Monograph:

Owens, D.H. (1982) Some Case Studies in Approximate Models in Multivariable Control. Research Report. ACSE Report 203 . Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield

Some Case Studies in Approximate Models in Multivariable Process Control

by

D. H. Owens, B.Sc.,A.R.C.S.,Ph.D.,A.F.I.M.A.,C.Eng.,M.I.E.E.

and

H. M. Wang*

and

A. Chotai, B.Sc.,Ph.D.

Department of Control Engineering
University of Sheffield
Mappin Street, Sheffield S1 3JD

Research Report No. 203

November 1982

This work is supported by SERC under grant GR/B/23250

*Visitor on leave from Shanxi Institute of Technology, China

1. Introduction

Closed-loop control system design are frequently based on the use of a simplified model either because the available model is regarded as being too complex for design work or because an accurate plant model is not available. In fact, approximate models of engineering plant are a fact of life as, although a detailed linear model may be a good approximation to observed plant dynamics, it never matches the plant exactly.

An approximate model can be of arbitrary dynamic complexity. But, generally speaking, it is desirable that the model is of low order and simple structure with the consequent benefits of reduced computational requirement and the possibility of achieving simple design to form the basis for further refinement and understanding.

Reference [1] gives both frequency domain and time domain design method using approximate model. These techniques require only a knowledge of a plant's open-loop response Y(t) from any source. A sufficient condition is given in [1] for ensuring that a controller designed on the basis of the approximate model G_A will also stabilize the real plant G. The statement of this condition is as following (ref. Fig.1).

If K stabilizes the model G_A, it will also stabilize the real plant G if

(a) the composite system GKF is both controllable and observable and

(b) $$\lambda_o \triangleq \sup_{S \in D} \gamma(s) < 1 \qquad \ldots(1)$$

where $\gamma(s)$ is any convenient real valued function satisfying

$$\gamma(s) \geq \gamma(\| (I_\ell + K(s)F(s)G_A(s))^{-1} K(s)F(s) \|_p \Delta(s)) \qquad \ldots(2)$$

for all $s \in D$

and $\Delta(s)$ is any available matrix-valued function satisfying

$$\Delta(s) \geq \| G(s) - G_A(s) \|_p \quad \text{for all } s \in D \qquad \ldots(3)$$

In formulas (2) and (3), D is the usual Nyquist 'infinite' semicircle in closed right-half complex plane and the notation $\| \cdot \|_p$ is defined as

$$\|A\|_p = \begin{pmatrix} |A_{11}| & \cdots\cdots & |A_{1m}| \\ \vdots & & \\ |A_{\ell 1}| & \cdots\cdots & |A_{\ell m}| \end{pmatrix} \qquad \ldots(4)$$

The spectral radius $\gamma(M)$ of an $\ell x \ell$ matrix M with eigenvalues $m_1, m_2 \ldots m_\ell$ is

$$\gamma(M) = \max_{1 \leq i \leq \ell} |m_i| \qquad \ldots(5)$$

[1] also gives a simplest possible choice of Δ, i.e. a frequency-independent bound:

$$\Delta = N_\infty^P(E) = \sup_{T \geq 0} (|E(0^+)| + \sum_{k=1}^{k^*} |E(t_k) - E(t_{k-1})| + |E(T) - E(t_{k^*})|) \qquad \ldots(6)$$

where t_k are time instant in which the local maxima or minima of $E(t)$ are achieved,

k^* is the largest integer k such that $t_k < T$,

$E(t)$ is the error function of open-loop response,

$$E(t) = Y(t) - Y_A(t) \qquad \ldots(7)$$

and $Y_A(t)$ is the unit step response of approximate model G_A.

In this report, we illustrate the application of these design techniques, and, in particular, methods for the choice of the approximate model G_A. Also a less conservative, frequency dependent form of Δ is derived and hence a new necessary condition for including integral action in controller is given. This condition is easier to satisfy than the similar conditions given by [1]. Meanwhile, by comparing the design results for two kinds of plant model, we will illustrate cases where the design techniques are easy to use and cases where problems can occur. Finally, the robustness of the design will be discussed.

2. Real plant and performance specification

Suppose that a multivariable plant has an "unknown" transfer function matrix (TFM) as:

$$G \left(\frac{g_{ij}}{T_{ij}S + 1} e^{-\tau_{ij}S} \right)_{\ell x m} \qquad \ldots(8)$$

A lot of plants (for example, distillation column with hydraulic time delay) can be denoted by such a TFM. The values of g_{ij} (open-loop gain), T_{ij} (time constant) and τ_{ij} (time delay) may be very different for different plant. But in point of view of using approximate model to design a controller, we devide them into two kinds as following:

(1) Plant is of less open-loop interactions (say, <60%) but of rather large time delay. As an example, we consider a real plant which has TFM

$$G_1(s) = \begin{bmatrix} \frac{119.3}{812.8s+1} e^{-18.6s} & \frac{-62.3}{904s+1} e^{-14.6s} \\ \frac{55.3}{766.3s+1} e^{-114s} & \frac{-109.7}{715s+1} e^{-17.5s} \end{bmatrix} \quad \ldots(9)$$

where the largest time delay is of about 15% value of time constant (in the same element), and the open-loop interaction is about 50%.

(2) Plant is of more open-loop interactions, but with small time delay. For example:

$$G_2(s) = \begin{bmatrix} \frac{119.3}{812.8s+1} e^{-3.7s} & \frac{-124.6}{904s+1} e^{-2.92s} \\ \frac{110.7}{766.3s+1} e^{-22.8s} & \frac{-109.7}{715s+1} e^{-3.5s} \end{bmatrix} \quad \ldots(10)$$

In other words, in $G_2(s)$, the open-loop interactions are twice of those in $G_1(s)$ but the time delays are $\frac{1}{5}$ of those in $G_1(s)$ (element by element).

Even though (9) and (10) are still approximate model relative to practice, we regard them as "real plant" in this paper.

We wish to design forward path controller K using a simple approximate model to guarantee that the real feed-back system (scheme of Fig.1(a))

(1) is stable,

(2) its response speed increases (say, at least 3 times) with respect to that of the open-loop,

(3) has small steady-state errors (e.g. less than 10%) in response to step demand,

(4) has acceptable interactions (e.g. less than 20%),

(5) is a robust design in sense that the plant G(s) changes over a period of time to the plant $\tilde{G}(s)$, stability will be retained provided that the change $\tilde{G}-G$ is small enough.

It is well known that the complexity of controller effects the cost of the plant. In the following sections, attention is focussed on the design of a proportional or proportional plus integral controller (with unity negative-feed-back) for an unknown plant.

As mentioned above, the open-loop response are assumed given from plant trials or model simulations. In fact, if the TFM is known as form of (8), the 'step response matrix' can be directly written as

$$Y(t) = Y(g_{ij}(1 - e^{-\frac{1}{T_{ij}}(t-\tau_{ij})})H(t-\tau_{ij}))_{\ell xm} \qquad \ldots(11)$$

where $H(t-\tau_{ij})$ is the unit-step function with time delay. $Y(t)$ is the only knowledge we assume about the plant.

In this paper, we confine our attention to two kinds of approximate model: a pseudo-diagonal model and first order model [3].

Throughout this paper we assume that $G(s)$ is both controllable and observable and $G(o)$ (or $Y(\infty)$) is nonsingular.

3. Proportional Control

3.1 Pseudo-diagonal model

When the open-loop interactions are small enough and can be neglected during the controller design (measured by $g_{ij}(i \neq j)/g_{ii}$), we can use diagonal model directly [1]. But when the open-loop interactions are not small enough, the design using diagonal model is necessarily too conservative because the error function matrix is of large value. In this case, the approximate model can be chosen as

$$G_A = \text{diag}\{ \frac{1}{1+\alpha_i s} \} P \qquad \ldots(12)$$

i.e. the dynamics of the plant are represented by a diagonal matrix, and the interactions are represented by a static interaction matrix. The α_i should be near the average value of T_{ij}, $j = 1,m$.

P, in general, can be chosen as G(o) or $Y(\infty)$ (i.e. the steady-state value matrix of open-loop response) to make the steady-state errors $E(\infty)$ are zero but it can be chosen by other means if desired.

In our example $G_1(s)$, choose $\alpha_1 = \alpha_2 = 800$ and $P = G_1(o)$, then

$$G_A = \begin{pmatrix} \frac{1}{1+800s} & 0 \\ 0 & \frac{1}{1+800s} \end{pmatrix} \begin{pmatrix} 119.3 & -62.3 \\ 55.3 & -109.7 \end{pmatrix} \qquad \ldots(13)$$

(Marked MD1)

The open-loop response errors are shown in Fig.2 and 3, and we get $N_\infty{}^P(E)$ by direct calculation:

$$N_\infty{}^P(E) = \begin{pmatrix} 5.36 & 6.33 \\ 14.56 & 11.09 \end{pmatrix}$$

Choose the controller $K = p^{-1}\text{diag}\{k_j\}$, where k_j are scalar constants. For simplicity, take $k_1 = k_2$, then controller is of form

$$K = p^{-1} \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \qquad \ldots(14)$$

The closed-loop TFM of approximate model feed-back system (assum F = I) is

$$H_c(s) = (I+G_AK)^{-1}G_AK = (I_2 + \frac{k}{1+800s} I_2)^{-1} \cdot \frac{k}{1+800s} I_2$$

$$= \frac{k}{800s+1+k} I_2 \qquad \ldots(15)$$

So, for any $k>-1$ system (Fig.1b) is stable.

The closed-loop response of approximate systems (Fig.1b) is

$$y_{A_j} = \frac{R_j k}{1+k}\left(1 - e^{-\frac{1+k}{\alpha}t}\right) \qquad \ldots(16)$$

where R_j is the magnitude of step demand. From (16) it is very clear that steady-state error is $\frac{1}{1+k}$ (for unit step demand) and the speed of response is as much as (1+k) times of that of open-loop response.

We next check condition (1) to decide the largest gain k that can be used. Obviously, for robustness of the controller, the maximum spectral radius should be less than unity and the smaller the spectral radius γ is, the more robust the design will be. But on the other hand, the higher the value of γ, (and hence the higher the gain), the faster the response speed. For simplicity, we will choose our design to ensure that $\gamma \leq 0.8$, i.e. (by choosing $\gamma(s) \equiv \gamma(\| (I_\ell+KFG_A)^{-1}KF\|_p \Delta(s))$

$$\lambda_o \triangleq \sup_{S \in D} \gamma(\| (I_\ell+KFG_A)^{-1}KF\|_p \Delta(s)) \leq 0.8 \qquad \ldots(17)$$

Noting that

$$(I+KG_A)^{-1} = (I + p^{-1}\text{diag}\{\frac{k}{800s+1}\} p)^{-1} = \frac{800s+1}{800s+1+k} I_2$$

so for any k>0 $\qquad \lim_{S\to\infty} \frac{800s+1}{800s+1+k} I_2 = I_2$

$$\therefore \lambda_o \triangleq \sup_{S \in D} \gamma(\| (I+KG_A)^{-1}K\|_p \Delta)$$

$$= \gamma(\|kP^{-1}\|_p N_\infty^P(E)) = k\gamma(\|P^{-1}\|_p N_\infty^P(E)) \qquad \ldots(18)$$

The largest gain k can be decided by

$$k \le \frac{0.8}{\gamma(\,\|P^{-1}\|_p \cdot N_\infty^{\,P}(E))} = \frac{0.8}{\gamma\left(\left\|\begin{pmatrix}119.3 & -62.3\\ 55.3 & -109.7\end{pmatrix}^{-1}\right\|_p \cdot \begin{pmatrix}5.36 & 6.33\\ 14.54 & 11.09\end{pmatrix}\right)}$$

$$\le \frac{0.8}{0.338} = 2.36$$

Choose k = 2.3, then the controller can be found

$$K = P^{-1}\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix} = \begin{pmatrix}119.3 & -62.3\\ 55.3 & -109.7\end{pmatrix}^{-1}\begin{pmatrix}2.3 & 0\\ 0 & 2.3\end{pmatrix} = \begin{pmatrix}0.02617 & -0.01485\\ 0.0132 & -0.02845\end{pmatrix}$$

The real plant closed-loop response (scheme Fig.1a) are shown in Fig.4 and 5. The max interaction is less than 20% but the steady-state errors is rather big (about 30%). The design is therefore unacceptable and we then redesign by improving the model to reduce $N_\infty^{\,P}(E)$ or including integral action in the controller. We consider the first possibility in section 3.2 and the second in part 4.

3.2 First order model

The first order model is of form [2]

$$G_A^{-1}(s) = A_o\, s + A_1$$

where A_o and A_1 are real constant matrices. The method of estimating them depends upon whether or not the real plant TFM is known. If it is then A_o and A_1 can be deduced directly from G(s) by using the formulas:

$$A_o^{-1} = \lim_{s\to\infty} sG(s) \quad (\text{neglecting } \tau_{ij})$$

$$\ldots(19)$$

$$A_1^{-1} = \lim_{s\to 0} G(s)$$

In other words, A_o^{-1} and A_1^{-1} represent the initial rate and steady-state values respectively of the real plant in response to unit step inputs. If the real plant TFM is unknown but the open-loop response matrix Y(t) is given, we can choose A_1^{-1} as $Y(\infty)$ and A_o^{-1} as $\left.\frac{dY}{dt}\right|_{t=0}$ (by neglecting time delay τ_{ij} as well) respectively.

In this way, the plant $G_1(s)$ can be shown to generate

$$A_o^{-1} = \begin{bmatrix} \frac{119.3}{812.8} & -\frac{62.3}{904} \\ \frac{55.3}{766.3} & -\frac{109.7}{715} \end{bmatrix}, \quad A_o = \begin{bmatrix} 8.79 & -3.95 \\ 4.128 & -8.37 \end{bmatrix}$$

$$A_1^{-1} = \begin{bmatrix} 119.3 & -62.3 \\ 55.3 & -109.7 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.01137 & -0.00646 \\ 0.00573 & -0.01237 \end{bmatrix}$$

giving the first order approximate model (MF1)

$$G_{A_1}(s) = \left(\begin{bmatrix} 8.79 & -3.95 \\ 4.128 & -8.37 \end{bmatrix} s + \begin{bmatrix} 0.01137 & -0.00646 \\ 0.00573 & -0.01237 \end{bmatrix} \right)^{-1} \qquad \ldots(20)$$

Using this model, the response errors are shown in Fig.6 and 7 and $N_\infty^P(E)$ can be obtained as

$$N_\infty^P(E) = \begin{bmatrix} 5.29 & 1.81 \\ 15.11 & 5.28 \end{bmatrix}$$

Following a first order design technique described by Owens [3], choose

$$K = kA_o - A_1$$

where k is the required closed-loop poles in each loop.

The closed-loop TFM for the approximate system (Fig.1b) is then

$$H_c(s) = \frac{k}{s+k} \{I - \frac{1}{k} A_o^{-1} A_1\} \qquad \ldots(21)$$

and closed-loop step response matrix is

$$\mathcal{L}^{-1} H_c(s) \frac{1}{s} = (1 - e^{-kt}) \{I_2 - k^{-1} A_o^{-1} A_1\} \qquad \ldots(22)$$

So, for any k>0, system 1(b) is stable. From (22) the steady-state error and degree of interactions for system Fig.1(b) can also be found.

Then choosing a value of k, we can check the validity of condition (1) at a selection of frequency points covering the bandwidth of interest. If $\lambda_o > 1$, we could then reduce k in an attempt to reduce $\| (I+KFG_A)^{-1} KF \|_p$ and hence λ_o. Repeat this procedure until $\lambda_o \leq 1$ (in our design, $\lambda_o \leq 0.8$, for robustness of design). In this way, we find k = 0.006 is a suitable value, and the spectral radius are shown in Fig.8 as a function of frequency.

The closed-loop response of the real plant are shown in Fig.9 and 10. By drawing a tangent in the initial point of closed-loop response we can see that the speed of closed-loop response is much faster than those of the open-loop (Closed-loop time constant is about 150sec, while the open-loop time constants are in range of 715-904 sec). The closed-loop interactions are less than 20% and the steady-state errors are about 20%. The design is robust as well because the maximum spectral radius is 0.7. Overall, the design is much better than that in Section 3.1 and we can say, from the viewpoint of controller design, a first order model is better than pseudo-diagonal model to approach the plant $G_1(s)$.

An important point we want to mention here is that, although using formula (19) can give a very good approximate model in many cases, sometimes the model chosen may be unstable. For example, in the case of $G_2(s)$, we obtain

$$A_o^{-1} = \lim_{s\to\infty} SG_2(s) = \begin{bmatrix} \frac{119.3}{812.3} & -\frac{124.6}{904} \\ \frac{110.7}{766.3} & -\frac{109.7}{715} \end{bmatrix}$$

and

$$A_1^{-1} = \lim_{s\to 0} G_2(s) = \begin{bmatrix} 119.3 & -124.6 \\ 110.7 & -109.7 \end{bmatrix}$$

That is the approximate model is

$$G_{A_2}(s) = [A_o s + A_1]^{-1} = \frac{1}{-383.8s^2+0.6228s+0.00142} \times$$

$$\times \begin{bmatrix} -56.31s+0.1690 & 52.85s-0.1765 \\ -55.43s+0.1568 & 58.84s-0.1554 \end{bmatrix}$$

which is clearly unstable.

In this case, we have to choose A_o and A_1 by other means or choose another form of model. One way of choosing A_o and A_1 is as follows (others are seen in Section 4.2 of this paper): in a frequency range of interest, calculate $G_2^{-1}(s)$ (neglecting the time delays τ_{ij}), and then take the general average value of their real part (element by element) as A_1, and take the rate of increase of their imaginary part (element by element) as A_o. In other words,

because $G_2^{-1}(i\omega)$ is to be represented by $A_o i\omega + A_1$

so Real $(G_2^{-1}(i\omega))$ should be represented by A_1

and Imag $(G_2^{-1}(i\omega))$ should be represented by $A_o\omega$.

