Monograph:
Xiong, Y.L. and Banks, S.P. (1980) Optimality Conditions for the Extreme Difference of Conjugate Functions. Research Report. ACSE Report 178 . Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield

Optimality Conditions for the Extreme Difference of Conjugate Functions

by

Y.L. Xiong[†] and S.P.Banks

Department of Control Engineering,
University of Sheffield,
Mappin Street,
Sheffield

Research Report No. 178

† Visiting from Huazhong Institute of Science and Technology.

Abstract

The conjugate strong duality theorem is extended by removing certain convexity assumptions with specific reference to the case where f=g. A practical optimization method is also presented.

1. Introduction

This paper is concerned with problems of optimality conditons and the solution of the extreme difference of conjugate functions . These problems are connected with conjugate dual optimization , but here we also consider the possibility that f=g , in which case the primal problem

$$h = \sup(f(x) - g(x))$$

has the trivial solution h=0.

The theory of conjugate functions was introduced by Fenchel (1) as early as 1949 , and developed in (2). Since then , there has been further intensive activity in the areas of theoretical development, applications and interpretations of existing ideas ; see for example (3)-(7). Apart from its value in nonlinear programming , the theory is interesting in its own right and has applications to the theory of inequalities and to the structure of convex sets.

In this paper we apply some basic properties of conjugate functions to the optimality problem of the extreme difference of such functions. We first review the basic definitions and state a fundamental theorem of this theory .

Recall that a function f defined on a convex set $X \subseteq E_n$ (n-dimensional Euclidean space) is convex if

$$f(\lambda x_2+(1-\lambda)x_1) \leq \lambda f(x_2)+(1-\lambda)f(x_1)$$

for any $x_1, x_2 \in X$ and any $\lambda \in [0,1]$. Concavity of f is defined similarly.

We shall use the standard notations

$$B(x^o,r)=\{x \in E_n : \|x-x^o\|<r\}$$

$$\overline{B}(x^o,r)=\{x \in E_n : \|x-x^o\|\leq r\}$$

$$S(x^o,r)= \partial \overline{B} = \{x \in E_n : \|x-x^o\|=r\},$$

and denote the convex hull of a set $Y \subseteq E_n$ by H(Y). If f is a convex function defined on a convex set $X \subseteq E_n$, a <u>subgradient</u> of f at $x \in X$ is a vector $\bar{x} \in E_n$ such that

$$f(x) \leq f(y)+(x-y)^T\bar{x} \quad , \text{ for all } y \in X.$$

The set of all subgradients of f at x is denoted by

$$\partial f(x).$$

If f(x) is a convex function on a set X, then we shall denote by

$$f'(x;y) = \lim_{h \to 0+} \frac{f(x+hy)-f(x)}{h}$$

the (upper) directional derivative of f in the direction y. Note that

$$f'(x;y) = \sup\{\xi^T y : \xi \in \partial f(x)\} .$$

Suppose , then, that X and Y are nonempty subsets of E_n, and let $f,g:E_n \to E_1$. Then we define the conjugate functions

$$\bar{f}^*(u) = \sup\{f(x)+x^T u : x \in X\} \tag{1.1}$$

$$\underline{g}^*(u) = \inf\{g(x)+x^T u : x \in Y\} \tag{1.2}$$

and the corresponding sets

$$U = \{u \in E_n : \bar{f}^*(u) < +\infty\} \tag{1.3}$$

$$V = \{u \in E_n : \underline{g}^*(u) > -\infty\} \tag{1.4}$$

It can be shown (2, 6) that

(1). $\bar{f}^*(u)$ is convex over U, $\underline{g}^*(u)$ is concave over V.

(2). The conjugate weak duality theorem

$$\sup\{f(x)-g(x):x \in X \cap Y\} \leq \inf\{\bar{f}^*(u)-\underline{g}^*(u):u \in U \cap V\} \tag{1.5}$$

holds , and we have

(3). The conjugate strong duality theorem .

