rangle$$

where $\phi \in \mathcal{D}(X,E)$, $\psi \in \mathcal{D}(Y,F)$ and $\phi\otimes\psi$ is the map in $\mathcal{D}(X\text{x}Y,E\text{x}F)$ defined by

$$(\phi\otimes\psi)(x,y) = \phi(x)\otimes\psi(y) \quad .$$

As usual we shall write

$$(L\psi)(x) = \int_Y K(x,y)\psi(y)d\mu_Y$$

where μ_Y is a volume measure on Y . It follows that we may write

$$(L\psi)(x) = \sum_{\beta=1}^{k} \int_{V_\beta} \rho_\beta(\phi_\beta^{-1}(y_1,\ldots,y_m))(K(x_1,\ldots,x_n,y_1,\ldots,y_m)\cdot$$
$$\phi_\beta^*\psi(y_1,\ldots,y_m)\rho(y)dy_1\ldots dy_m$$

where V_β is an open covering of Y with corresponding partition of unity ρ_β,ϕ_β is a local coordinate system on V_β and ϕ_β^* is the induced map . Note that, for simplicity, we have identified V_β with its image in $\mathbb{R}^n$. K is a qxr matrix of distributions on V_β, and ψ is an r dimensional vector function . (We have assumed that X and Y have dimensions n,m respectively and that the bundles have ranks q,r .)

We shall apply the above remarks to the Riccati equation (6.3) with E=F , $B = \delta_Y$ for some submanifold Y of X and , for simplicity, G,M,R are identity operators . If A is self-adjoint on X and we write

$$(Q(t)\xi)(x) = \int_X K(x,y,t)\xi(y)d\mu_X$$

for some distribution K , then we have

$$\int_{X\times X}\left\langle\left\{\frac{\partial K}{\partial t}(x,y,t) + A_x K(x,y,t) + A_y K(x,y,t) + \delta(x-y)\right.\right.$$

$$\left.\left. - \int_Y K(x,y\,|_Y,t)d\mu_Y^y \cdot \int_Y K(x\,|_Y,y,t)d\mu_Y^x\right\}h,k\right\rangle d\mu_X \times d\mu_X = 0$$

$$K(x,y,T) = \delta(x-y) \qquad (6.4)$$

where XxX is the product manifold with product measure $d\mu_X \times d\mu_X$, A_x refers to the operator A with respect to the 'variable' x and $\delta(x-y)$ is the distribution $\delta_Y \in \mathcal{D}'(X\times X;E)$ where Y is the diagonal submanifold of XxX . Also , $x|_Y$ indicates that , in the integration, x should be 'projected' along Y .

In order to solve (6.4) choose an open covering $\{U_i\}_{1\leq i\leq \alpha}$ of X consisting of local coordinate neighbourhoods and let $\{V_i\}_{1\leq i\leq\alpha}$ be a refinement of $\{U_i\}$ such that

(i) $U_i \subseteq V_i$

(ii) $V_i \cap V_j = \phi$ if $i \neq j$

(iii) $\overline{V}_i \cap \overline{V}_j$ is empty or an n-1 dimensional manifold with boundary.

$$\text{(iv)}\ X = \bigcup_{i=1}^{\alpha} \overline{V}_i \ .$$

(This is effectively a cellular decomposition of X.)
Then , if x^i, y^j are local coordinates in $V_i x V_j$ and K^{ij} is the distribution K expressed in these coordinates , then K^{ij} can be chosen to satisfy

$$\frac{\partial K^{ij}}{\partial t}(x^i,y^j,t) + A_{x_i} K^{ij}(x^i,y^j,t) + A_{y_j} K^{ij}(x^i,y^j,t) + \delta(x^i-y^j)$$

$$- \int_Y K^{ij}(x^i,y|_Y,t)d\mu_Y^y \int_Y K^{ij}(x|_Y,y^j,t)d\mu_Y^x = 0 \qquad (6.5)$$

$$K^{ij}(x^i,y^j,T) = \delta(x^i-y^j)$$

together with the boundary conditions

$$K^{ij}\Big|_{\partial(V_i x V_j)} = K^{k\ell}\Big|_{\partial(V_k x V_\ell)} \qquad (6.6)$$

for all $k\ell$ for which

$$\partial(V_i x V_j) \cap \partial(V_k x V_\ell) \neq \phi \ .$$

(7) Examples

In this section we shall apply the theory above to the generalised Laplacian Δ on the sphere and the torus . In the first example we shall show that the heat flow problem on the sphere is not approximately controllable with a finite number of point controllers.