In this way, we find

$$A_o = \begin{pmatrix} 57.97 & -52.16 \\ 54.59 & -55.68 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0.1654 & -0.1640 \\ 0.1608 & -0.1681 \end{pmatrix}$$

yields the model

$$G_{A_2} = \frac{1}{380.35s^2+1.614s+0.00143} \begin{pmatrix} 55.68s+0.1681 & -52.16s-0.1640 \\ 54.59s+0.1608 & -57.97s-0.1654 \end{pmatrix}$$

which is stable with step response errors as shown in Fig.11 and 12. (This model is marked by MF2). From there $N_\infty{}^P(E)$ can be found to be

$$N_\infty{}^P(E) = \begin{pmatrix} 2.93 & 10.89 \\ 5.8 & 6 \end{pmatrix}$$

In the similar manner to the above, we can predict that the largest k can be used is k = 0.0037 and the corresponding closed-loop response of the real plant is illustrated in Fig.13.

The controller makes the system response rate increase by about 3 times (closed-loop time constant is about 270 sec while the open-loop time constants are in range of 715-904 sec). But the steady-state error is too big (60%) and hence the design is unacceptable.

Comparing Fig.13 with Fig.9 and 10, we can get an impression that the controller for plant $G_1(s)$ (less interaction but relative large time delay) is easy to design using an approximate model, and $G_2(s)$ (small time delay but large interaction) is difficult. The same things can be seen in next part (see Section 4.3.2).

To summarize the three controllers designed above, the common defect is that the steady-state error exceeds the performance specification. This is because those controllers are 'low gain' type. In other words, this is because that the theory used is of 'low gain' type guaranteeing stability for all controllers of low enough gain [1].

To offset the steady-state error, the integral action should be included in controller. This is considered in the next section.

4. Proportional Plus Integral Control

4.1 Necessary condition

Reference [1] indicates that the possibility of including integral action is related to the magnitude of the modelling error and the steady-state performance of the model G_A. When K includes integral action, $\lim_{s\to 0} K = \infty$ then

$$\lim_{s\to 0} \|(I+KG_AF)^{-1}KF\|_p = \|G_A^{-1}(o)\|_p$$

So the necessary condition for including integral action is that [1]

$$\gamma(\|G_A^{-1}(o)\|_p \, {}_pN_\infty^p(E)) < 1 \qquad \ldots(23)$$

Even though condition (23) can allow large model errors in most cases, problems can occur if $G_A^{-1}(o)$ is badly conditioned (i.e. has a large spread of eigenvalues and skew eigenvectors), condition (23) can be a very difficult condition to satisfy. For example, in plant $G_2(s)$ we have

$$G_A(o) = G_2(o) = \begin{bmatrix} 119.3 & -124.6 \\ 110.7 & -109.7 \end{bmatrix}$$

and suppose that choose approximate model to make

$$N_\infty^{\ P}(E) = \begin{bmatrix} 1 & 1.2 \\ 2 & 2.5 \end{bmatrix}$$

Although the elements of $N_\infty^{\ P}(E)$ are much less than the elements of $G_A(o)$, condition (23) is not satisfied. To deduce this, note that

$$G_A(o)^{-1} = \begin{bmatrix} -0.1554 & 0.1765 \\ -0.1568 & 0.1690 \end{bmatrix}$$

and hence

$$\gamma(\|G_A(o)^{-1}\|_p N_\infty^{\ P}(E)) = 1.18 > 1$$

In such a case, we must modify the theory in condition (1) if integral action is required.

Noting

$$\lim_{t\to\infty} E(t) = E_\infty$$

we obtain

$$\lim_{t\to\infty} (E(t)-E_\infty) = 0$$

and, as E(t) is exponentially bounded, so

$$\int_o^\infty \|E(t) - E_\infty\|_p \, dt < +\infty \qquad \ldots(24)$$

On the other hand

$$\mathcal{L}(E(t)-E_\infty) = \frac{1}{s}(G(s)-G_A(s)) - \frac{E_\infty}{s}$$

$$= \int_o^\infty (E(t)-E_\infty)e^{-st}dt$$

so that

$$|s^{-1}|\cdot \|G(s)-G_A(s)-E_\infty\|_p \leq \int_o^\infty \|E(t)-E_\infty\|_p dt \quad , \quad Res \geq 0 \qquad ...(25)$$

or equivalently

$$\|G(s)-G_A(s)\| \leq \|E_\infty\|_p + |s| \int_o^\infty \|E(t)-E_\infty\|_p dt = \Delta_o(s) \qquad ...(26)$$

It follows that $\|G(s)-G_A(s)\|$ is also bounded as follows

$$\|G(s)-G_A(s)\|_p \leq \min(\Delta_o(s), N_\infty{}^p(E)) \qquad ...(27)$$

and hence we can replace $\Delta = N_\infty{}^p(E)$ by $\Delta = \min(\Delta_o(s), N_\infty{}^p(E))$ in (2). The necessary condition for including integral terms in controller is then

$$\gamma(\|G_A{}^{-1}(o)\|_p \Delta(o)) < 1 \qquad ...(28)$$

or equivalently

$$\gamma(\|(G_A{}^{-1}(o)\|_p \cdot \|E_\infty\|_p) < 1 \qquad ...(29)$$

If we choose $G_A{}^{-1}(o) = G^{-1}(o)$, then $\|E_\infty\|_p = 0$. The condition (29) is then always satisfied, i.e. proportional plus integral controllers can always be used. For the convenience of use, rewrite the condition (2) as

$$\gamma(s) = \gamma(\,\|(I_\ell + K(s)F(s)G_A(s))^{-1}K(s)F(s)\|_p \Delta(s)) \qquad \ldots(30)$$

and $\quad \Delta(s) = \min(\Delta_o(s),\ N_\infty^{\,p}(E))$ (element by element)

where $\quad \Delta_o(s) = \|E_\infty\|_p + |s| \int_o^\infty \|E(t) - E_\infty\|_p dt$

That is the condition used in the following sections.

4.2 Pseudo-diagonal model

In the similar manner to Section 3.1, the approximate model for either $G_1(s)$ or $G_2(s)$ can be chosen as

$$G_A(s) = \begin{pmatrix} \frac{1}{1+\alpha_1 s} & 0 \\ 0 & \frac{1}{1+\alpha_2 s} \end{pmatrix} G(o) \qquad \ldots(31)$$

When open-loop response is known only, replace G(o) by $Y(\infty)$ in (31). The difference between (31) and (12) is that in (31) the static interaction matrix have to be chosen as G(o) such that $\|E_\infty\|_p = 0$ and hence (29) is always satisfied.

For plant $G_1(s)$, take $\alpha_1 = 850$, $\alpha_2 = 750$, so

$$G_{A_1}(s) = \begin{pmatrix} \frac{1}{1+850s} & 0 \\ 0 & \frac{1}{1+750s} \end{pmatrix} \begin{pmatrix} 119.3 & -62.3 \\ 55.3 & -109.7 \end{pmatrix} \qquad \text{(MD2)}$$

The response errors are shown in Fig.14 and 15. By calculating we get

$$N_\infty^{\,p}(E) = \begin{pmatrix} 7.22 & 3.72 \\ 15.4 & 6.73 \end{pmatrix}$$

and

$$I_\infty{}^p(E) = \int_o^\infty ||E-E_\infty||_p dt = \begin{pmatrix} 0.296 & 0.436 \\ 0.729 & 0.226 \end{pmatrix} \times 10^4$$

Suppose the controller K is of form

$$K = G(o)^{-1} \text{ diag}\left\{k_j + \frac{c_j}{s}\right\} \qquad \ldots(31)^*$$

The closed-loop TFM for G_A is (assume F = I)

$$H_c(s) = \{I+G_AK\}^{-1}G_AK = \left\{I+\begin{pmatrix} \frac{1}{1+\alpha_1 s} & 0 \\ 0 & \frac{1}{1+\alpha_2 s} \end{pmatrix}\begin{pmatrix} k_1+\frac{c_1}{s} & 0 \\ 0 & k_2+\frac{c_2}{s} \end{pmatrix}\right\}^{-1}\begin{pmatrix} \frac{1}{1+\alpha_1 s} & 0 \\ 0 & \frac{1}{1+\alpha_2 s} \end{pmatrix} \times$$

$$\times \begin{pmatrix} k_1+\frac{c_1}{s} & 0 \\ 0 & k_2+\frac{c_2}{s} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{k_1 s+c_1}{\alpha_1 s^2+(1+k_1)s+c_1} & 0 \\ 0 & \frac{k_2 s+c_2}{\alpha_2 s^2+(1+k_2)s+c_2} \end{pmatrix} \qquad \ldots(32)$$

or

$$H_c(s) = \text{diag}\left\{\frac{k_j s+c_j}{\alpha_j s^2+(1+k_j)s+c_j}\right\}$$

So, for any $k_j>-1$, $c_j>0$, approximate system Fig.1(b) is stable and non-interacting.

The closed-loop unit step response of approximate system is

$$Y_{A_j}(t) = \frac{k_j}{\alpha_j}\left\{\frac{d_j}{a_j^2-b_j^2} + \frac{1}{2b_j}\,\frac{b_j-a_j-d_j}{b_j-a_j}\,e^{-(a_j-b_j)t} - \frac{1}{2b_j}\,\frac{b_j+a_j-d_j}{b_j+a_j}\,e^{-(a_j+b_j)t}\right\}$$

$j = 1,2$ ...(33)

where

$$a_j = \frac{1+k_j}{2\alpha_j}$$

$$b_j = \frac{\sqrt{(1+k_j)^2-4c_j}}{2\alpha_j} \qquad \text{...(34)}$$

$$d_j = \frac{c_j}{k_j}$$

Now, we choose values of k_j and c_j, and check the validity of condition (1) at a selection of frequency range covering the bandwidth of interest. When $k_1 = k_2 = 2.8$ and $c_1 = c_2 = 0.002$ the spectral radius of (30) is plotted against ω in Fig.16. The largest value is about 0.8. These values of k and c are therefore largest gain we can use to retain the required robustness of design.

The closed-loop responses of the real system (scheme Fig.1(a)) are illustrated in Fig.17 and 18. The steady-state errors are of course zero because integral action are included. The maximum interaction is less than 20%. From Fig.17 and 18, we can say that the speed of response is good as, by plotting a tangent in initial point of response (neglecting time delay), we can find that the 'equivalent closed-loop time constant' of the real system which is about $\frac{1}{3}$ of that of the open-loop time constant.

So the controller designed above is good enough. The success of the design is also indicated by the similar stability and overall dynamic characteristics and identical steady-states of the real and approximating feedback schemes.

4.3 First order model

There are several ways to choose a first order approximate model. Two of them are as described in Section 3.2. However, we now give two other methods to choose a first order model.

4.3.1 Multiplication factor method

The method is as follows: Choose $A_1 = G^{-1}(o)$ or $A_1 = Y^{-1}(\infty)$ to ensure that $||E_\infty||_p = 0$ and hence integral action can always be included.

Then choose

$$A_o = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} A_1$$

where α_1 and α_2 are constants.

It follows that

$$G_A = (A_o s+A_1)^{-1} = \left\{\left(\begin{bmatrix} \alpha_1 s & 0 \\ 0 & \alpha_2 s \end{bmatrix} + I\right) A_1\right\}^{-1}$$

$$= A_1^{-1} \begin{pmatrix} \alpha_1 s+1 & 0 \\ 0 & \alpha_2 s+1 \end{pmatrix}^{-1} = G(o) \begin{bmatrix} \frac{1}{1+\alpha_1 s} & 0 \\ 0 & \frac{1}{1+\alpha_2 s} \end{bmatrix} \qquad \ldots(35)$$

Although it is similar to (30), we here regard it as first order model and hence use the first order design method described by [3]. For plant $G_1(s)$, we take

$\alpha_1 = 770$ (near the average of T_{11} and T_{21})

$\alpha_2 = 800$ (near the average of T_{12} and T_{22})

and hence

$$A_o = \begin{bmatrix} 770 & 0 \\ 0 & 800 \end{bmatrix} \begin{bmatrix} 119.3 & -62.3 \\ 55.3 & -109.7 \end{bmatrix}^{-1} = \begin{bmatrix} 8.750 & -4.974 \\ 4.584 & -9.896 \end{bmatrix}$$

The response errors are shown in Fig.19 and 20, and from there we get

$$N_\infty^P(E) = \begin{pmatrix} 7.23 & 6.38 \\ 15.06 & 11.04 \end{pmatrix} \quad \text{and} \quad I_\infty^P(E) = \begin{pmatrix} 0.72 & 0.74 \\ 0.63 & 0.70 \end{pmatrix} \times 10^4$$

Suppose that the controller is of first order form (ref. [3])

$$K(s) = \left\{ k + c + \frac{kc}{s} \right\} A_o - A_1 \qquad \ldots(36)$$

The closed-loop TFM for the approximate system (Fig.1a) is

$$H_c(s) = \frac{1}{(s+k)(s+c)} \{kcI+s((k+c)I-A_o^{-1}A_1)\} \qquad \ldots(37)$$

So the approximate system is stable if and only if k>0 and c>0. The validity of condition (1) is checked by the same way as that described in Section 4.2. When k = 0.004, c = 0.0003, the spectral radius of (30) is plotted against ω in Fig.21. The largest value is approch 0.83. So k = 0.004 and c = 0.0003 are the largest gains that can be used to satisfy the robustness requirement. The closed-loop response of the real system (scheme Fig.1a) are shown in Fig.22 and Fig.23. The steady-state errors are zero and interactions are less than 20%. Because 0<c<<k, we can still use k to measure the speed of response. So the speed is more than 3 times of that of open-loop (closed-loop time constant is $\frac{1}{0.004}$ = 250 sec, and the open-loop time constants are in range 715 - 904 sec). The design is successful as all of the performance specification is achieved.

For comparison, we choose $G_{A_1}(s)$ as MF1 (ref. section 3.2) and design a proportional plus integral controller. The spectral radius and closed-loop response of real systems are shown in Fig.24, 25 and 26. The suitable gains are k = 0.006, c = 0.0002. The main difference is that the speed of responses of Fig.25 and 26 is quicker than those in Fig.22 and 23. Interactions and steady-state errors in two designs are about the same. From this viewpoint, we can say MF1 is a more successful model than MF3 to approximate $G_1(s)$.

4.3.2 <u>Iterative model choice</u>

Sometimes difficulties might occur when we use the methods described above to choose a first order model (either because the model is unstable or because the response errors are too big). So another method (even though it is inconvenient) is given as follows: Choose A_1 as $G(o)^{-1}$ or $Y(\infty)^{-1}$, then choose a number of A_o and calculate for checking if its response errors E(t) are small enough. Repeat this procedure we might find a better model than those described by (19) and (35).

For example, to real plant $G_2(s)$ choose

$$A_1 = \begin{pmatrix} 119.3 & -124.6 \\ 110.7 & -109.7 \end{pmatrix}^{-1}$$

and get A_o by iterative choice as

$$A_o = \begin{pmatrix} -50 & 52 \\ -55 & 50 \end{pmatrix} \qquad \text{(marked MF4)}$$

The response errors are shown in Fig.27 and 28. Compared with their steady-state values the errors are very small. We can also obtain

$$N_\infty^{\ P}(E) = \begin{bmatrix} 5.51 & 4.62 \\ 9.66 & 11.41 \end{bmatrix} \quad \text{and} \quad I_\infty^{\ P}(E) = \begin{bmatrix} 0.53 & 0.55 \\ 0.47 & 1.2 \end{bmatrix} \times 10^4$$

Compared with the error matrices obtained before, these values are not big. So this model can also approach the real plant.

For the purpose of comparing two kinds of real plant, the controller is designed using MF4 by the similar means with section 4.3.1. The spectral radius and closed-loop response are illustrated in Fig.29 and 30. The largest gain that can be used to satisfy the required robustness is k = 0.0037, c = 0.0001.

From Fig.30, we can see, although the model error is not big, the closed-loop response of real system is very bad. The main problem is that the interaction is too big (more than 80%!). Even though we make a great effort (for example, choose controller K is of more general form

$$K = \text{diag}\left\{k_j + c_j + \frac{k_j c_j}{s}\right\}A_o - A_1$$

or choose other first order models) the situation is little improved.

5. Summary and Discussion

1. In this paper, several methods for choosing approximate models are given. Those models can closely represent the real plant which is assumed to be of form (6). Corresponding to these models, some simple form of controllers are given. Using these models and controller forms, the stability and performance of the approximate systems are easily decided. We give table (1) to summarize them.

2. In section 4.1 of this paper, a more general measure of model/plant mismatch Δ is derived (ref. formula 27). Consequently a less conservative necessary condition for including integral action is obtained. This is an important addition to the results of [1]. It means that the integral action can easily be used, particularly if $||E_\infty||_p = 0$, the integral action can always be included in the controller.

3. The paper has considered two examples. One of them is of large time delay but less open-loop interactions (G_1). The other is of more open-loop interactions but with small time delay (G_2). The results of the design indicates that in the first case no control difficulties exist but in the second case some problems occur, (ref. section 3.2 and 4.3.2). This appears to be due to the bad-condition of steady-state matrix $G_2(o)$. (The eigenvectors of $G_2(o)$ are more skew than those of $G_1(o)$).

4. The designs are robust in the sense that if plant change to a "new plant" $\tilde{G}$ with step response $\tilde{Y}$, stability of scheme Fig.1(a) will be retained provided $(\tilde{Y}-Y)$ is small enough.