This states that if f is concave , g is convex , $\text{int}X \cap \text{int}Y \neq \emptyset$ and inf on the right is actually attained.

If the assumption of the conjugate strong duality theorem is not completely satisfied , there may be a duality gap; for example, in the case where f=g, and f is not affine , then it cannot be convex and concave at the same time and so (1.6) may not hold (since the right hand side may not be zero).

The main interest in this paper is in the case where f=g and when X and Y have empty interiors ,e.g. $X,Y \subseteq Z_n$ (the space of n-tuples of integers). Then $\text{int}X \cap \text{int}Y = \emptyset$.

2. Optimality Conditions for the Extreme Difference of Conjugate Functions

We consider the following problem (which we refer to as the problem of the extreme difference of conjugate functions):

$$\text{Minimize} \quad \bar{f}^*(u) - \underline{g}^*(u) = \varepsilon(u)$$

$$\text{subject to } u \in U \cap V$$

and suppose that the sets X and Y over which the primal functions f and g are defined are compact . Note that , if f and g are continuous , then $U=V=E_n$, so that there is no constraint on u. Although the compactness assumption is quite strong , it is necessary for the proofs to follow.With this assumption , the sets

$$\bar{X}(u) = \{y:\ y \text{ maximizes } f(x)+x^T u \text{ over } X\} \tag{2.1}$$

$$\underline{X}(u) = \{y:\ y \text{ minimizes } g(x)+x^T u \text{ over } Y\} \tag{2.2}$$

are compact . If f and g are also continuous, it can be shown[9,11] that

$$\partial \bar{f}^*(u^0) = H[\bar{X}(u^0)] \tag{2.3}$$

$$\partial \underline{g}^*(u^0) = H[\underline{X}(u^0)] \tag{2.4}$$

In order to develop the optimality condition , the following lemma is needed.

Lemma 2.1 (Fundamental Inequalities)

Let X be a nonempty compact set in E_n , $x^0 \in X$ and $u \in E_n$. Then,

$$\sup\{(x-x^0)^T u : x \in X\} \geq \|u\| \inf\{\|x-x^0\| : x \in \partial X\} \tag{2.5}$$

$$\inf\{(x-x^0)^T u : x \in X\} \leq -\|u\| \inf\{\|x-x^0\| : x \in \partial X\} . \tag{2.6}$$

(Recall that ∂X denotes the boundary $\bar{X} \cap \overline{(E_n - X)}$ of X.)

Proof It is obvious that

$$\sup\{(x-x^0)^T u : x \in S(x^0,r)\} = \|u\| r$$
$$\inf\{(x-x^0)^T u : x \in S(x^0,r)\} = -\|u\| r.$$

Let $r=\inf\{\|x-x^0\|: x \in \partial X\}$. Then, $S(x^0,r) \subseteq X$ and so

$$\begin{aligned}\sup\{(x-x^0)^T u : x \in X\} &\geq \sup\{(x-x^0)^T u : x \in S(x^0,r)\}\\ &= \|u\| r\\ &= \|u\| \inf\{\|x-x^0\|: x \in \partial X\}\\ \inf\{(x-x^0)^T u : x \in X\} &\leq \inf\{(x-x^0)^T u : x \in S(x^0,r)\}\\ &= -\|u\| \inf\{\|x-x^0\|: x \in \partial X\} . \quad \square\end{aligned}$$

Since a convex function achieves a maximum over a compact convex set at an extreme point , and a concave function achieves a minimum over a compact convex set at an extreme point[9,10], we have

$$\sup\{(x-x^0)^T u : x \in X\} = \sup\{(x-x^0)^T u : x \in H[X]\}.$$
$$\inf\{(x-x^0)^T u : x \in X\} = \inf\{(x-x^0)^T u : x \in H[X]\}.$$