Example 7.1 We shall consider the equation

$$\frac{\partial \xi}{\partial t} = -\Delta\xi + \sum_{i=1}^{d} \delta_{P_i} u_i \qquad (7.1)$$

on $H^2(X,\Lambda^2 T^*(X))$, for some points $P_1,\ldots,P_d$ on the sphere . By lemma 4.3 , $\delta_{P_i} \in H^{-1-\varepsilon}(\Lambda^2 T^*(X))$. Now , $\Lambda^2 T_x^*(X)$ is one dimensional and generated locally by $dx_1 \wedge dx_2$. It is easy to see that , again locally ,

$$-\Delta f = \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\right) dx_1 \wedge dx_2 \quad .$$

We require to find the eigenvalues of $-\Delta$ on the sphere . However, it is well known that , in the usual (non-global) coordinate system these are just the spherical harmonics

$$Y_{\ell m}(\theta,\phi) = \pm\left[\frac{2\ell+1}{4\pi}\,\frac{(\ell-|m|)!}{(\ell+|m|)!}\right]^{\frac{1}{2}} P_\ell^m(\cos\theta)e^{im\phi}$$

with multiplicity $2\ell+1$ for each eigenvalue $\ell(\ell+1)$,(c.f. Flanders, 1963) . The equation (7.1) is therefore not approximately controllable at a finite number of points by theorem 5.2.

Example 7.2 In this example we shall consider again the heat flow problem on the torus and show that on surfaces of genus greater than zero , numerical methods are likely to be required for the solution of controllability and optimal control problems . In toroidal coordinates (c.f Moon and Spencer , 1971) we have

$$\nabla^2\phi = \frac{(\cosh\eta - \cos\theta)^3}{a^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\cosh\eta - \cos\theta}\frac{\partial\phi}{\partial\theta}\right) + \frac{(\cosh\eta - \cos\theta)^2}{a^2\sinh^2\eta}\frac{\partial^2\phi}{\partial\psi^2}$$

Now ∇^2 is negative definite and so to find the spectrum we put

$$\nabla^2\phi = -k^2\phi$$

and let

$$\phi = (\cosh\eta - \cos\theta)^{\frac{1}{2}}\Theta(\theta)\Psi(\psi) \ .$$

Then we obtain the equation

$$\alpha + \beta\left(\left\{\frac{d^2\Theta}{d\theta^2}\Big/\Theta\right\}\cdot(\cosh\eta - \cos\theta)^{\frac{1}{2}} + \frac{1}{\sinh^2\eta}\frac{d^2\Psi}{d\psi^2}\Big/\Psi\right) = -k^2$$

where

$$\alpha(\theta) = \left\{-\frac{3}{4}\frac{\sin^2\theta}{a^2} + \frac{1}{2}\frac{\cos\theta(\cosh\eta - \cos\theta)}{a^2}\right\}$$

$$\beta(\theta) = \frac{(\cosh\eta - \cos\theta)^{3/2}}{a^2} \ .$$

Hence ,

$$- \frac{1}{\sinh^2\eta} \frac{\Psi''}{\Psi} = \ell^2 \text{ , say}$$

and

$$\Psi = A\cos(\ell\sinh\eta\psi) + B\sin(\ell\sinh\eta\psi) \text{ .}$$

Since Ψ is continuous at $\psi = 0, 2\pi, 4\pi, \ldots$, we have

$$\ell\sinh\eta = \text{integer .}$$

Moreover , we have

$$\frac{d^2\Theta}{d\theta^2} + \gamma(\theta)\Theta = 0 \tag{7.2}$$

where

$$\gamma(\theta) = \ell^2 - \frac{\alpha}{\beta} - \frac{k^2}{\beta} \quad (\cosh\eta - \cos\theta)^{-\frac{1}{2}} \text{ .}$$

Equation (7.2) must have two linearly independent solutions depending on ℓ and k , say $f_1(\theta,k,\ell)$, $f_2(\theta,k,\ell)$ and we must determine the spectral values of k in terms of ℓ from the continuity of Θ at $\pm(2n+1)\pi/2$. We shall study this numerical problem and the associated controllability problem in a future paper .