Suppose the error matrix $(\tilde{Y}-Y_A)$ for the "new plant" is $\tilde{E}$ and their norm matrix is $N_\infty^p(\tilde{E})$, then the criterion of stability is simply

$$\lambda_o = \sup_{S\in D} \gamma(\,||(I+KG_A)^{-1}K||_p N_\infty^p(\tilde{E})) < 1 \qquad \ldots(38)$$

Assume $N_\infty^p(\tilde{E}) \leq \eta \cdot N_\infty^p(E)$ where η is a scalar constant, then

$$\gamma(\,||(I+KG_A)^{-1}K||_p\, N_\infty^p(\tilde{E})) \leq \eta\cdot\gamma(\,||(I+KG_A)^{-1}K||_p\, N_\infty^p(E))$$

So the largest value of η to satisfy (38) can be found by

$$\eta = \frac{1}{\sup\limits_{S \in D} \gamma(\| (I+KG_A)^{-1} K \|_p \, N_\infty{}^P(E))} \qquad \ldots(39)$$

In our design, we keep $\gamma_{S \in D} (\| (I+KG_A)^{-1} K \|_p \, N_\infty{}^P(E)) < 0.8$, so

$$\eta = \frac{1}{0.8} = 1.25$$

It is clear that stability of "new plant" will be retained if

$$N_\infty{}^P(\tilde{E}) \leq 1.25 \; N_\infty{}^P(E) \qquad \text{(element by element)}$$

or the general form

$$N_\infty{}^P(\tilde{E}) \leq \frac{1}{\lambda_o} N_\infty{}^P(E) \qquad \ldots(40)$$

This condition is related to the condition (38) of [1] but has a much simpler form.

Table (1)

Approximate model	Controller form	Condition of stability for approximate system	Performance of approximate system	Possibility of including integral action
Pseudo-diagonal model $G_A = \text{diag}\left\{\frac{1}{1+\alpha_j s}\right\} P$	Proportional control (P) $K = P^{-1}\text{diag}\ \{k_j\}$	$k_j > -1$		Check needed
$G_A = \text{diag}\left\{\frac{1}{1+\alpha_j s}\right\} G(o)$	P + integral control (I) $K = G(o)^{-1}\text{diag}\left\{k_j + \frac{c_j}{s}\right\}$	$k_j > -1, \quad c_j > 0$		No check needed
First order model $G_A = (A_o s + A_1)^{-1}$ $A_o^{-1} = \lim_{s\to\infty} sG(s)$ $A_1^{-1} = \lim_{s\to o} G(s)$	P or P+I $K = kA_o - A_1$ or $K = \{k+c+\frac{c}{s}\}A_o - A_1$	$k > 0 \quad c > 0$	identical response speed with time constant k^{-1} (assume $k >> c$)	No check needed
Real $(G^{-1}(i\omega)) \to A_1$ Imag $(G^{-1}(i\omega)) \to A_o\omega$	"	"	"	Check needed
$A_1 = G(o)^{-1}$ $A_o = \begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix} A_1$	"	"	"	No check needed
$A_1 = G(o)^{-1}$ A_o by iterative choice	"	"	"	"

References

[1] Owens, D.H., and Chotai, A.: 'Robust controller design for linear dynamic systems using approximate models', Research Report No. 194, Dept. of Control Engineering, University of Sheffield, 1982.

[2] Edwards, J.B., and Owens, D.H.: '1st-order-type models for multivariable process control', Proc.IEE, Vol.124, No.11, Nov. 1977.

[3] Owens, D.H.: 'Feedback and multivariable systems', Peter Peregrinus, Stevenage, 1978.