We therefore have

<u>Corollary 2.1</u> Let X be a nonempty compact set in E_n and let $u \in E_n$. If $x^0 \in H[X]$, then we have

$$\sup\{(x-x^0)^T u : x \in X\} \geq \|u\| \inf\{\|x-x^0\|: x \in \partial H[X]\} \qquad (2.7)$$
$$\inf\{(x-x^0)^T u : x \in X\} \leq -\|u\| \inf\{\|x-x^0\|: x \in \partial H[X]\} \qquad (2.8)$$

<u>Theorem 2.1</u> (Optimality Conditions)

Let X and Y be nonempty compact sets in E_n , and let $f,g:E_n \to E_1$ be continuous , $\bar{f}^*(u)$ and $\underline{g}^*(u)$ be defined by (1.1) and (1.2) . Then u^0 is a solution of the problem

$$\text{minimize } \bar{f}^*(u)-\underline{g}^*(u)= \varepsilon(u), \; u \in E_n$$

if and only if

$$H[\bar{X}(u^0)] \cap H[\underline{X}(u^0)] \neq \emptyset.$$

<u>Proof</u> . <u>Sufficiency</u>: Since $H[\bar{X}(u^0)] \cap H[\underline{X}(u^0)] \neq \emptyset$, there exists $x^0 \in H[\bar{X}(u^0)] \cap H[\underline{X}(u^0)]$. Then, for any $u \in E_n$, the definition (1.1) implies that

$$\bar{f}^*(u)=\sup\{f(x)+x^Tu \ : \ x\in X\}$$
$$=\sup\{f(x)+x^Tu^0+(x-x^0)^T(u-u^0) \ : \ x\in X\}+x^{0T}(u-u^0)$$
$$\geq \sup\{f(x)+x^Tu^0+(x-x^0)^T(u-u^0) : x\in\bar{X}(u^0)\}+x^{0T}(u-u^0),$$

since $\bar{X}(u^0)\subseteq X$. Since $f(x)+x^Tu^0$ is constant on $\bar{X}(u^0)$, we have

$$\sup\{f(x)+x^Tu^0+(x-x^0)^T(u-u^0) : x\in\bar{X}(u^0)\}$$
$$=\bar{f}^*(u^0)+\sup\{(x-x^0)^T(u-u^0) \ : \ x\in\bar{X}(u^0)\},$$

and by corollary 2.1, since $x^0\in H[\bar{X}(u^0)]$, we have

$$\sup\{(x-x^0)^T(u-u^0) \ : \ x\in\bar{X}(u^0)\}\geq \|u-u^0\|\inf\{\|x-x^0\| : x\in\partial H[\bar{X}(u^0)]\}.$$

Therefore, we obtain

$$\bar{f}^*(u)\geq \bar{f}^*(u^0) + x^{0T}(u-u^0) + \|u-u^0\|\inf\{\|x-x^0\| : \ x\in\partial H[\bar{X}(u^0)]\}.$$

Similarly,

$$\underline{g}^*(u)\leq \underline{g}^*(u^0) + x^{0T}(u-u^0) - \|u-u^0\|\inf\{\|x-x^0\| : \ x\in\partial H[\underline{X}(u^0)]\},$$

and so

$$\bar{f}^*(u)-\underline{g}^*(u) \geq \bar{f}^*(u^0)-\underline{g}^*(u^0)+\|u-u^0\|(\inf\{\|x-x^0\| : \ x\in\partial H[\underline{X}(u^0)]\}$$
$$+\inf\{\|x-x^0\| : \ x\in\partial H[\underline{X}(u^0)]\}).$$
$$\geq \bar{f}^*(u^0)-\underline{g}^*(u^0).$$

Since this is true for all $u\in E_n$, u^0 is an optimal point.

