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DYADIC EXPANSIONS AND MULTIVARIABLE FEEDBACK DESIGN

by

D.H. Owens B.Sc., A.R.C.S., Ph.D., A.F.I.M.A., C.Eng., M.I.E.E.

Department of Control Engineering,
University of Sheffield,
Mappin Street, Sheffield S1 3JD.

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DYADIC EXPANSIONS AND MULTIVARIABLE FEEDBACK DESIGN

Consider an m-input/m-output linear, time-invariant system described by the mxm transfer function matrix G(s) and the design of a unity negative feedback system of the form indicated in Fig. 1 to produce desired stability and performance characteristics. The forward path control system has mxm transfer function matrix denoted by K(s).

Frequency domain design techniques for such feedback schemes (Rosenbrock, 1974; Owens, 1978; MacFarlane, 1980; Patel and Munro, 1982), are, almost without exception, based upon the reduction of the design process to the analysis of m non-interacting single-input/single;output systems whose closed-loop stability and performance characteristics are manipulated to produce the required stability and performance for the real multivariable plant. For example, the direct and inverse Nyquist array design techniques use the notion of precompensation to achieve the system state of diagonal dominance where interaction effects can be ignored in stability assessment. The characteristic locus design methods achieve similar objectives by transforming the system transfer function matrix to exactly diagonal form using similarity transformations. The use of dyadic expansions in design (Owens, 1975, 1978, 1979) has its origins and physical motivations in the field of nuclear reactor spatial control design (Owens, 1973a, 1973b) where the identified scalar systems can be associated with well-defined modes of system behaviour. It does however apply quite generally and is related to the inverse Nyquist array and characteristic locus methodologies in that it achieves a reduction of the design process to m scalar designs using a combination of diagonal dominance and equivalence transformation ideas. The simplest situation where dyadic concepts are of value lies in the consideration of systems with dyadic transfer function matrices.

Dyadic Systems

A system with mxm transfer function matrix G(s) is dyadic if (Owens 1978, 1979) it has a dyadic transfer function matrix

$$G(s) = P_1 \begin{pmatrix} g_1(s) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & g_m(s) \end{pmatrix} P_2 \qquad (1)$$

expressed as the series connection of the non-interacting system diag $\{g_j(s)\}_{1\le j\le m}$ and the constant matrices P_1, P_2. Without loss of generality (Owens 1979) it can be assumed that the pair (P_1, P_2) is permissible in the sense that both matrices are nonsingular and, writing P_1 and P_2 in the form

$$P_1 = [\alpha_1, \ldots, \alpha_m] \quad , \quad P_2 = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} \quad , \qquad (2)$$

there is a permutation ℓ of the integers $\{1,2,\ldots,m\}$ such that

$$\bar{\alpha}_j = \alpha_{\ell(j)} \quad , \quad \bar{\beta}_j = \beta_{\ell(j)} \quad , \quad 1 \le j \le m \qquad (3)$$

$$\ell(\ell(j)) = j \quad , \quad 1 \le j \le m \qquad (4)$$

i.e. the rows (resp. columns) of P_2 (resp P_1) exist in similar complex conjugate pairs. A consequence of this observation is that the scalar systems $\{g_j(s)\}_{1\le j\le m}$ may contain complex coefficients but satisfy the 'reality constraint'

$$\overline{g_j(s)} \equiv g_{\ell(j)}(\bar{s}) \quad , \quad 1 \le j \le m \qquad (5)$$

The block diagram interpretation of a dyadic system is illustrated in Fig. 2 where it is seen that the transfer function matrix relating

the 'internal inputs' $\hat{u} = P_2 u$ to
the 'internal outputs' $\hat{y} = P_1^{-1} y$ is non-interacting. The interaction effects in G can be described in terms of the non-dynamic elements P_1 and P_2 only, the interactions in the plant being the result of the choice of inputs u and outputs y for system description rather than the 'natural' choice of input $\hat{u}$ and output $\hat{y}$. Examples of dyadic systems include the 2x2 systems

$$G(s) = \frac{1}{(s+1)^2} \begin{pmatrix} 1-s & 2-s \\ \frac{1}{3} - s & 1-s \end{pmatrix}$$

$$\equiv \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{(s+1)} & 0 \\ 0 & \frac{1}{(s+1)^2} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ \frac{2}{3} & 1 \end{pmatrix} \qquad (6)$$

and

$$G(s) = \begin{pmatrix} G_1(s) & G_2(s) \\ G_2(s) & G_1(s) \end{pmatrix}$$

$$\equiv \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} G_1(s) + G_2(s) & 0 \\ 0 & G_1(s) - G_2(s) \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad (7)$$