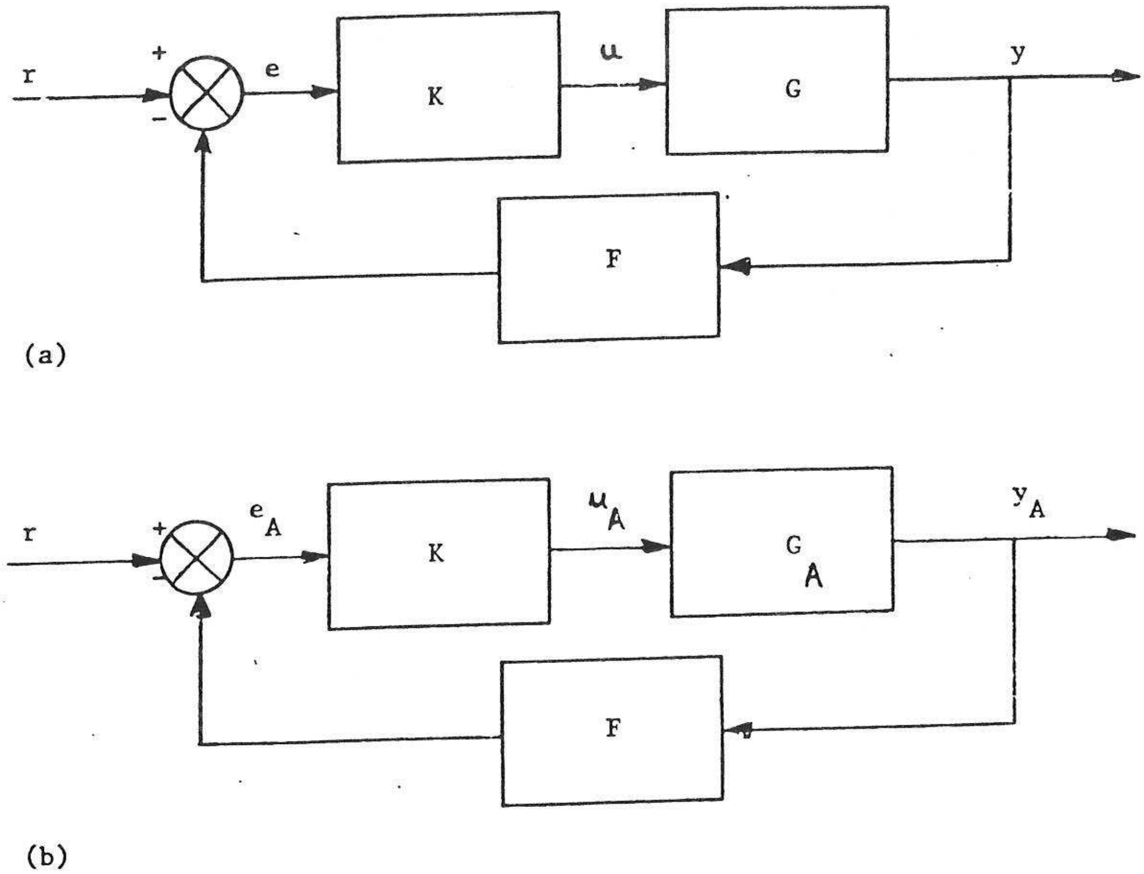


Fig. 1 (a) Real and (b) Approxmating Feedback Systems

MD1

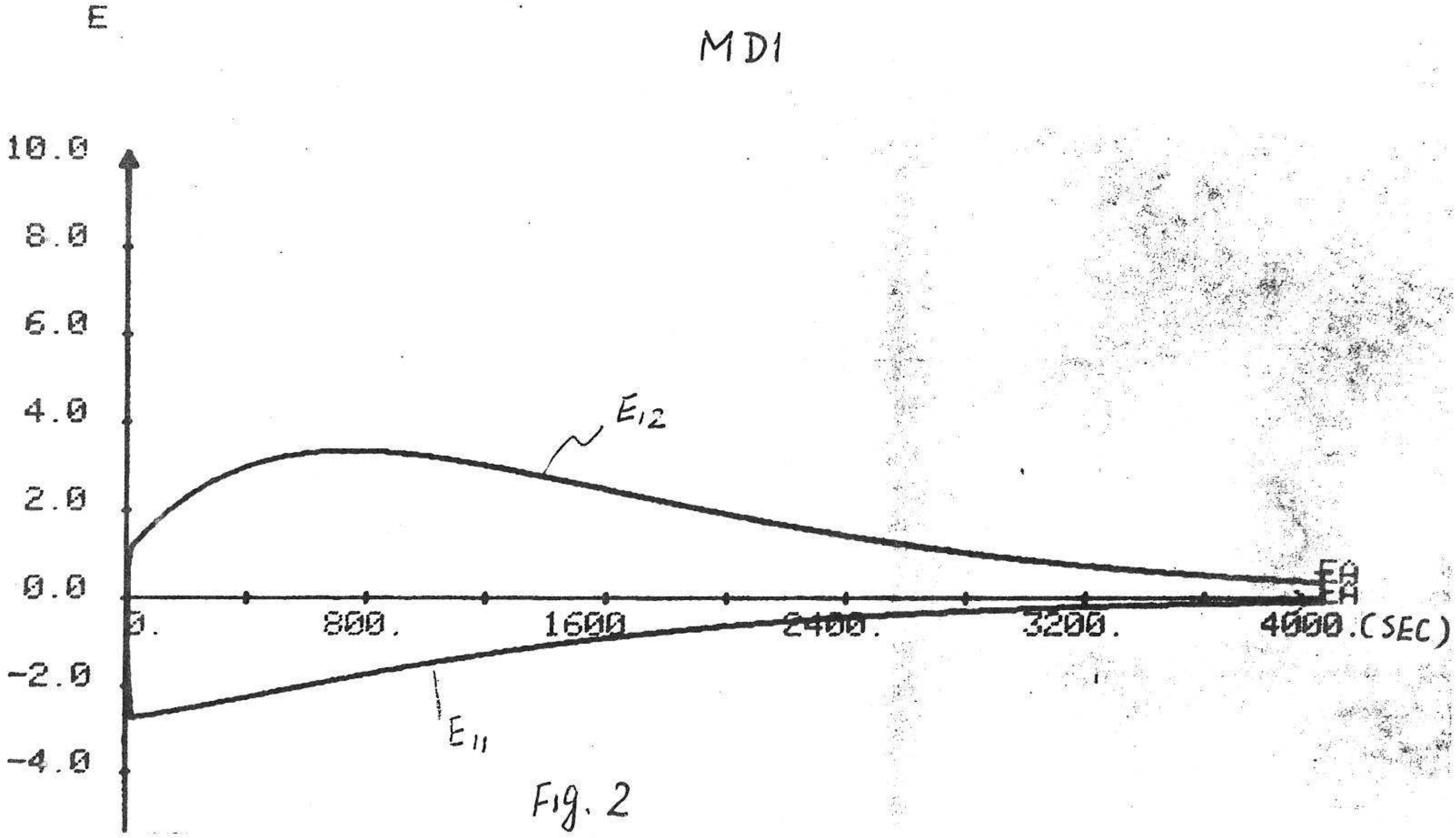


Fig. 2

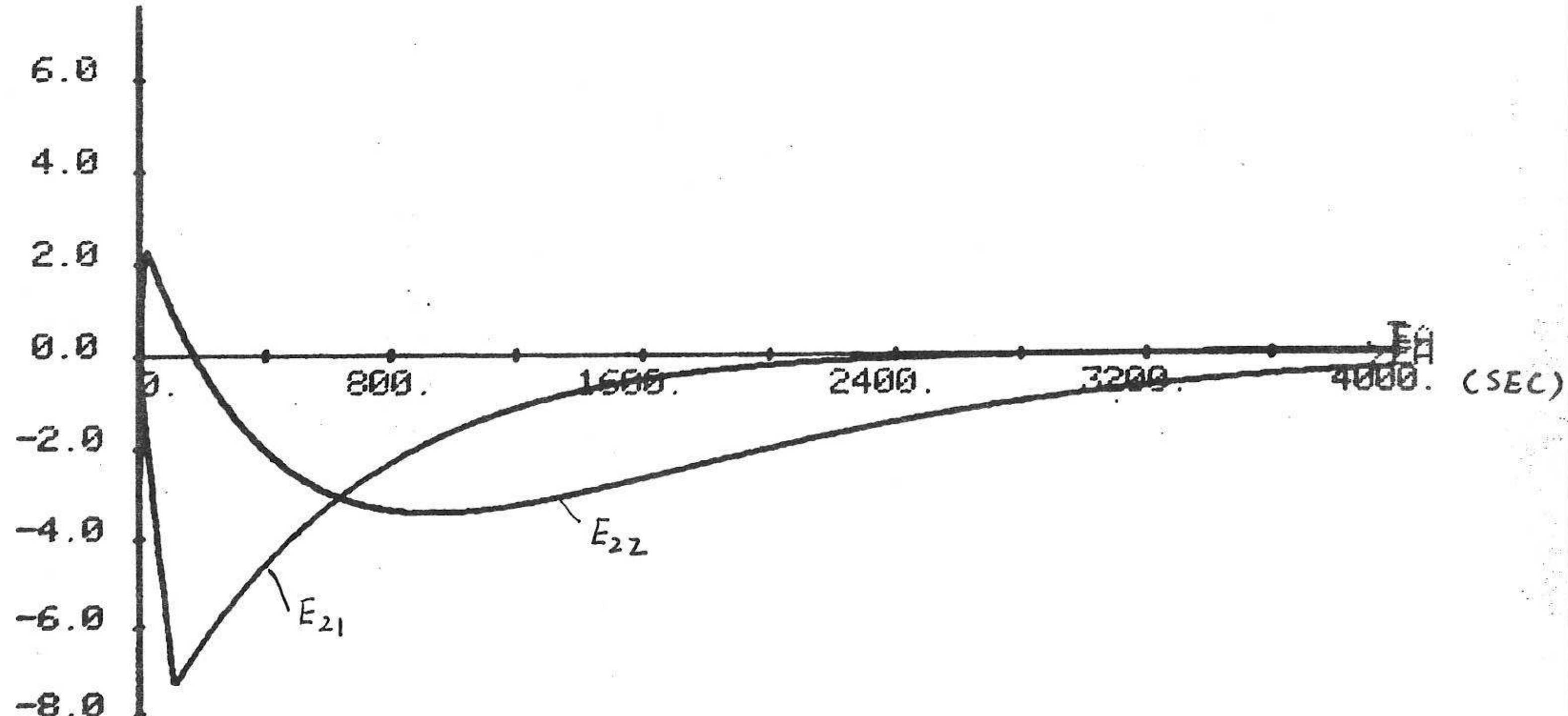


Fig. 3 RESPONSE ERROR

Y MD1 $k = 2.3$

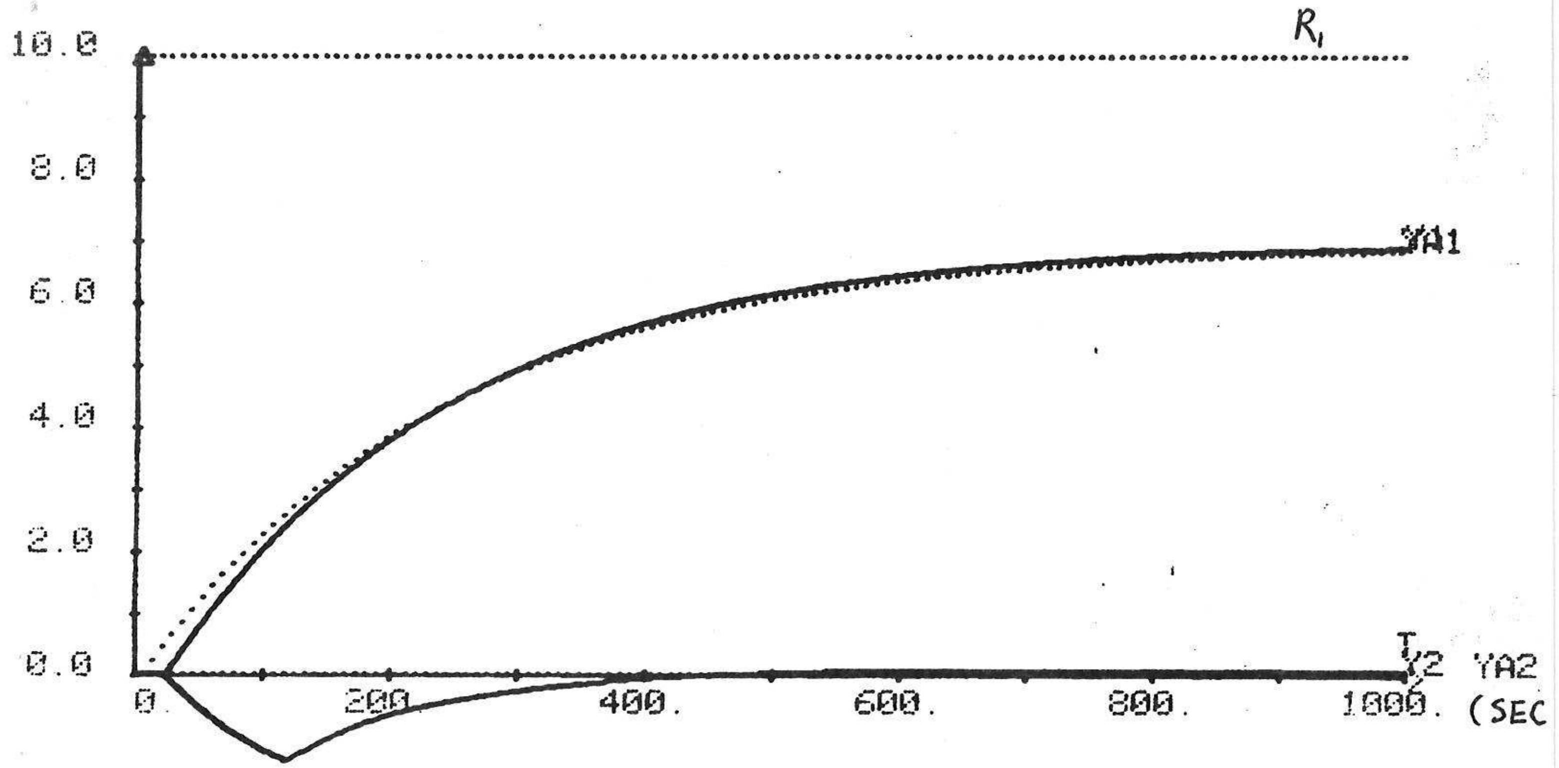
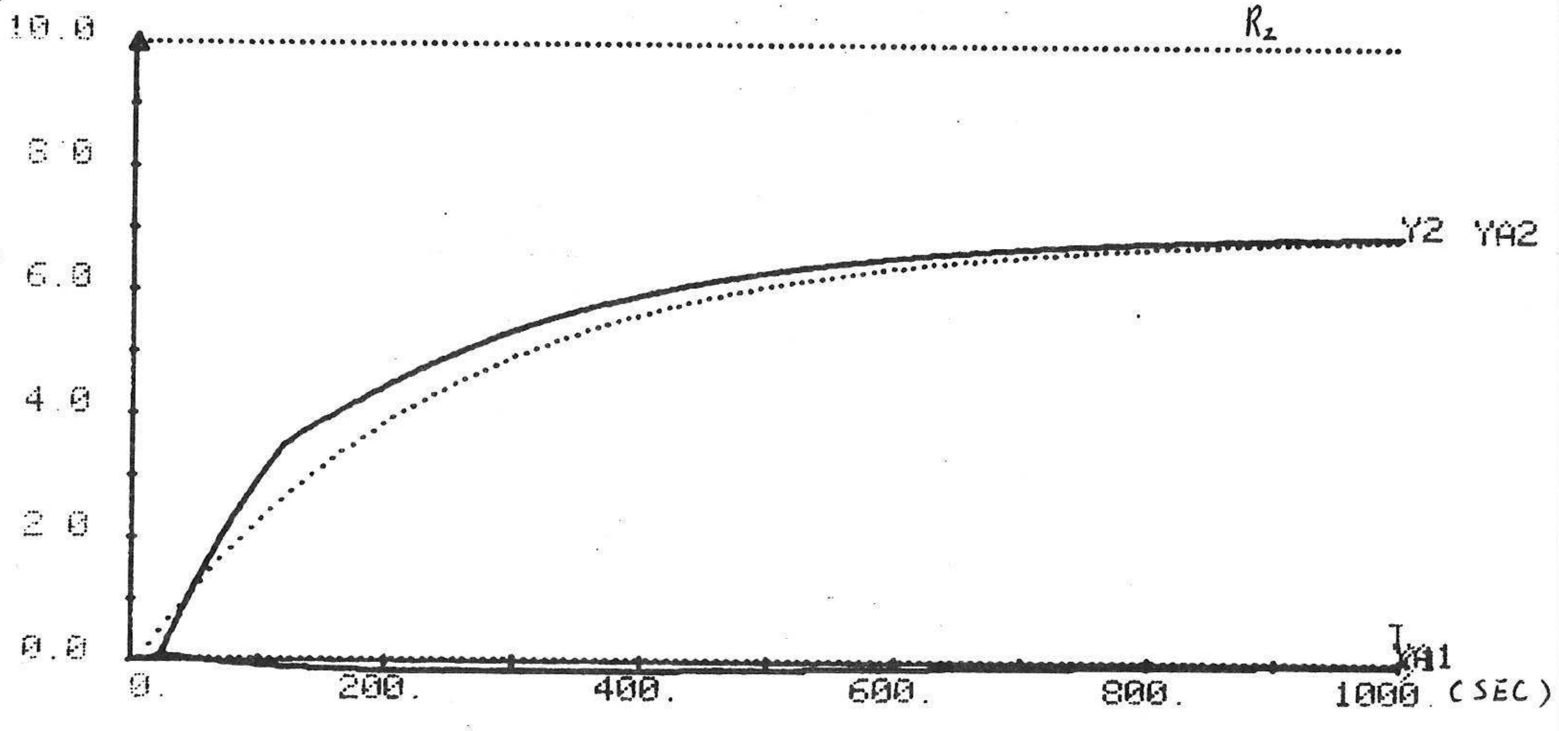