<u>Necessity</u>: Suppose $H[\bar{X}(u^0)]\cap H[\underline{X}(u^0)]=\phi$. Then, since $H[\bar{X}(u^0)]$ and $H[\underline{X}(u^0)]$ are compact (i.e. closed and bounded), the strong separation theorem(9) implies that there exists a nonzero vector p and $\varepsilon>0$ such that

$$\inf\{p^Tx \ : \ x\in\partial\underline{g}^*(u^0)\}\geq\varepsilon+\sup\{p^Tx \ : \ x\in\partial\bar{f}^*(u^0)\}. \text{ (using (2.3),(2.4))}.$$

Now the continuity of f and g imply, by an elementary argument, that the (upper) directional derivatives $\bar{f}^{*\prime}(u^0;p)$, $\underline{g}^{*\prime}(u^0;p)$ of $\bar{f}^*$ and $\underline{g}^*$ at u^0 in the direction p exist and so we have

$$\underline{g}^{*\prime}(u^0;p)\geq\varepsilon+\bar{f}^{*\prime}(u^0;p).$$

Therefore, p is a descent direction of $\varepsilon=\bar{f}^*-\underline{g}^*$ at u^0, so u^0 is not a minimum point of ε. (Note again by continuity of f and g and the compactness of X and Y that $U=V=E_n$, so u^0 is not a boundary point.)

3. Solution of the Extreme Difference Problem

The extreme difference problem, as we have seen, is concerned with the minimization of $\varepsilon(u)$. Given a point u^0, theorem 2.1 shows that u^0 is a solution to the problem if

$$H[\overline{X}(u^0)] \cap H[\underline{X}(u^0)] \neq \emptyset ,$$

i.e. $$0 \in H(u^0) \triangleq H[\underline{X}(u^0)] - H[\overline{X}(u^0)] = \{x-y:\ x \in H[\underline{X}(u^0)], y \in H[\overline{X}(u^0)]\} \quad (3.1)$$

If this condition does not hold, then we have seen that there exists a descent direction and so we would like to find the direction of steepest descent.

Definition 3.1 A vector $\overline{d}$ is called a direction of steepest descent of ε at u if

$$\varepsilon'(u;\overline{d}) = \overline{f}^{*\prime}(u;\overline{d}) - \underline{g}^{*\prime}(u;\overline{d}) = \min_{\|d\|\leq 1}(\overline{f}^{*\prime}(u;d) - \underline{g}^{*\prime}(u;d)).$$

Theorem 3.1

Let X be a nonempty compact set in E_n and suppose that $f,g:E_n \rightarrow E_1$ are continuous; then the direction of steepest descent $\overline{d}$ of ε at u is given by

$$\overline{d} = \begin{cases} 0 & \text{if } \overline{p}=0 \\ \dfrac{-\overline{p}}{\|\overline{p}\|} & \text{if } \overline{p}\neq 0 \end{cases}$$

where $\overline{p} = \min_{p \in H[u]} \|p\|$.

Proof. First note that

$$\partial\varepsilon(u) = H[\overline{X}(u)] - H[\underline{X}(u)] . \quad (3.2)$$

For, if $\overline{\xi} \in \partial\overline{f}^*(u)$, $\underline{\xi} \in \partial \underline{g}^*(u)$, then

$$\overline{f}^*(u') \geq \overline{f}^*(u) + \overline{\xi}^T(u'-u)$$

$$\underline{g}^*(u') \leq \underline{g}^*(u) + \underline{\xi}^T(u'-u)$$

for all $u' \in E_n$. Thus,

$$\varepsilon(u') \geq \varepsilon(u) + (\overline{\xi}-\underline{\xi})^T(u'-u)$$

and so

$$H[\overline{X}(u)] - H[\underline{X}(u)] \subseteq \partial\varepsilon(u).$$

To prove the reverse inclusion, suppose that $\xi \in \partial\varepsilon(u)$, but $\xi \notin H[\overline{X}(u)] - H[\underline{X}(u)]$ $(=H(u))$. Then, since $H(u)$ is compact and convex, the strong separation theorem implies that there exists a vector d and $\varepsilon > 0$ such that

$$d^T\xi > \varepsilon,$$
$$d^T\xi' \leq 0, \text{ for all } \xi' \in H[u].$$

Hence,

$$\varepsilon'(u;d) > \varepsilon,$$

and

$$\begin{aligned}\varepsilon'(u;d) &= \overline{f}^{*\prime}(u;d) - g^{*\prime}(u;d)\\ &= \sup\{d^T\overline{\xi} : \overline{\xi} \in \partial\overline{f}^*(u)\} - \inf\{d^T\underline{\xi} : \underline{\xi} \in \underline{g}^*(u)\}\\ &= \sup\{d^T\xi' : \xi' \in H(u)\}\\ &\leq 0,\end{aligned}$$

which is a contradiction.