In both cases the dyadic structure is not obvious by inspection of elements of G. The dyadic structure of G lies deeper than the mere numerical structure as illustrated by (7) where G is dyadic independent of the detailed dynamics in G_1 and G_2. The structure of dyadic systems can be reflected by equivalent modal or invariance definitions of dyadic transfer function matrices given in Owens (1978) which goes some way to explaining why dyadic systems frequently occur in systems with a degree of dynamic symmetry and in systems with simple first and second order dynamic form (Owens 1978).

Control of Dyadic Systems

Controller design for a dyadic system $G(s)$ can proceed (Owens, 1978, 1979) by initially ignoring the causes of interaction P_1 and P_2 and designing controllers $k_j(s)$ for each scalar subsystem $g_j(s)$ by feedback of $\hat{y}_j$ to $\hat{u}_j$ as illustrated in Fig. 3. The scalar feedback systems have transfer functions

$$h_j(s) = \frac{g_j(s)k_j(s)}{1 + g_j(s)k_j(s)} \quad , \quad 1 \le j \le m \tag{8}$$

Given these designs, a multivariable controller $K(s)$ for $G(s)$ can be constructed of the dyadic form

$$K(s) = P_2^{-1} \begin{pmatrix} k_1(s) & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & k_m(s) \end{pmatrix} P_1^{-1} \tag{9}$$

The effect of this controller on the plant $G(s)$ can be assessed from the return-difference relation

$$|I_m + G(s)K(s)| \equiv \prod_{j=1}^{m} (1 + g_j(s)k_j(s)) \tag{10}$$

and the closed-loop transfer function matrix identity

$$H_c(s) \overset{\Delta}{=} (I_m + G(s)K(s))^{-1}G(s)K(s) \equiv P_1 \begin{pmatrix} h_1(s) & & \\ & \ddots & \\ & & h_m(s) \end{pmatrix} P_1^{-1} \tag{11}$$

Equation (10) indicates that the stability characteristics of the multivariable feedback system of Fig. 1 are identical to those of the m scalar feedback systems of Fig. 3 in the sense that they have identical pole distributions i.e. stabilization of the multivariable plant G can be

achieved by separate stabilization of m scalar systems $g_j(s)$, $1 \le j \le m$.
Equation (11) indicates that the input-output behaviour of the multivariable feedback scheme is closely related to the dynamics of the scalar feedback systems via the plant matrix P_1. The details of this relationship depend crucially on the nature of $\{g_j\}, \{k_j\}$ and P_1. It is not, in general, true that acceptable dynamic characteristics of $h_1, h_2, \ldots, h_m$ will guarantee acceptable dynamic characteristics of the multivariable feedback scheme. In general the controls of $k_j(s)$, $1 \le j \le m$, must be designed to ensure 'compatible' dynamics for $h_1, h_2, \ldots, h_m$. This is not necessary if P_1 is diagonal as the closed-loop transfer function matrix $H_c(s)$ is then diagonal with diagonal elements $h_1, h_2, \ldots, h_m$. If, however P_1 is not diagonal, the situation is not so well-defined. In particular, if a closed-loop system with small interaction effects is required, it is normally necessary (Owens, 1978, 1979) to design in such a way that all scalar systems h_j have similar step response characteristics.

The resultant design K(s) is obtained from (9) which imposes a few simple constraints on the choice of the $\{k_j\}$. More precisely, if K(s) is to be realizable in the sense that it contains only transfer functions with real coefficients then it is necessary that

$$\overline{k_j(s)} \equiv k_{\ell(j)}(\bar{s}) \quad , \quad 1 \le j \le m \qquad (12)$$

but this is easily achieved in practice. For example, in the frequently encountered case when both P_1 and P_2 are real, then $\ell(j) = j$ $(1 \le j \le m)$ and (12) reduces to the requirement that each control transfer function $k_j(s)$ must contain only real coefficients.