(8) Conclusions

In this paper we have presented a theory of controllability and optimal control of parabolic distributed parameter systems defined on compact manifolds . Particular emphasis has been given to the heat flow problem on compact orientable surfaces described by the Laplace-Beltrami operator . It has been seen that one is usually led to numerical spectral determination problems in order to determine controllability criteria .

Optimal control theory leads to local integral Riccati equations which again must be solved numerically in order to determine the feedback control . The numerical problems associated with this theory will be examined in a future paper .

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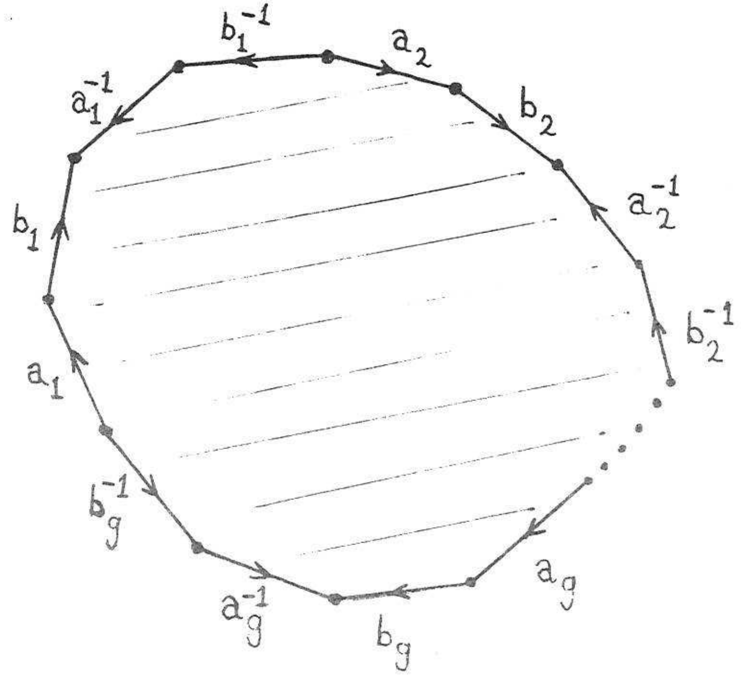

fig. 5.1.

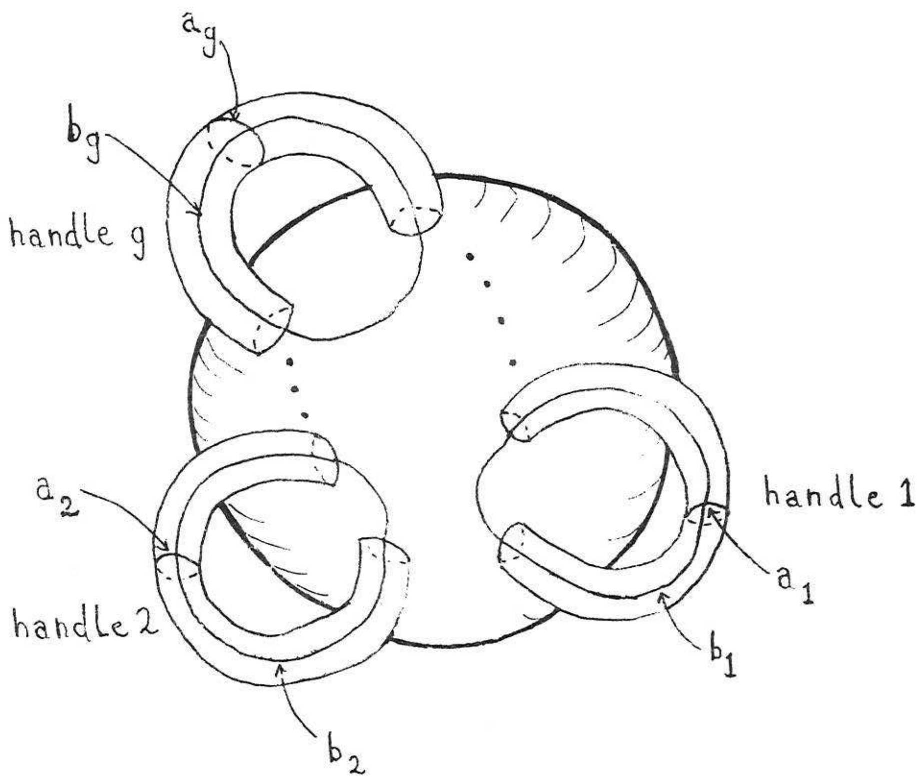

fig 5 2.