To prove the theorem, note that if $\overline{d}$ is the direction of steepest descent then

$$\begin{aligned}\varepsilon'(u;\overline{d}) &= \min_{\|d\|\leq 1} \varepsilon'(u;d) = \min_{\|d\|\leq 1} \sup_{p\in\partial\varepsilon(u)} d^Tp\\ &\geq \sup_{p\in\partial\varepsilon(u)} \min_{\|d\|\leq 1} d^Tp = \sup_{p\in\partial\varepsilon(u)} -\|p\|\\ &= -\|p\|\end{aligned}$$

by (3.2). Hence, all we have to do is to show that there exists a direction $\overline{d}$ such that

$$\varepsilon'(u;\overline{d}) = -\|\overline{p}\|;$$

$\overline{d}$ will then be the direction of steepest descent. If $\overline{p}=0$, then $\varepsilon'(u;0)=0=-\|\overline{p}\|$, so $\overline{d}=0$. If $\overline{p} \neq 0$, put $\overline{d} = -\overline{p}/\|\overline{p}\|$. Then,

$$\begin{aligned}\varepsilon'(u;\overline{d}) &= \sup\{\overline{d}^Tp : p \in \partial\varepsilon(u)\}\\ &= \frac{1}{\|\overline{p}\|} \sup\{-\|\overline{p}\|^2 + \overline{p}^T(\overline{p} - p) : p \in \partial\varepsilon(u)\}\\ &= -\|\overline{p}\| + \frac{1}{\|\overline{p}\|}\sup\{\overline{p}^T(\overline{p} - p) : p \in \partial\varepsilon(u)\}.\end{aligned}$$

Since $\overline{p}$ is the shortest vector in $\partial\varepsilon(u)$, and $\partial\varepsilon(u)$ is closed and convex, then $\overline{p}^T(\overline{p} - p) \leq 0$, for all $p \in \partial\varepsilon(u)$. Hence $\varepsilon'(u;\overline{d}) = -\|\overline{p}\|$. □

The minimization of $\varepsilon(u)$ for $u \in E_n$ can now be obtained by applying theorem 3.1 which leads to the algorithm shown in fig.3.1, and described by the following steps:

(1). Choose $u \in E_n$.

(2). Solve the problems:

$$\overline{f}^*(u) = \max\{ f(x)+x^T u : x \in X \}$$
$$\underline{g}^*(u) = \min\{ g(x)+x^T u : x \in Y \}$$

and find $\overline{X}(u)$ and $\underline{X}(u)$.

(3). Solve the subproblem:

$$\text{minimize } \|p\| \text{ subject to } p \in H[\underline{X}(u)] - H[\overline{X}(u)] \quad .$$

If $\|p\|=0$ stop; there is no descent direction , and u is an optimal solution.

Otherwise $\|p\|>0$ and put $d=p/\|p\|$.

(4). Solve the following subproblems:

(a). minimize $\lambda_1 = \dfrac{\overline{f}^*(u) - (f(x)+x^T u)}{d^T(x-\overline{x}_0)}$

subject to $d^T(x-\overline{x}_0) > 0,\ x \in X \cap Y$

(b). minimize $\lambda_2 = \dfrac{\underline{g}^*(u) - (g(x)+x^T u)}{d^T(x-\underline{x}_0)}$

subject to $d^T(x-\underline{x}_0) < 0,\ x \in X \cap Y$

where

$$d^T\overline{x}_0 = \max\{d^T x : x \in \overline{X}(u)\}$$
$$d^T\underline{x}_0 = \min\{d^T x : x \in \underline{X}(u)\}.$$

Let $\lambda=\min(\lambda_1,\lambda_2)$, $u=u+\lambda d$.