The control design procedure can be extended (Owens 1973b, 1978) to relate the scalar designs to the integrity of the final design to sensor or actuator failures. Also, by consideration of special classes of dyadic systems representing multivariable generalizations of first and second order systems (Owens 1978, 1981a), it is possible to use dyadic systems as approximate plant models to simplify the design process.

Dyadic Expansions of G at a Specific Frequency

Even if the mxm plant transfer function matrix G(s) is not dyadic in the sense defined above, it is , in general, possible to associate a dyadic structure (or dyadic expansion) with G(s) at a specific frequency $s = i\omega'$. More precisely (Owens 1978, 1979) if $G(i\omega')$ is nonsingular and

$$M(\omega') \triangleq G(-i\omega')G^{-1}(i\omega') \tag{13}$$

has a complete set of eigenvectors, then there exists a permissible transformation $(P_1(\omega'), P_2(\omega'))$ and complex scalars $\gamma_1, \gamma_2, \ldots, \gamma_m$ such that

$$G(i\omega') = P_1(\omega') \begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_m \end{pmatrix} P_2(\omega') \tag{14}$$

In particular $P_1(\omega')$ cal be taken to be an eigenvector matrix of $M(\omega')$ with suitable column scaling to ensure properly (3) and $P_2(\omega')$ taken to be $P_1^{-1}(\omega')\ G(i\omega')$ with suitably scaled rows.

Manipulation of Characteristic Loci at a Specific Frequency

The dyadic expansion of G at $s = i\omega'$ has an important application in the construction of controllers K(s) such that the forward path transfer function $Q(s) = G(s)K(s)$ has desired eigenvalues at $s = i\omega'$. This problem is fundamental to the characteristic locus design methodology (Owens 1978; MacFarlane, 1980) which bases its design strategies on manipulation of the eigenstructure of Q by suitable choice of K.

The dyadic structure on the right-hand-side of (14) suggests the dyadic control system

$$K(s,\omega') \triangleq P_2^{-1}(\omega') \begin{bmatrix} k_1(s,\omega') & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & k_m(s,\omega') \end{bmatrix} P_1^{-1}(\omega') \qquad (15)$$

After a little computation, it can be verified (Owens 1978, 1979) that the eigenvalues $q_j(i\omega')$, $1 \leq j \leq m$, of $Q(i\omega')$ are then given by

$$q_j(i\omega') = \gamma_j \; k_j(i\omega',\omega') \quad , \quad 1 \leq j \leq m \qquad (16)$$

and that $P_1(\omega')$ is an eigenvector matrix of $Q(i\omega')$.
Suitable choice of compensation networks $k_j(s,\omega')$, $1 \leq j \leq m$, will clearly enable the designer to produce desired eigenvalues $q_j(i\omega')$, $1 \leq j \leq m$, for $Q(i\omega')$ whilst ensuring physical realizability of $K(s)$ by satisfying the reality constraint (12)

Manipulation of Characteristic Loci over a Frequency Interval

Although manipulation of characteristic loci at a specific frequency is vitally important, it is probably more important to be capable of assessing the effect of the proposed controllers on the loci at other frequencies. With $P_1(\omega')$ and $P_2(\omega')$ as above a transformed plant $H(s,\omega')$ can be defined by the relation (Owens 1978, 1979)

$$G(s) = P_1(\omega') \; H(s,\omega') \; P_2(\omega') \qquad (17)$$

The identity (14) indicates that H is diagonal at $s = i\omega'$ with

$$H_{jj}(i\omega',\omega') = \gamma_j \quad , \quad 1 \leq j \leq m \qquad (18)$$

Noting that the characteristic loci of $Q = GK$ are identical to those of $H \operatorname{diag}\{k_j\}$, it is natural to use the diagonal terms as rational approximation to the eigenvalues i.e.