Fig. 4 CLOSED LOOP RESPONSE



. . . . Y_A

——— Y

Fig 5 CLOSED LOOP RESPONSE

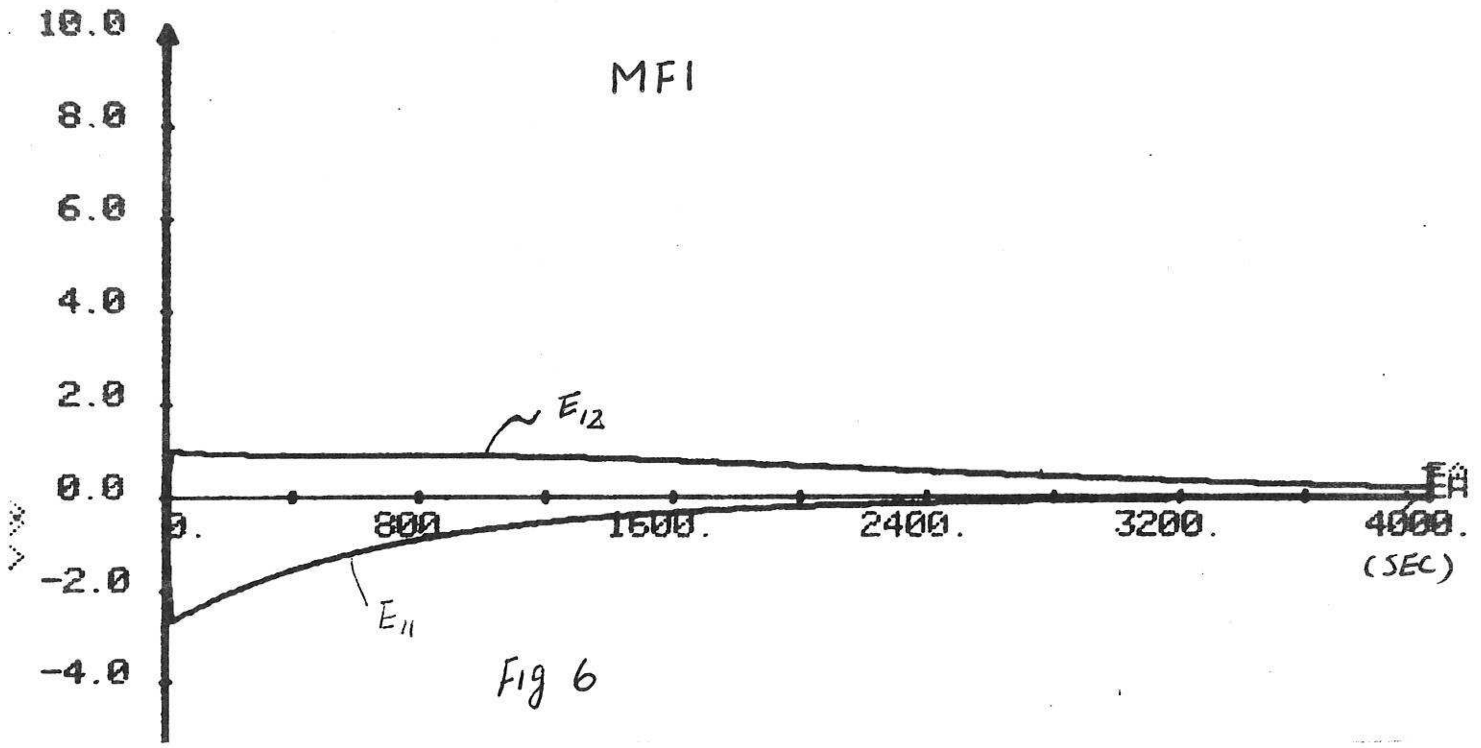


Fig 6

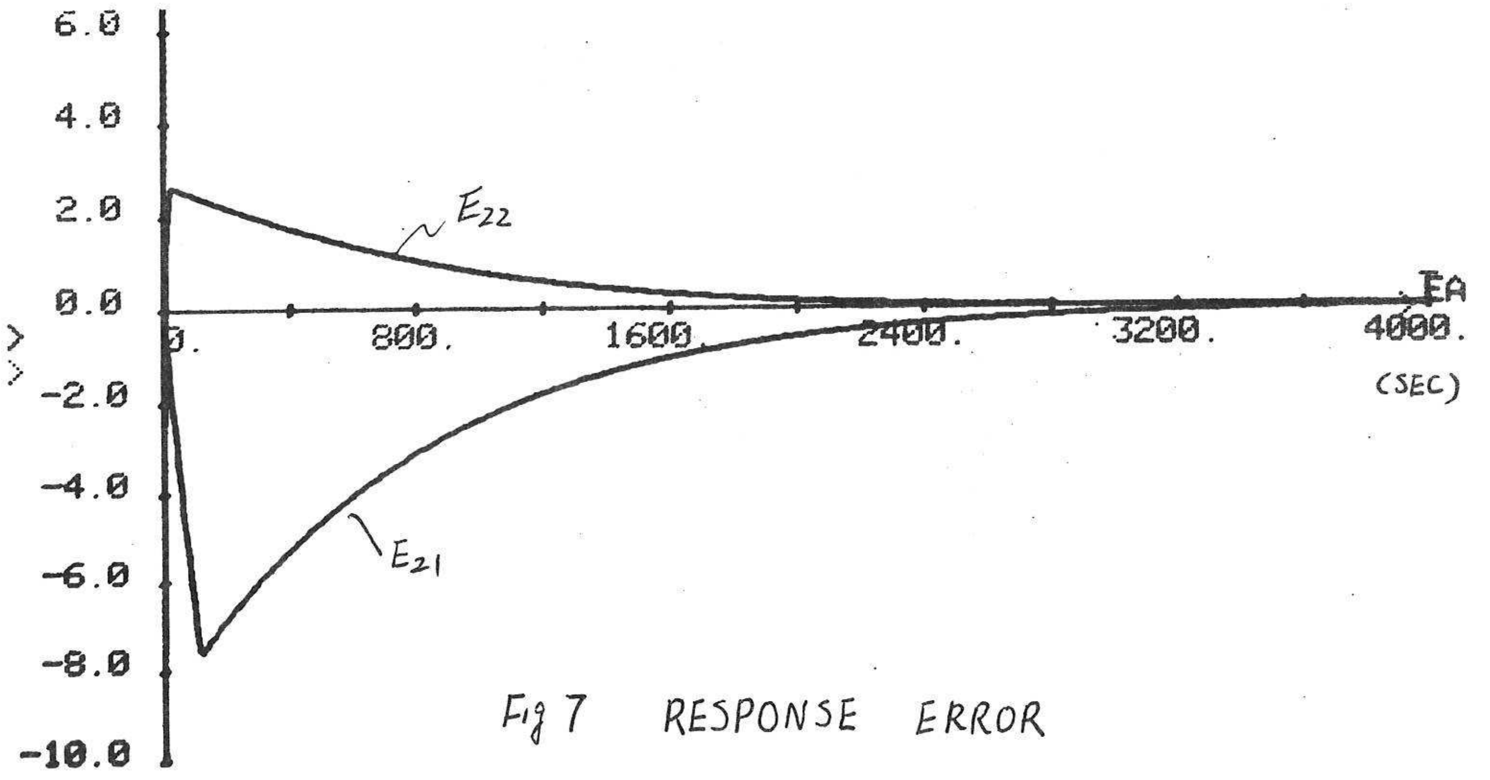


Fig 7 RESPONSE ERROR

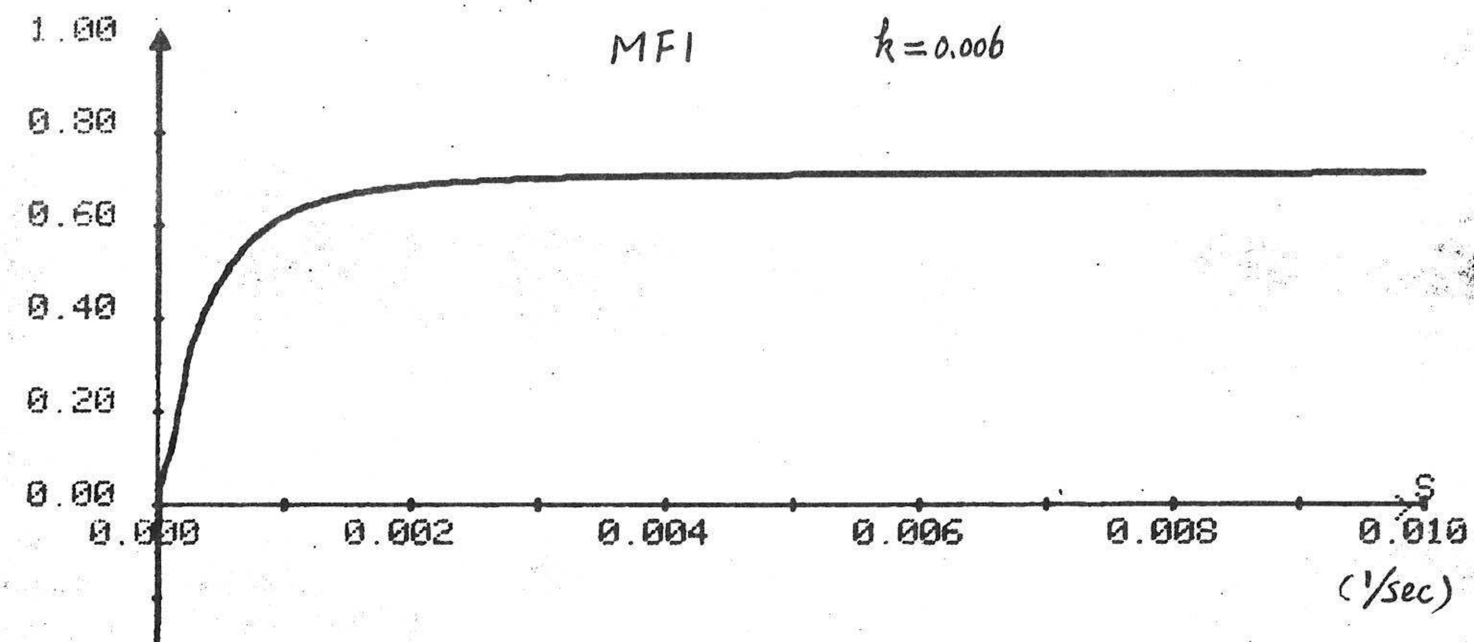


Fig 8 SPECTRAL RADIUS

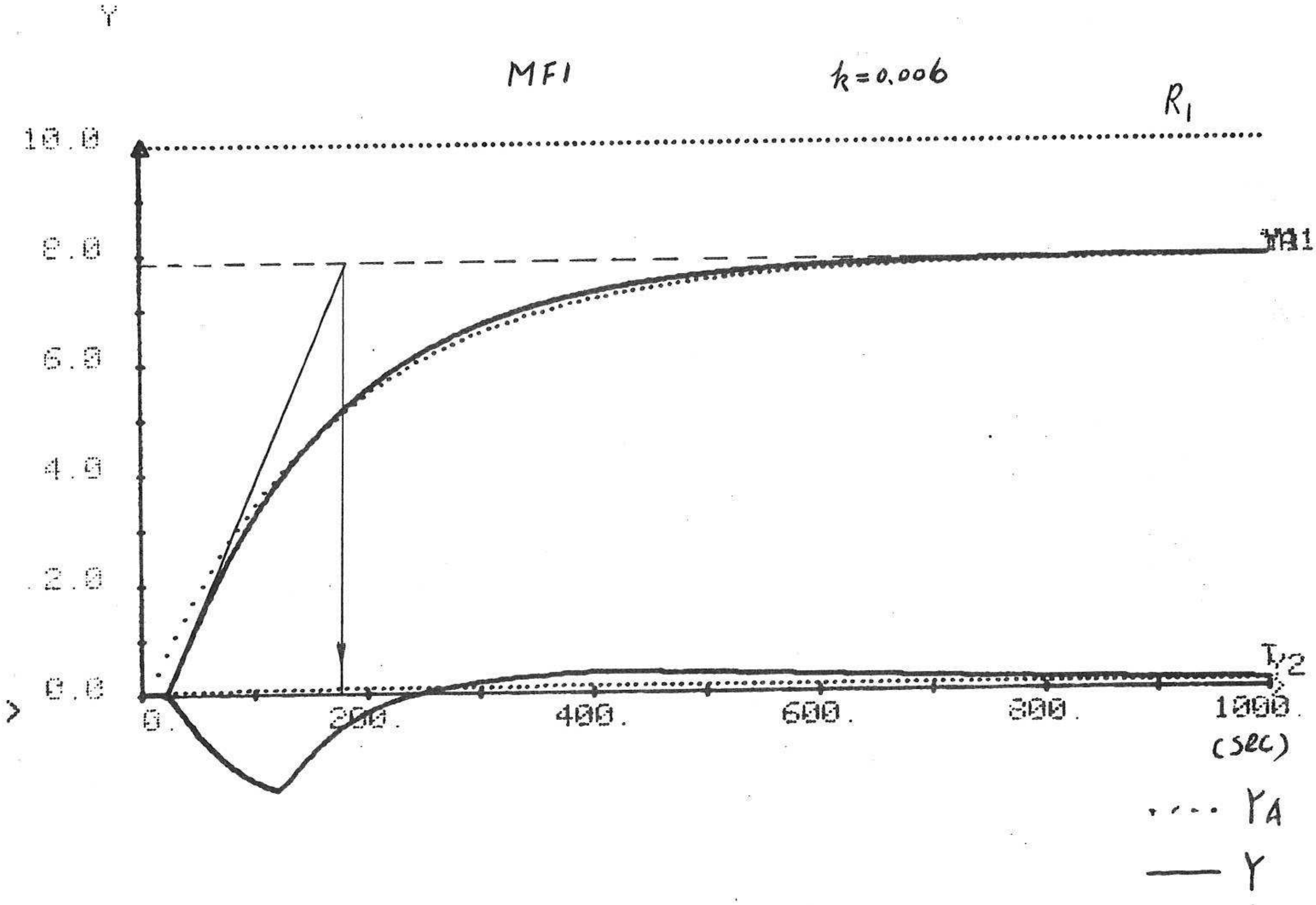


Fig 9 CLOSED LOOP RESPONSE

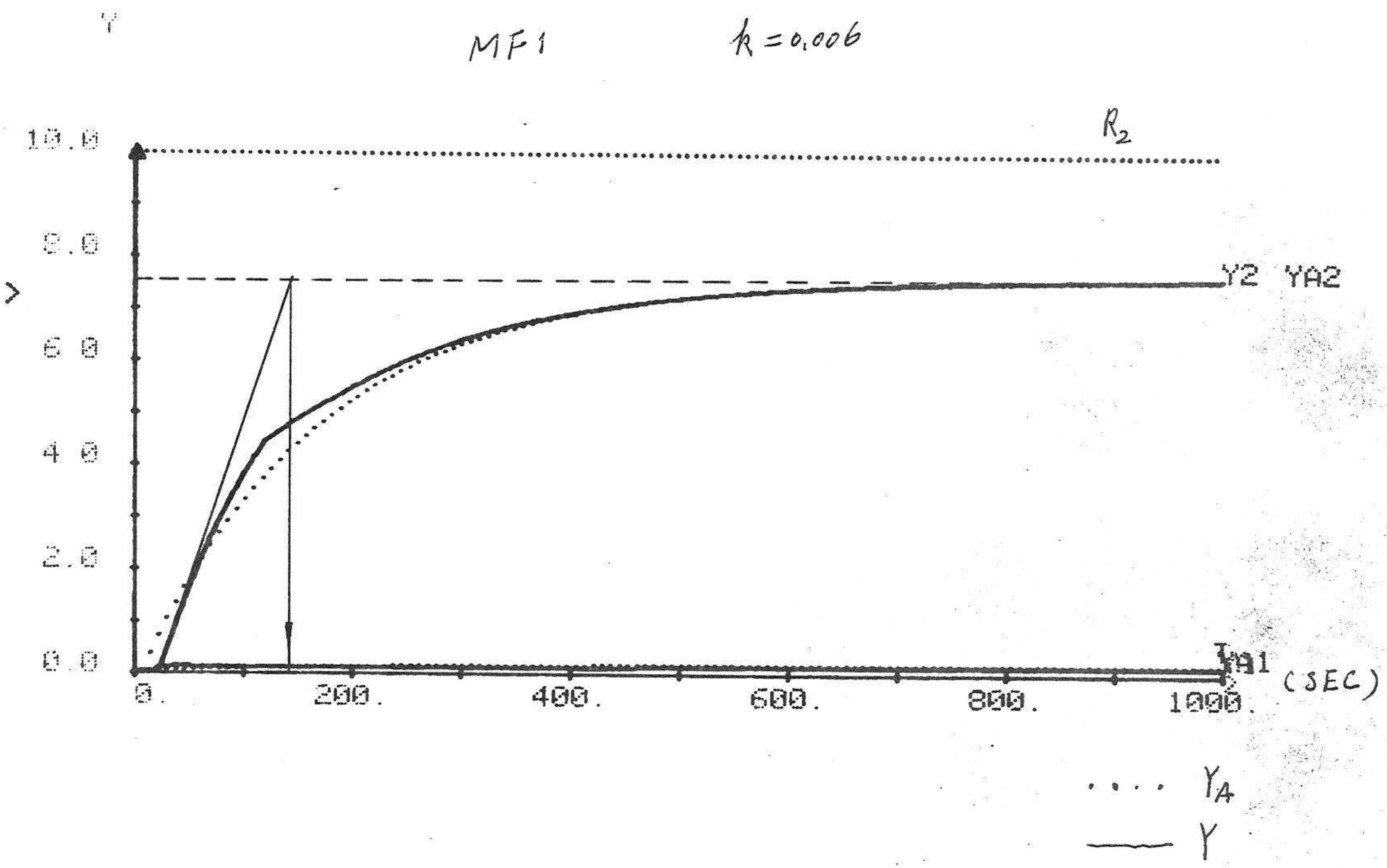


Fig 10 CLOSED LOOP RESPONSE

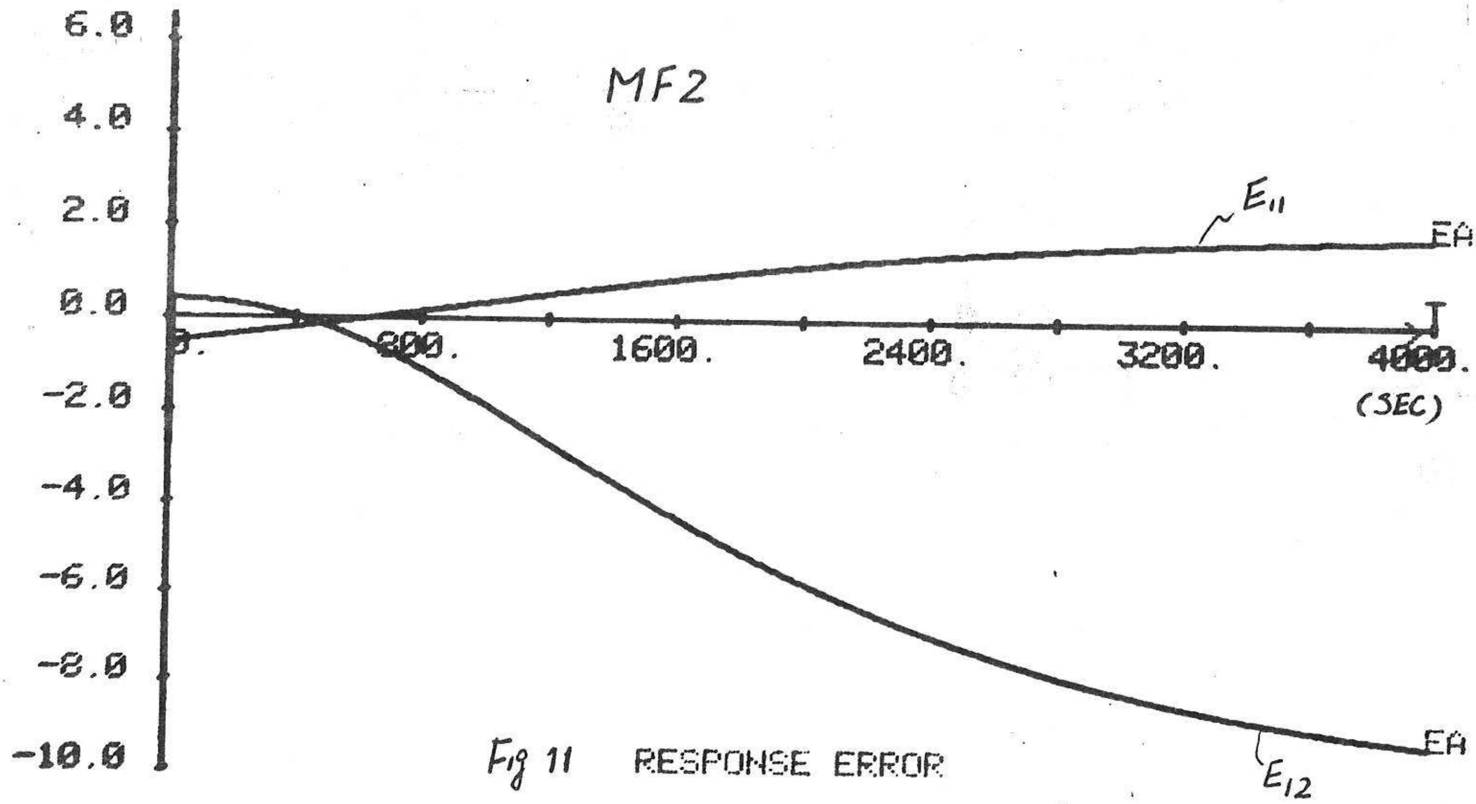