$$q_j(s) \simeq H_{jj}(s,\omega')k_j(s,i\omega') \quad , \quad 1 \leq j \leq m \tag{19}$$

the approximation being exact at $s = i\omega'$ due to (16) and (18). It is exact at all other frequencies only if G is dyadic. If G is not dyadic, then the errors in the approximation can be quantified using Gershgorins theorem (Rosenbrock, 1974; Owens, 1978; Patel and Munro, 1982) which reveals that the eigenvalues $q_j(s)$ all lie in the union of m 'Gershgorin circles' $B_j(s,\omega')$ of centre $H_{jj}(s,\omega')\ k_j(s,\omega')$ and radius

$$r_j(s,\omega') \triangleq |k_j(s,\omega')| \sum_{q\neq j} |H_{qj}(s,\omega')| \quad , \quad 1 \leq j \leq m \tag{20}$$

The circles have zero radius at $s = i\omega'$ and small radius in the vicinity of $s = \omega'$.

The behaviour of $q_j(i\omega)$, $\omega \geq o$, can be estimated by plotting $H_{jj}(i\omega,\omega')k_j(i\omega,\omega')$ as a Nyquist diagram in the complex plane and superimposing at each frequency a circle of radius $r_j(i\omega,\omega')$ as illustrated in Fig. 4. Invoking the working rule of thumb that each circle, in general, contains an eigenvalue, it follows that $q_j(i\omega)$ lies in $B_j(i\omega,\omega')$ and the characteristic locus $q_j(i\omega)$, $\omega \geq o$, lies in the Gershgorin band generated by the union of the $B_j(i\omega,\omega')$ as ω increases from zero to infinity. This is illustrated in Fig. 4.

The importance of these ideas in design is realized by using eqn. (19) as a working approximation to the characteristic loci during the design phase and choosing the compensator k_j based on classical analysis of the scalar transfer function H_{jj} to produce the required gain/phase portrait over a frequency range of interest containing ω'. The success of the design can then be checked by superimposing the Gershgorin band on the Nyquist plot to bound the errors in prediction of the actual characteristic locus.

Although both the Nyquist plot of $H_{jj}k_j$ and its Gershgorin band depend on the choice of compensator k_j, the fractional prediction error

$$\frac{|q_j(i\omega) - H_{jj}(i\omega,\omega')k_j(i\omega,\omega')|}{|H_{jj}(i\omega,\omega')k_j(i\omega,\omega')|} \leq \frac{\sum_{q\neq j} |H_{qj}(i\omega,\omega')|}{|H_{jj}(i\omega,\omega')|} \qquad (21)$$

is bounded by a controller-independent quantity. The likely errors in prediction can therefore be estimated at the beginning of the design by the normal 'dry-run' with $k_j(s,\omega') \equiv 1,\ 1 \leq j \leq m$.

Dyadic Expansion and Diagonal Dominance

The Gershgorin bands provide estimates of the position of the characteristic loci, and the estimates are very good in the vicinity of $s = i\omega'$, but there is no guarantee that they will be useful over the whole frequency range $o \leq \omega < +\infty$ if the Gershgorin circles have 'large' radius. There is however one important special case (Owens 1978, 1979) where wide Gershgorin bands can be accommodated in stability analysis, namely the situation when a 'diagonal dominance' condition holds such that the familiar (-1,0) point of the complex plane does not lie in or on any Gershgorin band as illustrated in Fig. 5. Under this condition, it is unnecessary to estimate the position of the characteristic loci as it can be proved (Owens 1978, 1979) that, if the Nyquist diagram of $H_{jj}k_j$ encircles the (-1,0) point n_j times in a clockwise manner (anti-clockwise encirclements being counted as negative) then K stabilizes G if the following Nyquist-like stability condition holds

$$n_o + n_1 + n_2 + \ldots + n_m = 0 \qquad (24)$$

where n_o is the number of poles of GK in the interior of the Nyquist contour. In effect, the diagonal dominance condition enables stability predictions to be made on the basis of the m scalar systems H_{jj}, $1 \leq j \leq m$, whilst ignoring the off-diagonal interaction terms in H.

An alternative technique based on inverse Nyquist ideas is given in Owens (1978,1979) and has strong connections with the inverse Nyquist array design technique.

The Method of Dyadic Expansion

If the diagonal dominance condition does not hold, design is undertaken in a piece-wise frequency sense using a controller of the form

$$K(s) = K(s,\omega_m)\ \{I_m + \frac{1}{s} K(\omega_\ell)\} \qquad (22)$$

where $K(s,\omega_m)$ is a dyadic controller for $G(s)$ constructed to produce the required gain and phase characteristics for the characteristic loci of $G(s)\ K(s,\omega_m)$ in the vicinity of an intermediate frequency ω_m of interest and $K(\omega_\ell)$ is a proportional dyadic controller for $s^{-1}G(s)K(s,\omega_m)$ ensuring the required gain and phase characteristics of the characteristic loci of $s^{-1}G(s)K(s,\omega_m)K(\omega_\ell)$ in the vicinity of a low frequency of interest ω_ℓ. The combination of $K(s,\omega_m)$ and $K(\omega_\ell)$ to form $K(s)$ as in (22) will produce the required characteristic loci for GK provided that the integral gains are not too high and that the desired characteristics in the vicinity of ω_m and ω_ℓ are compatible. This can always be checked by calculation of the exact characteristic loci.

A Unified Design Theory

The use of diagonal dominance and eigenvalue methods in the method of dyadic expansion suggest that the method of dyadic expansion the characteristic locus method and the inverse and direct Nyquist array techniques can be unified into one general design algorithm involving transformations, pre-and post-compensation, diagonal dominance checks and eigenvalue manipulation and estimation. This has been demonstrated in Owens (1981b).

D.H. Owens
The University of Sheffield.

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List of Figure Captions

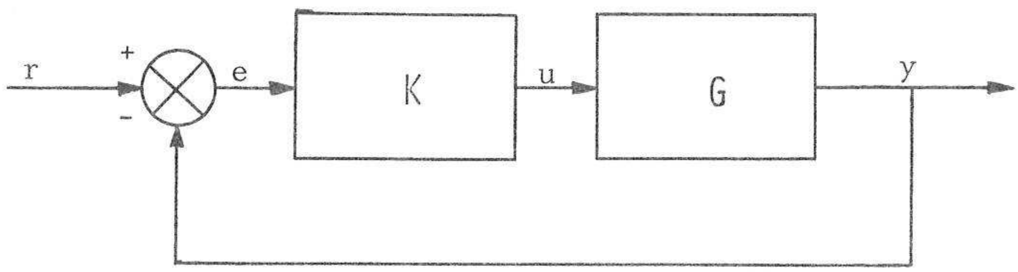
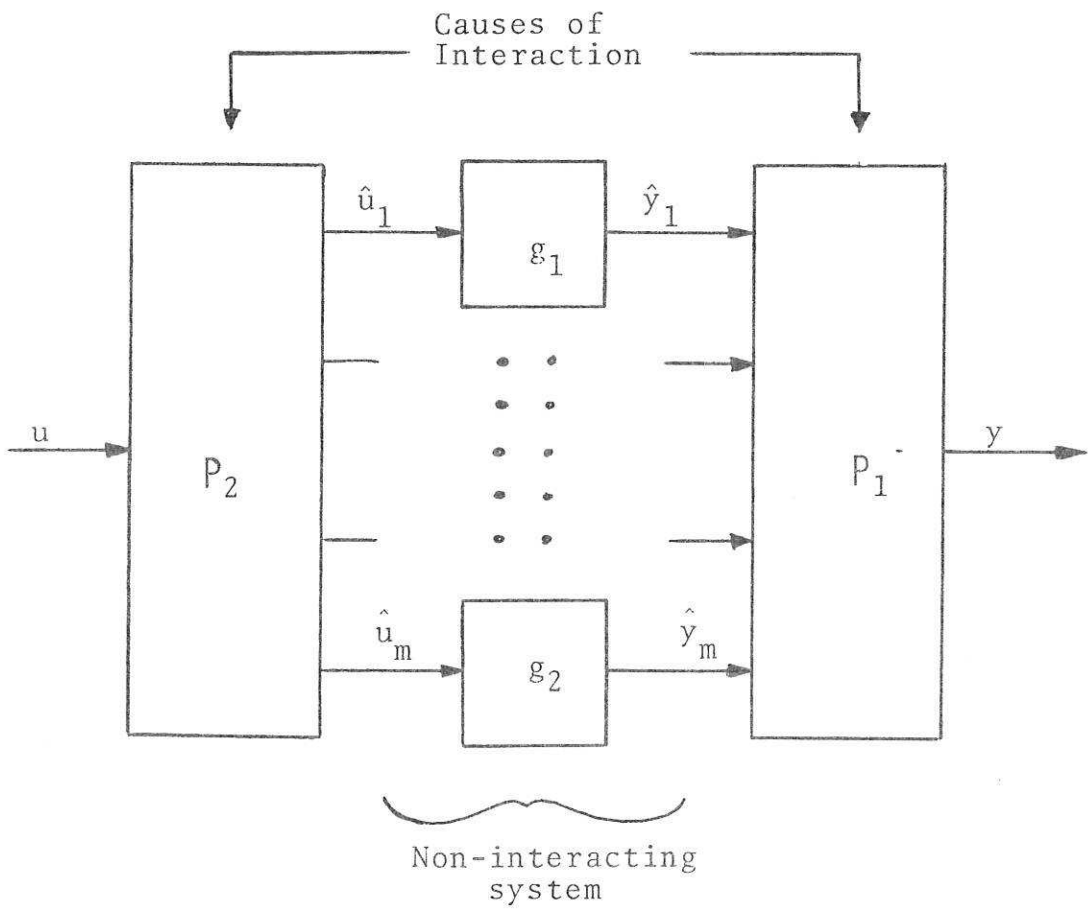