Fig 11 RESPONSE ERROR

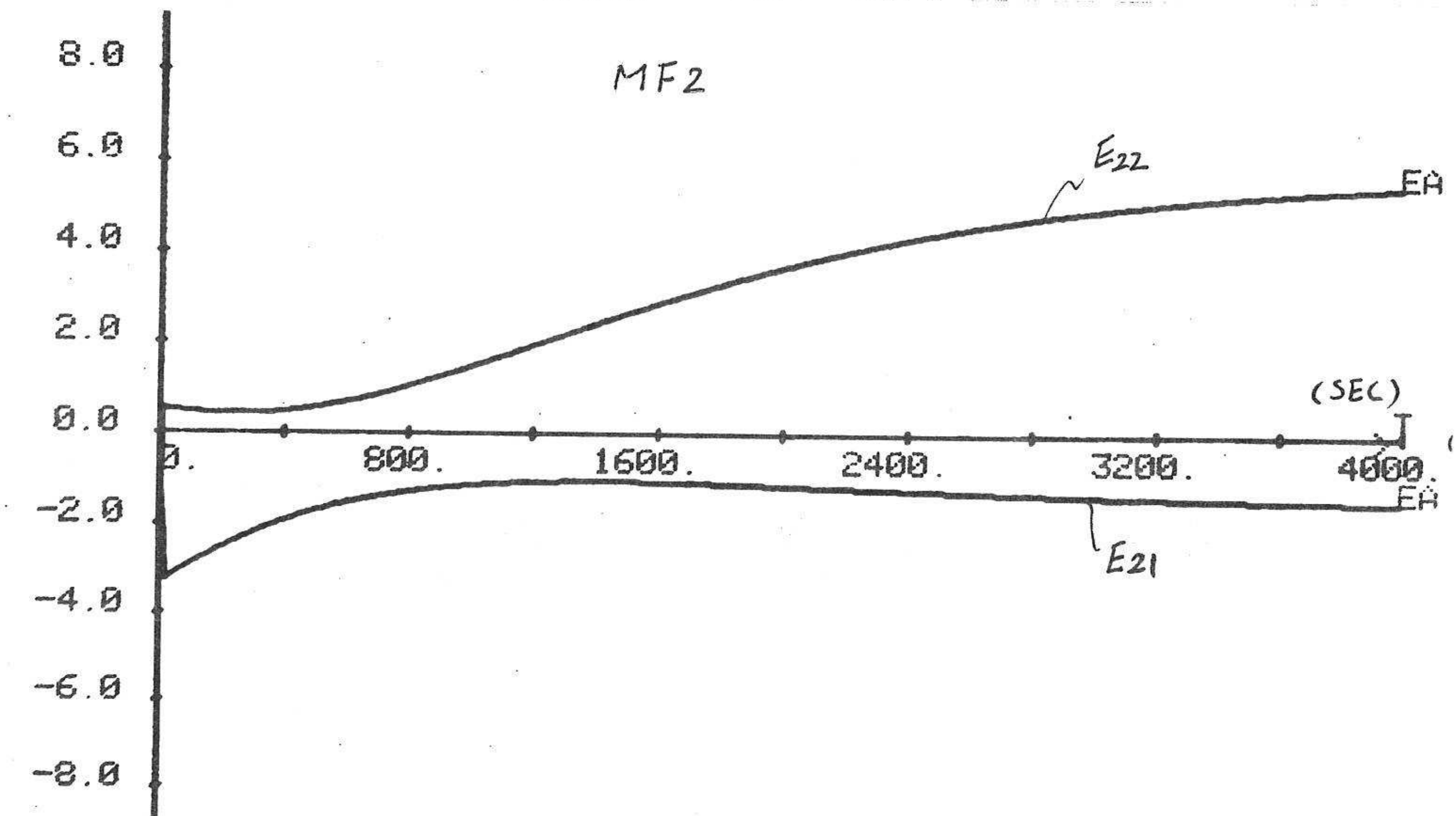


Fig 12 RESPONSE ERROR

MF 2 $k = 0{,}0037$

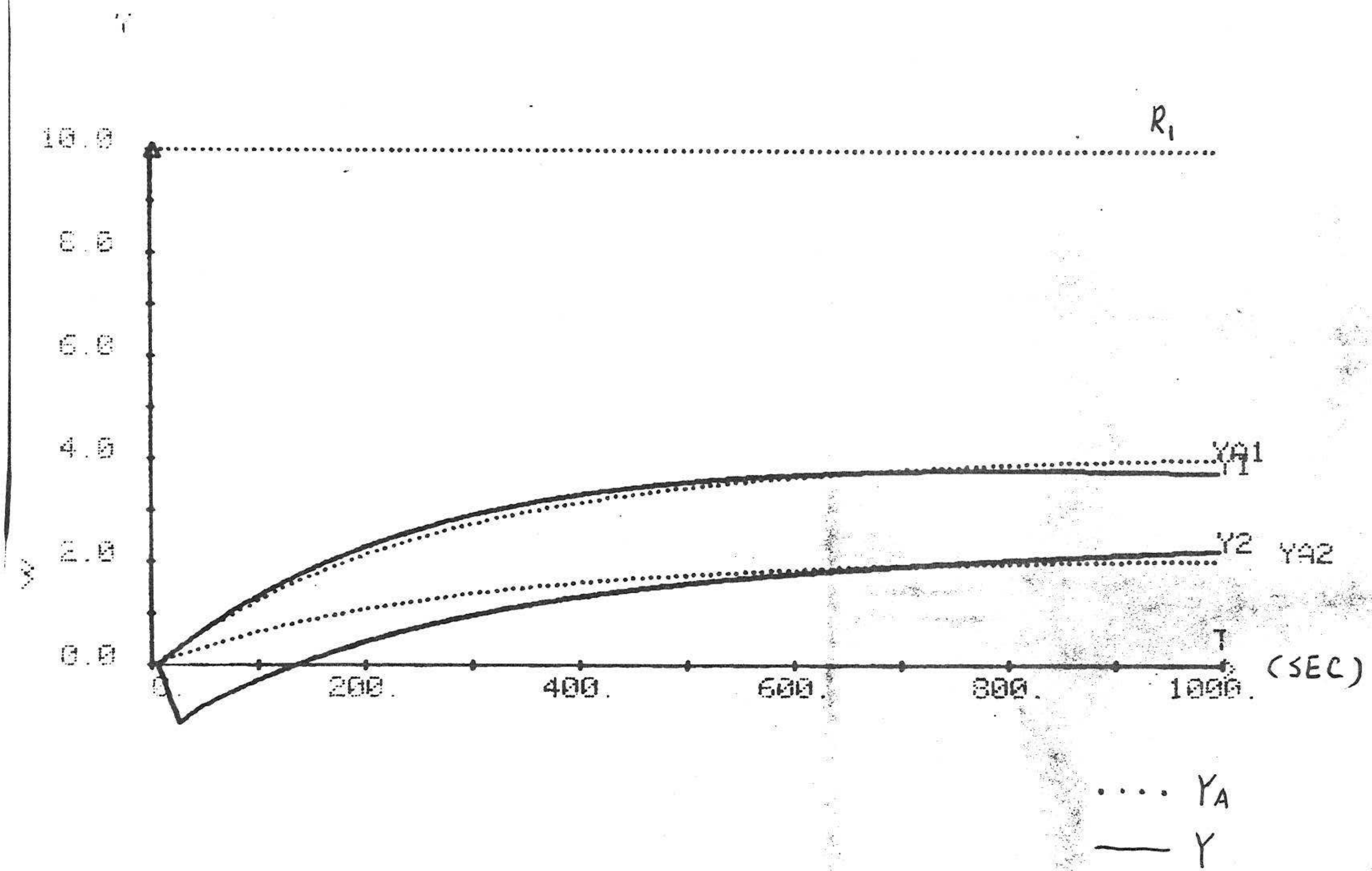


Fig 13 CLOSED LOOP RESPONSE

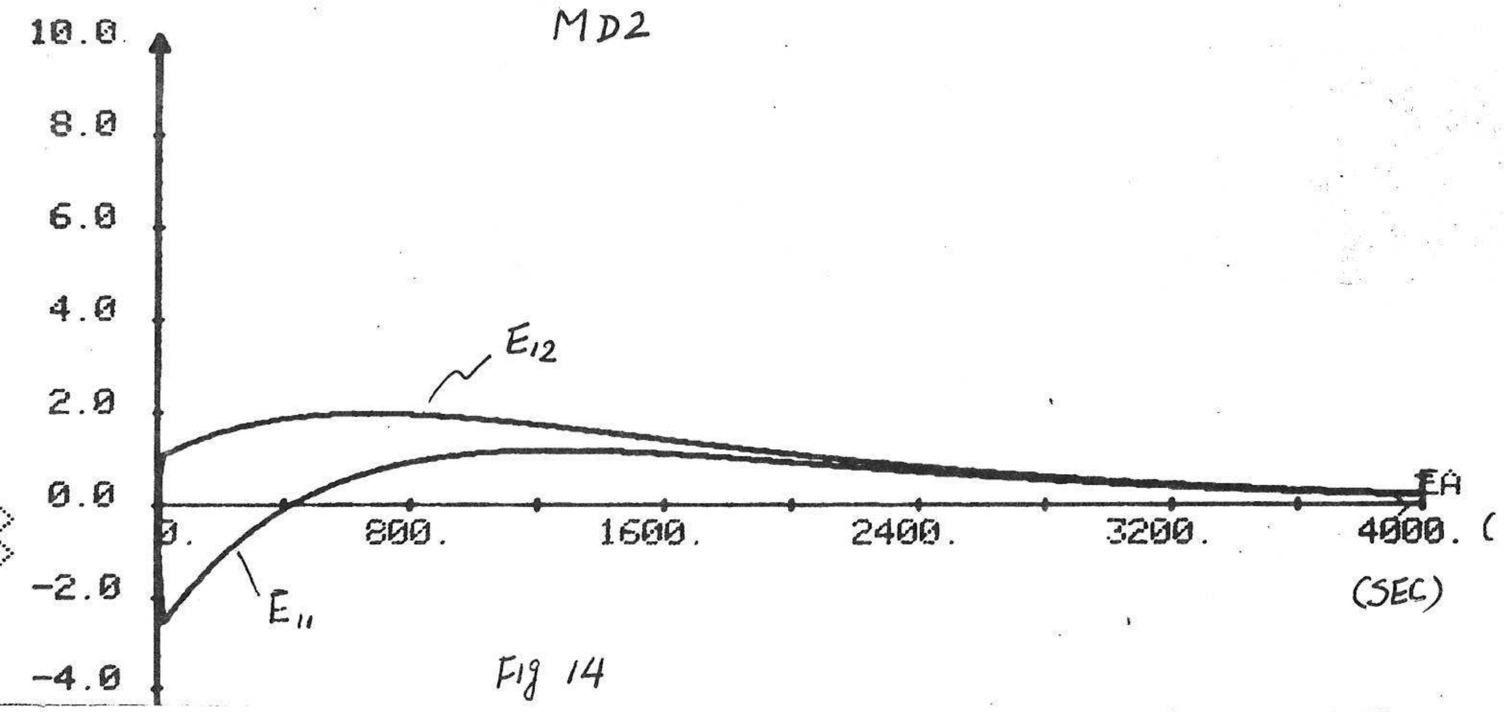


Fig 14

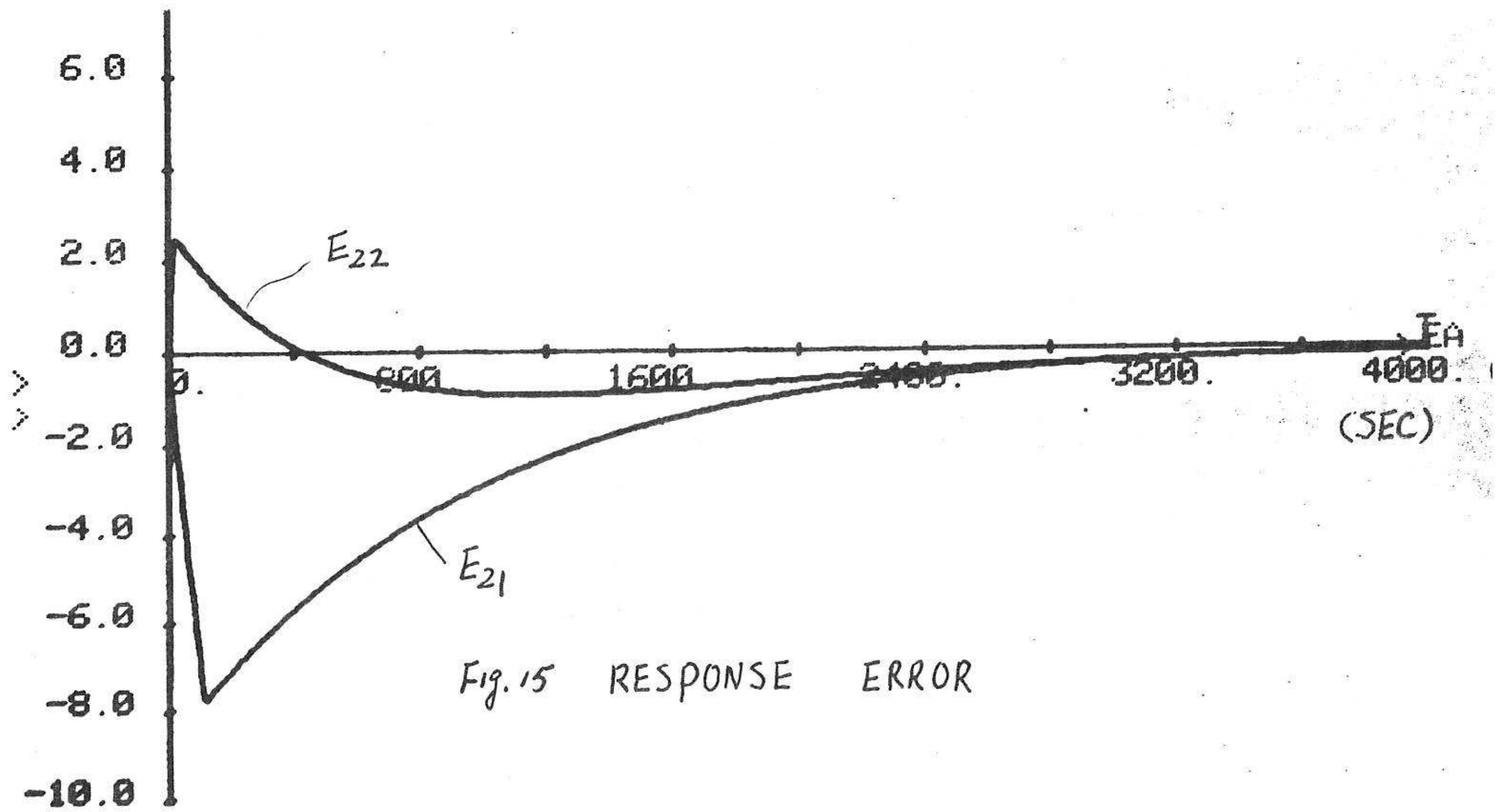


Fig. 15 RESPONSE ERROR

MD2 $k=2.8$, $c=0.002$

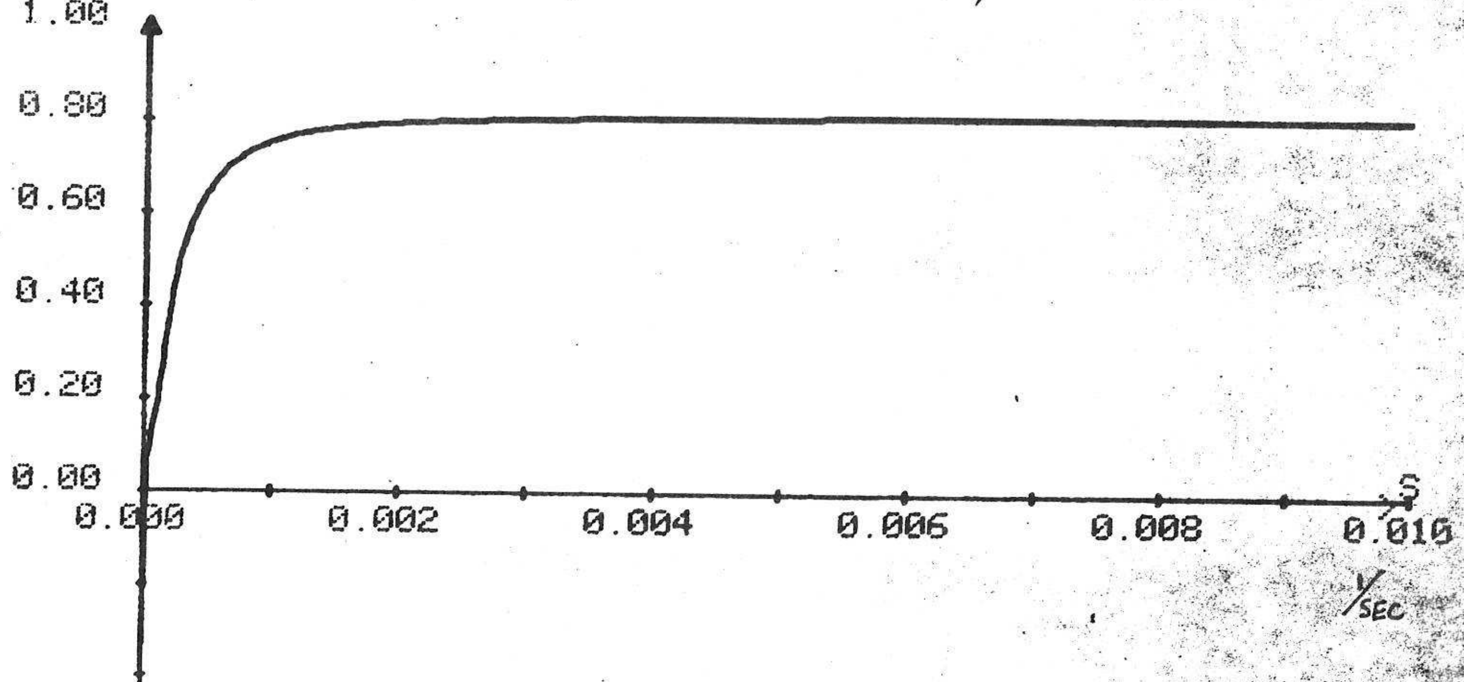


Fig.16. SPECTRAL RADIUS

MD2 $k=2.8$, $c=0.002$