FIGURE 1


Causes of Interaction
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Non-interacting system

FIGURE 2

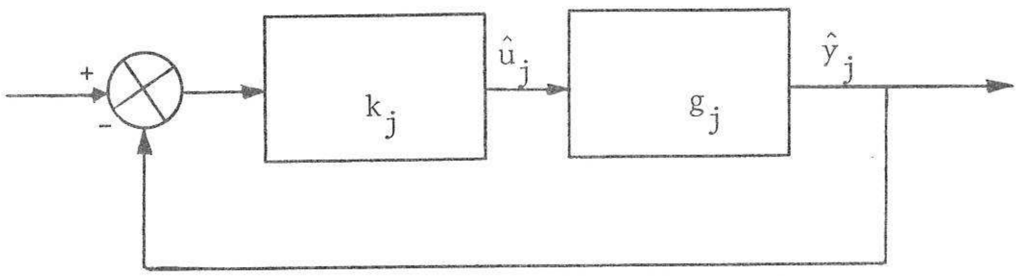
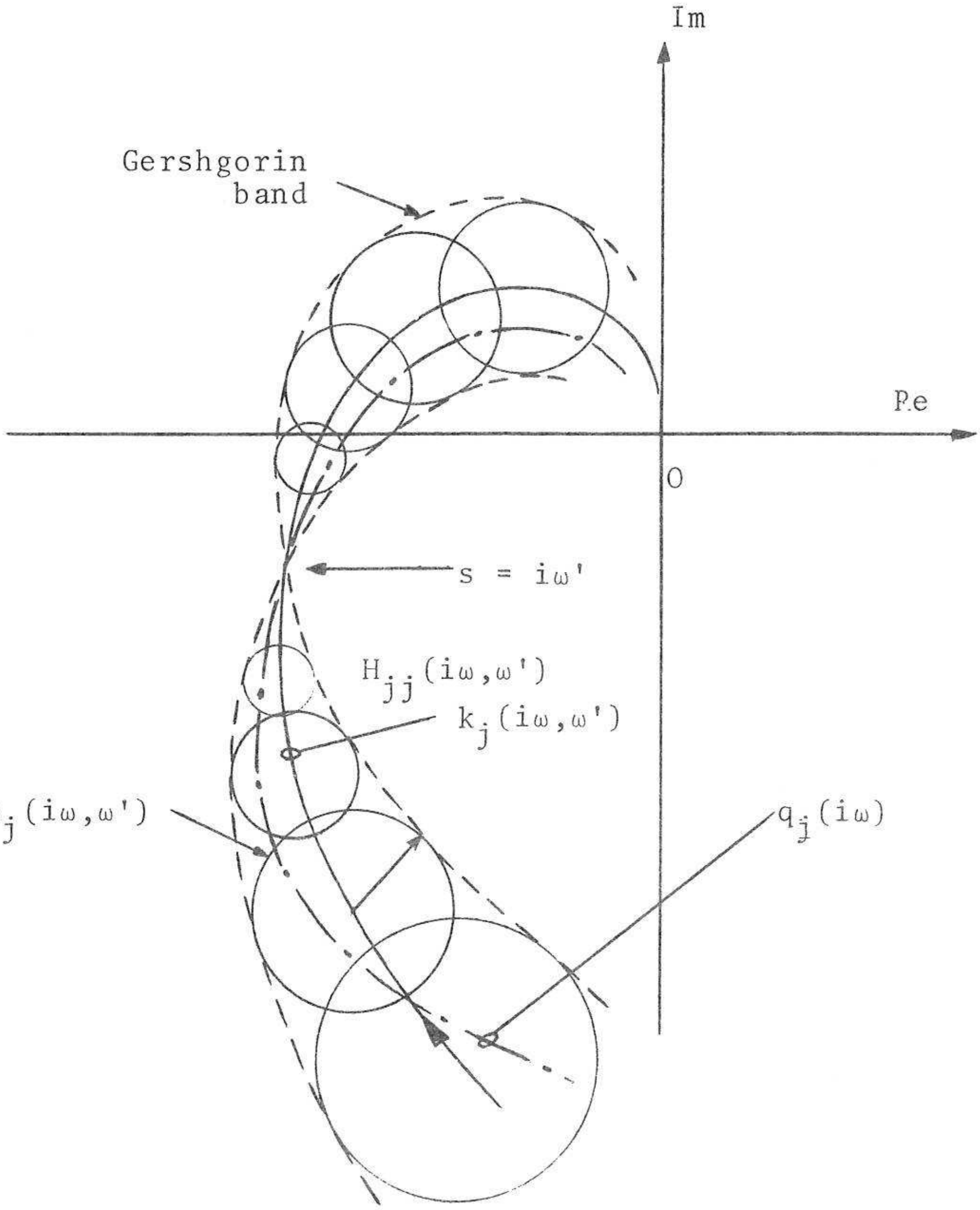