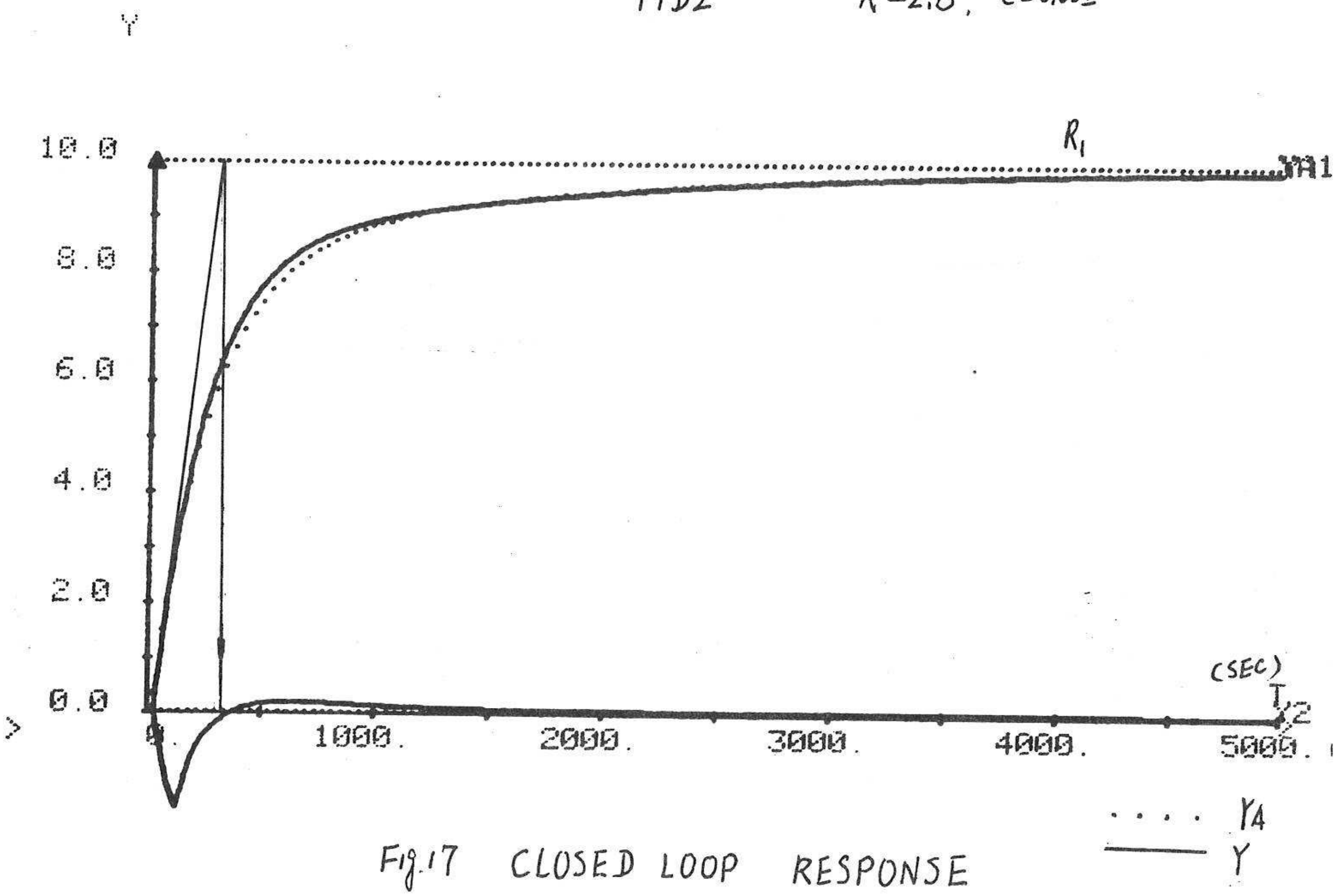


Fig.17 CLOSED LOOP RESPONSE

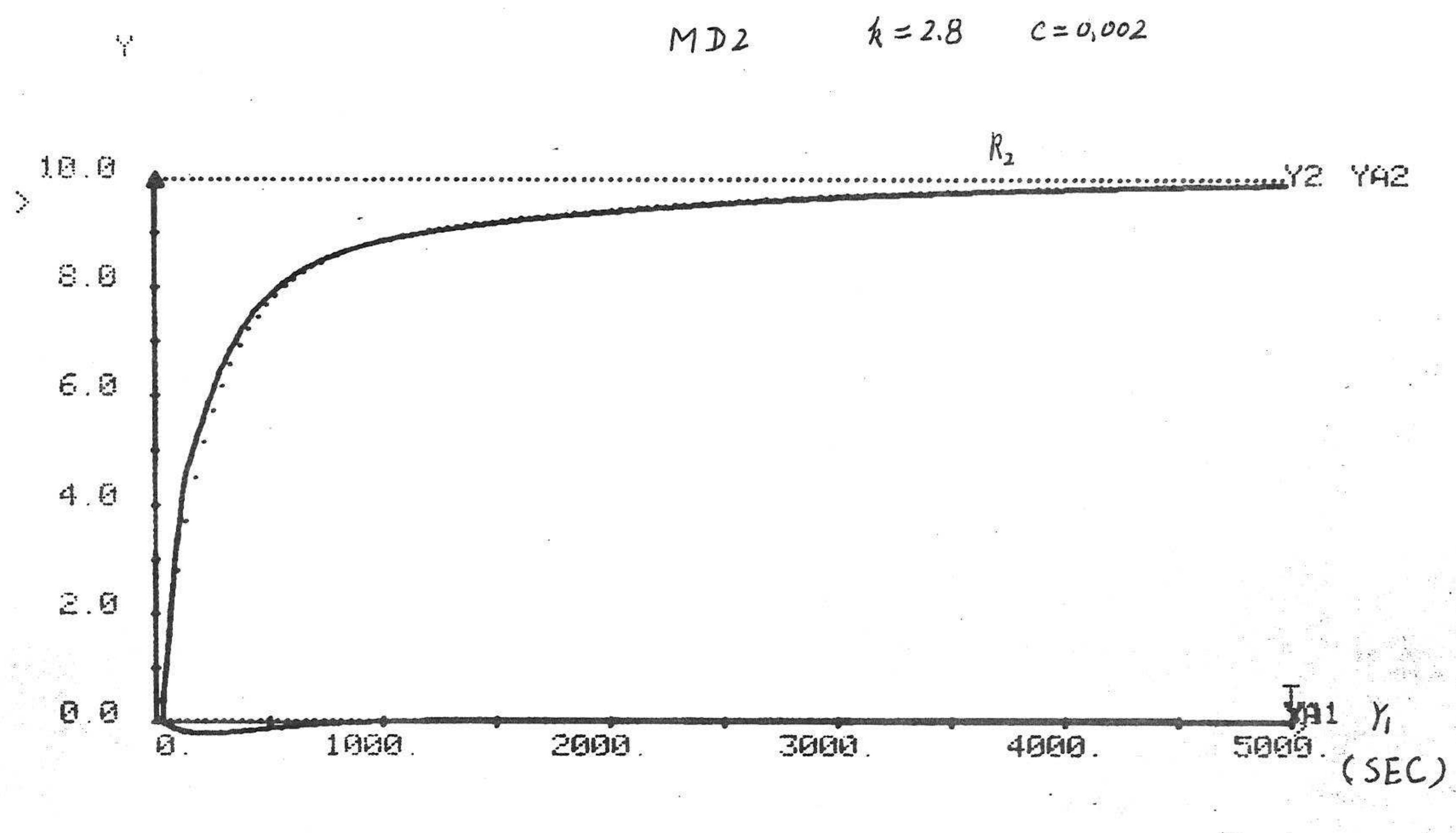


Fig. 18 CLOSED LOOP RESPONSE

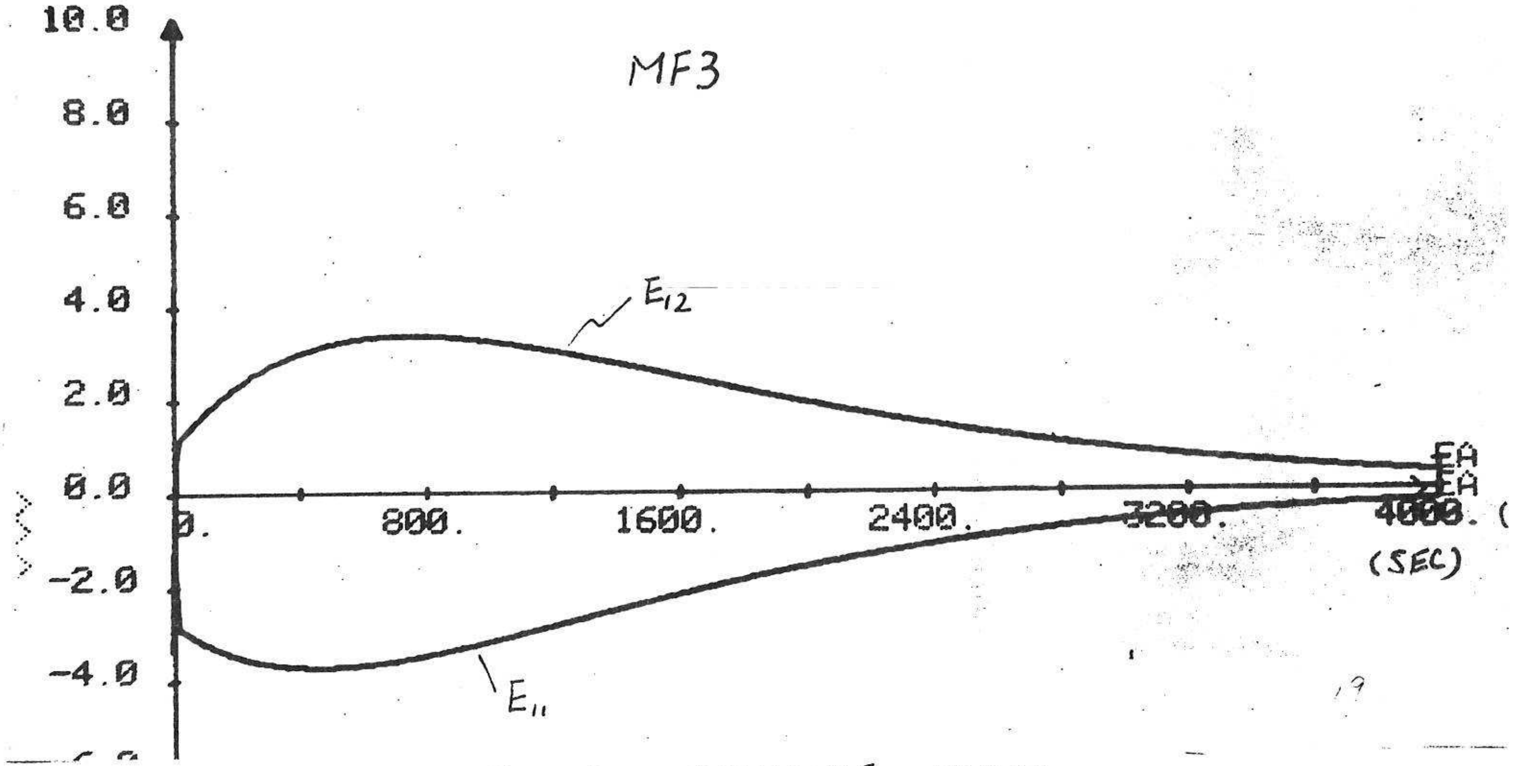


Fig. 19 RESPONSE ERROR

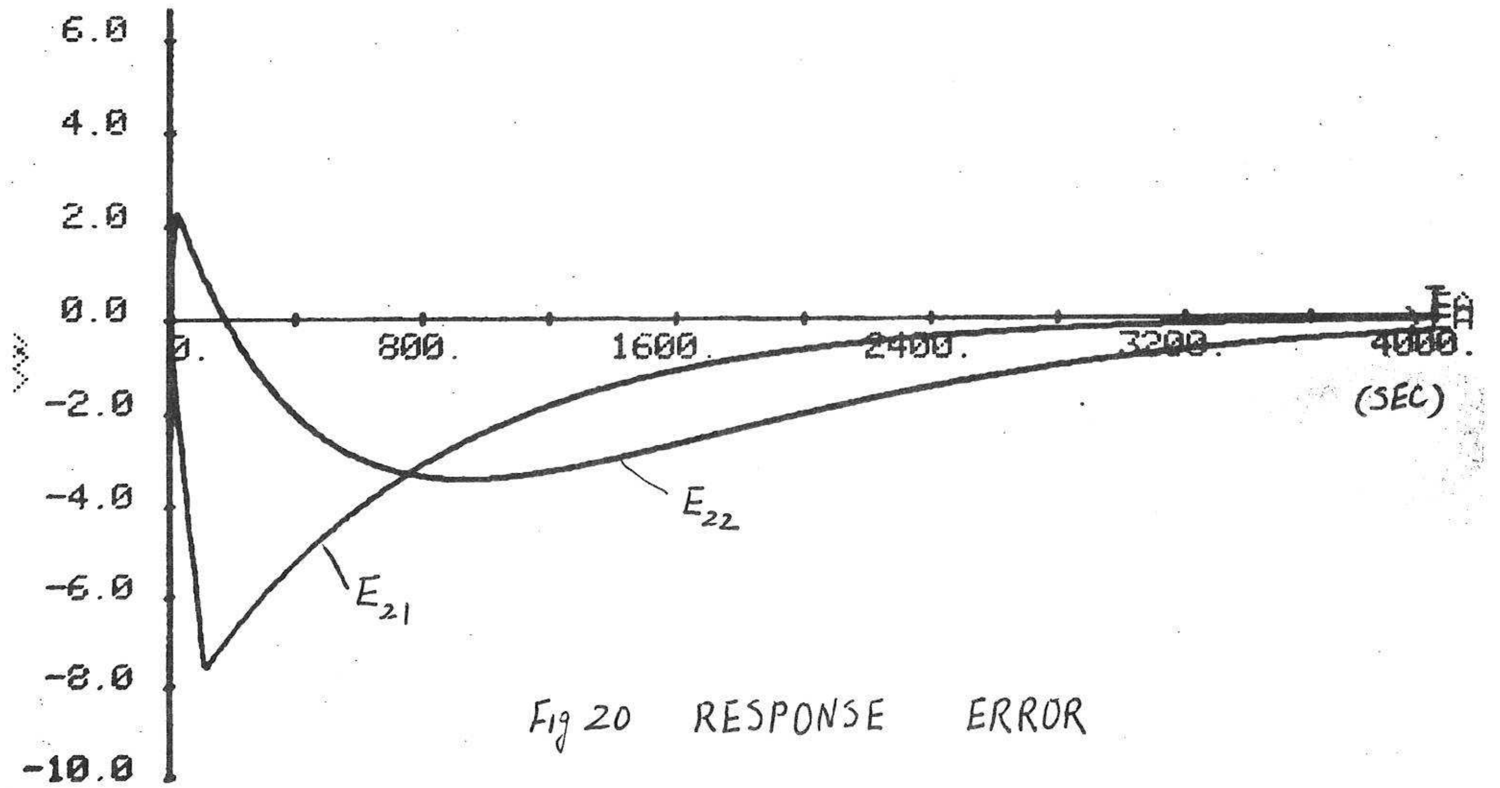


Fig 20 RESPONSE ERROR

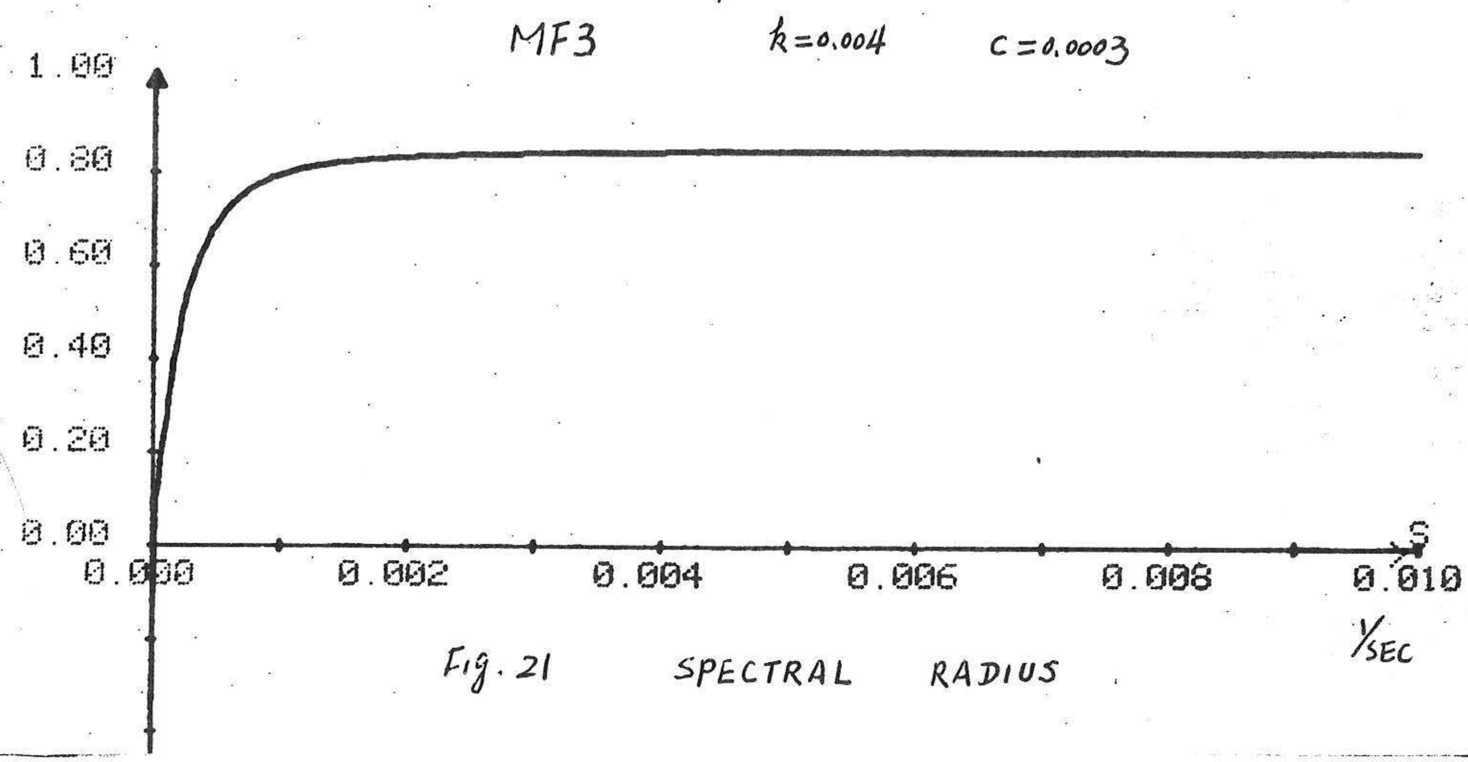


Fig. 21 SPECTRAL RADIUS

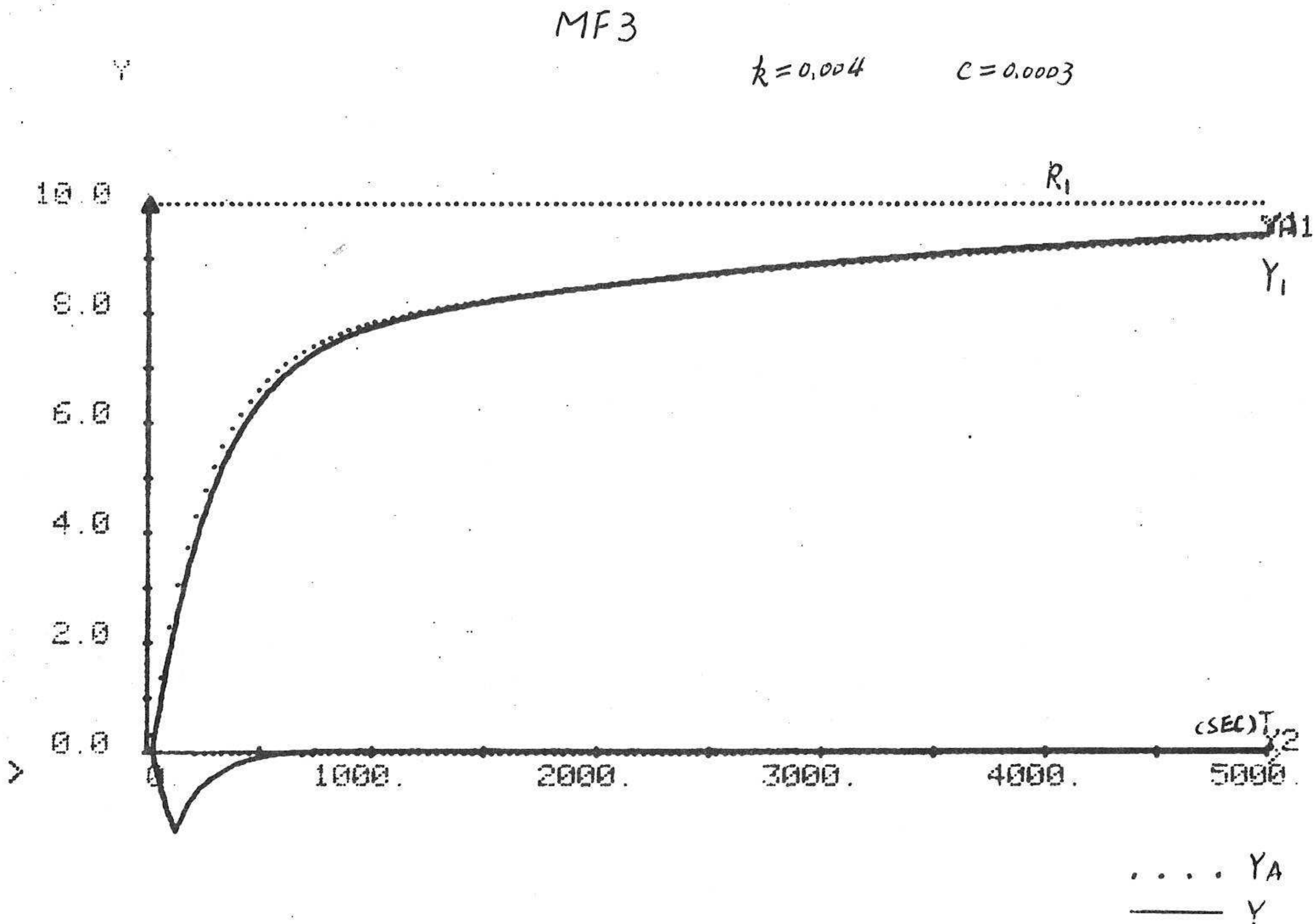


Fig. 22 CLOSED LOOP RESPONSE