FIGURE 3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s = iω'
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$q_j(i\omega)$

FIGURE 4

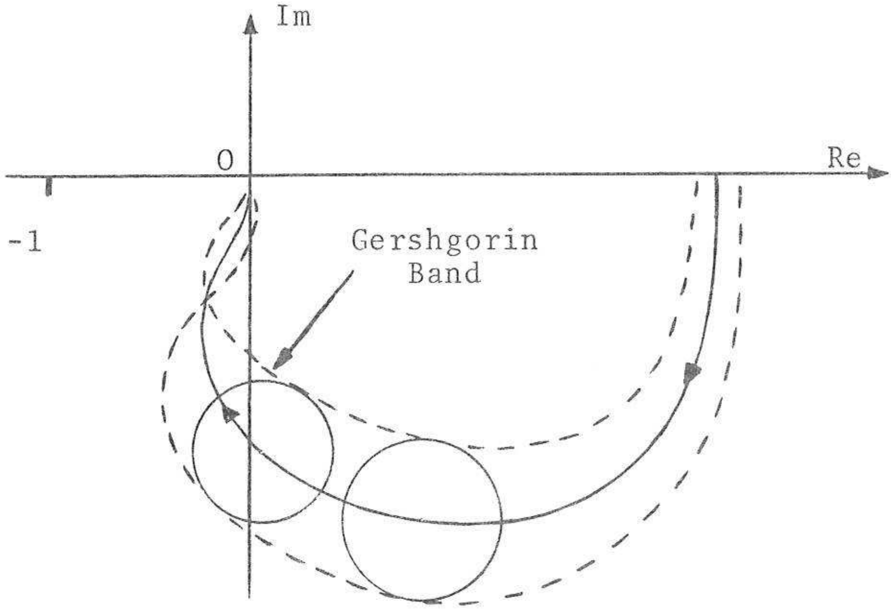

FIGURE 5