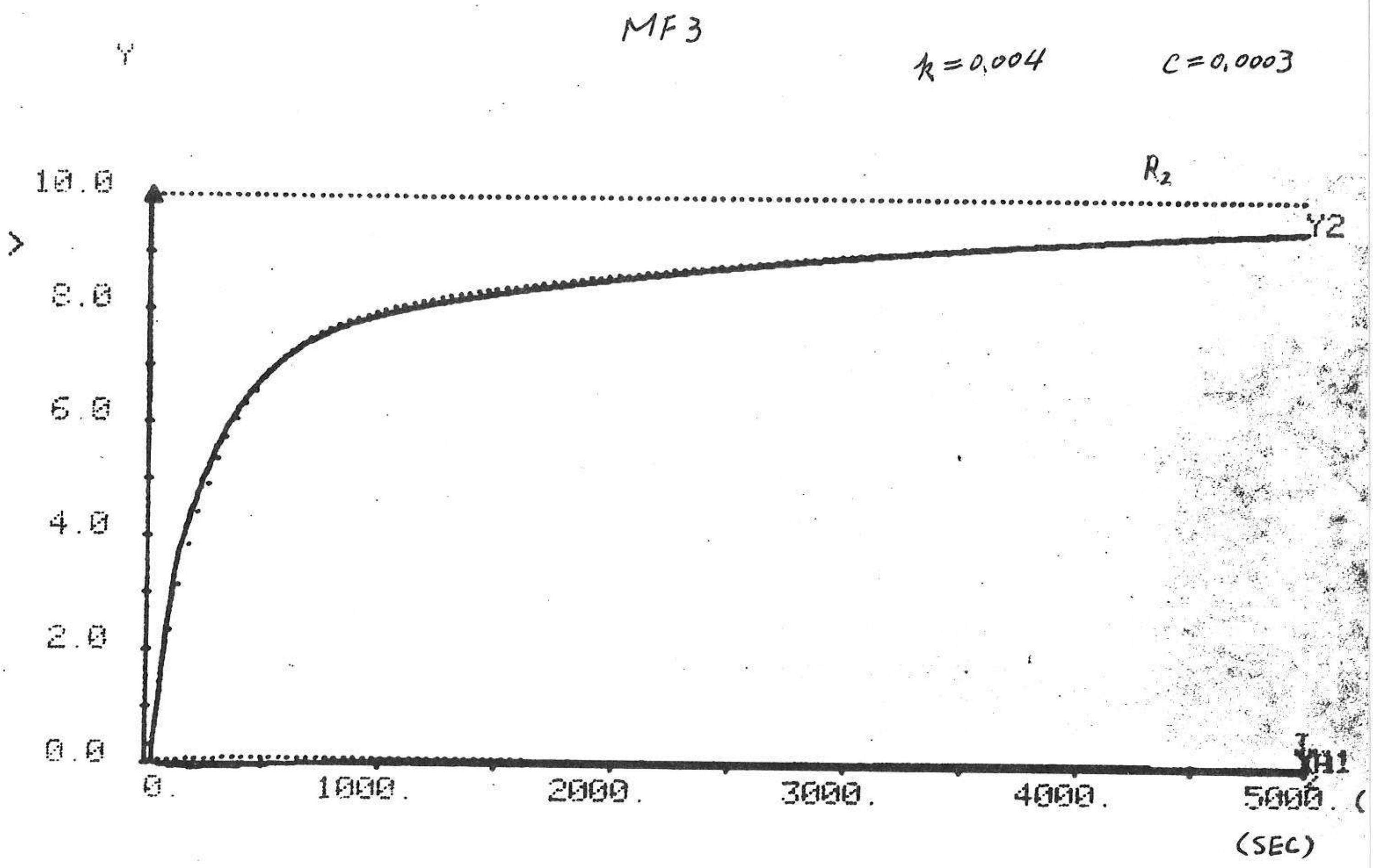


Fig. 23 CLOSED LOOP RESPONSE

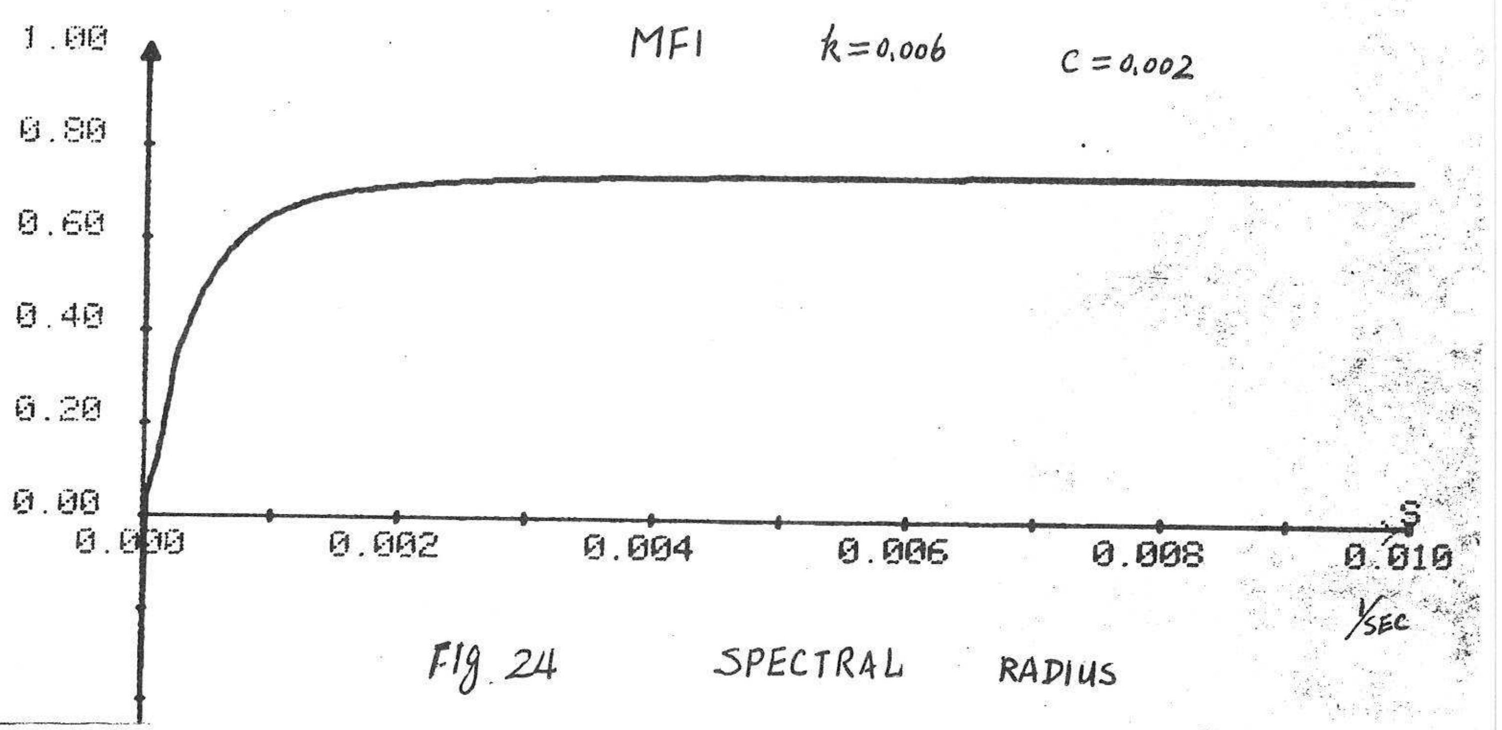


Fig. 24 SPECTRAL RADIUS

MF1 $k=0.006$ $c=0.0002$

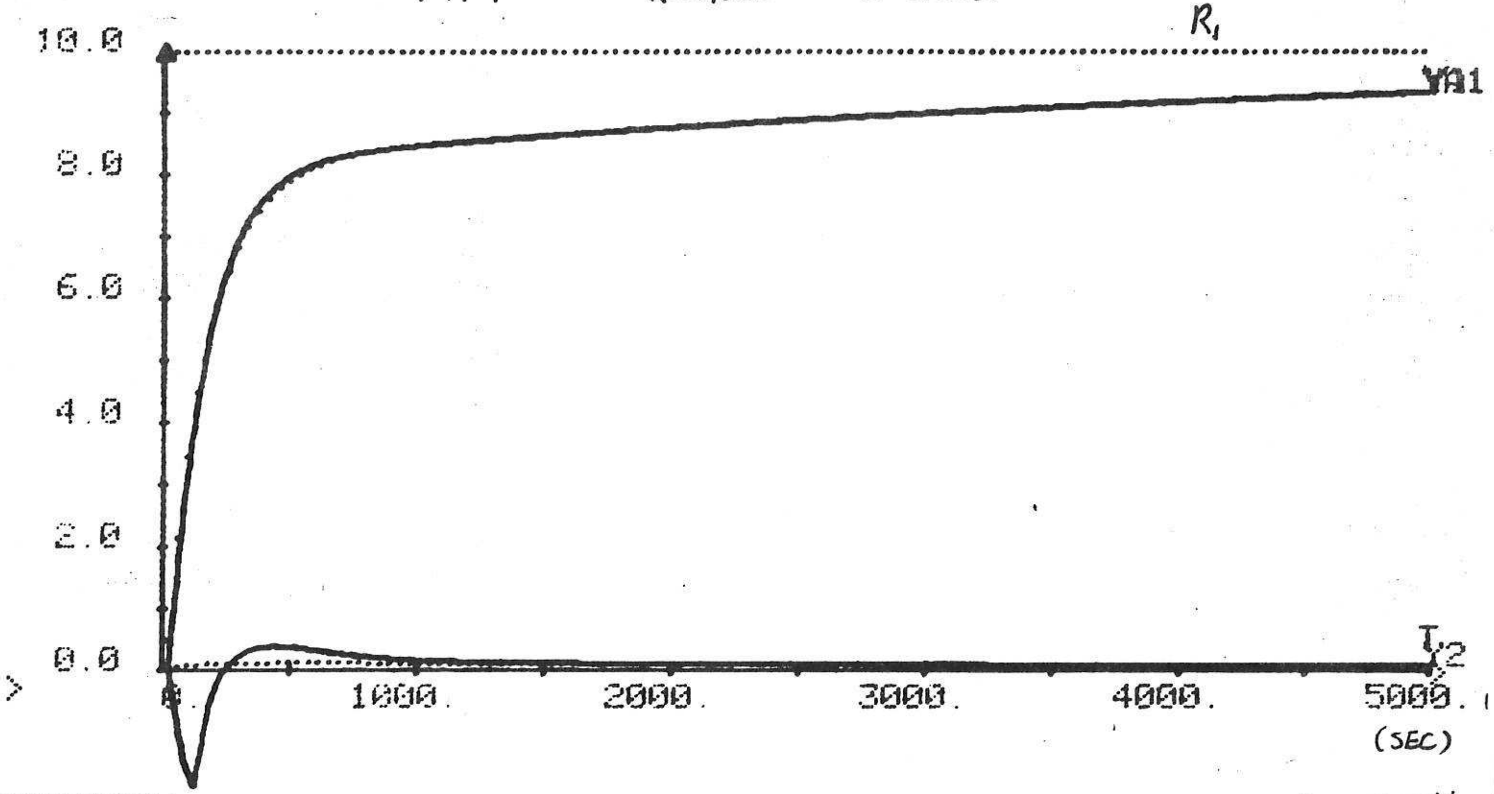


Fig 25. CLOSED LOOP RESPONSE

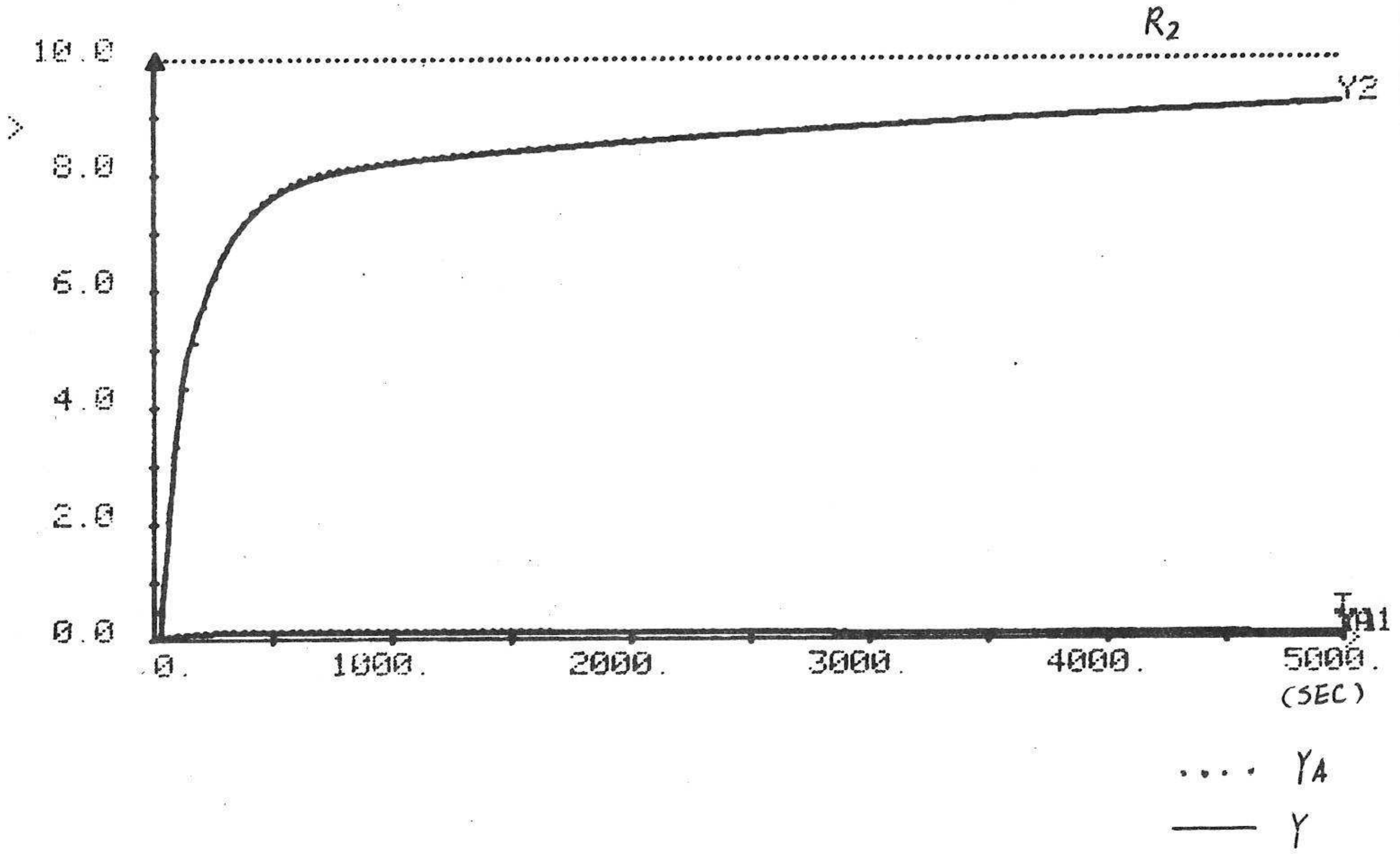


Fig. 26 CLOSED LOOP RESPONSE

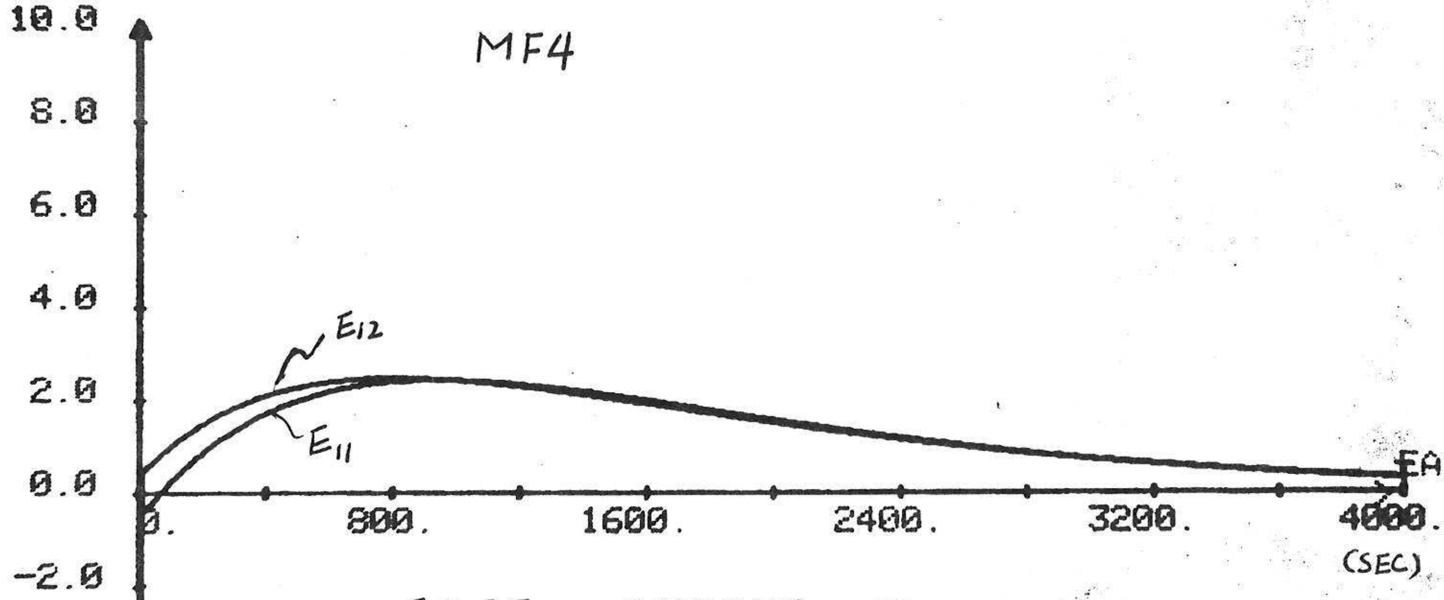


Fig 27 RESPONSE ERROR

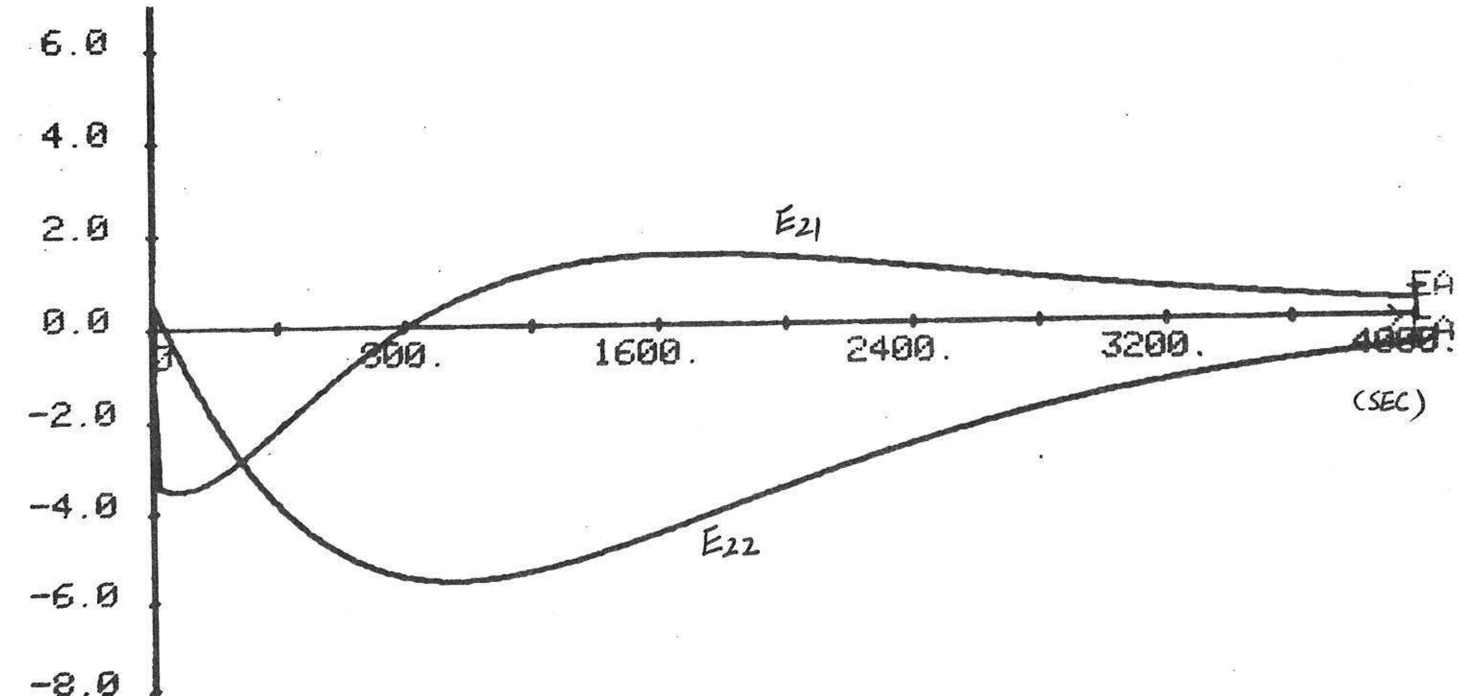


Fig.28 RESPONSE ERROR