(5). Go to step (1).

<u>Remark</u>. It can be shown that the value of λ obtained in (4) guarantees a decrease in $\varepsilon(u)$ in the direction d. It should be noted that such a λ does not necessarily minimize $\varepsilon(u+\lambda d)$. We are therefore using a steepest descent method with inexact line search .

4. Example[11].

Consider the following problem :

$$\text{minimize} \quad \varepsilon(u) = \overline{f}^*(u) - \underline{f}^*(u) \ , \ u \in E_n \tag{4.1}$$

where

$$\begin{aligned} \overline{f}^*(u) &= \max\{ f(x)+x^Tu : x_1,x_2=0,1,2 \text{ or } 3\} \\ \underline{f}^*(u) &= \min\{ f(x)+x^Tu : x_1,x_2=0,1,2 \text{ or } 3\} \end{aligned} \tag{4.2}$$

and $f(x)$ $(x=(x_1,x_2)^T)$ is given as a two-dimensional array in table4.1 (i.e. the sets X,Y introduced above are equal and discrete subsets of E_1).

Table 4.1

3	-6	-3	0	-4
2	-11	-10	-2	-4
1	-8	-10	-5	-4
0	-6	-10	-7	-12
	0	1	2	3

Note that

(1). $X,Y=\{0,1,2,3\}$.

(2). f will be regarded as a continuous function of x, whose values at the points $x_1,x_2=0,1,2,3$ are specified.

(3). f=g , and it is clear from table 4.1 that f is not affine. Hence f could not be convex and concave at the same time and so the conditions of the conjugate strong duality theorem do not hold .

We shall solve the problem (4.1) by the descent procedure starting with $u^0=(0,0)^T$. The sets $H[\overline{X}(u^0)]$ and $H[\underline{X}(u^0)]$ are found from (4.2) , and it is clear that

$$\overline{X}(u^0)=\{(2,3)^T\}, \ \underline{X}(u^0)=\{(3,0)^T\} .$$

Since $H[\overline{X}(u^0)] \cap H[\underline{X}(u^0)] = \emptyset$, it follows from theorem 2.1 that u^0 is not an optimal solution of the problem (4.1) . Thus , as we have seen , there exists a descent direction given by minimizing the norms of the elements in

$$H(u^0) = H[\underline{X}(u^0)] - H[\overline{X}(u^0)] = (1,-3)^T.$$

Hence , $d=(1,-3)^T$ is the direction of maximum descent .(cf. fig.4.1). In order to find an appropriate multiplier λ , we could use a one-dimensional search[12] ; however , it is easier to use the method of step (4) given in the above algorithm . Solving this subproblem , we find that $\lambda^0=1/9$ and so the next value of u is given by

$$u^1=u^0 + \lambda^0 d^0 = (1/9,-1/3)^T .$$

Repeating this procedure we find that the solution of (4.2) with $u=u^1$ gives

$$\overline{X}(u^1)= \{(2,3)^T\}, \quad \underline{X}(u^1)= \{(3,0)^T,(0,2)^T\}.$$

(cf.fig.4.2).

Again , $H[\overline{X}(u^1)] \cap H[\underline{X}(u^1)] = \emptyset$, and so u^1 is not optimal . By theorem 3.1 , we find that the steepest descent direction is

$$d^1=(-2,-3)^T.$$

Hence , solving the subproblem in step (4) again we find $\lambda^1=47/117$, and so $u^2=u^1+\lambda^1 d^1 = (-9/13,-20/13)^T$.

Returning to (4.2) , with $u=u^2$ we find that

$$\overline{X}(u^2) = \{(0,0)^T,(2,3)^T\} \text{ and } \underline{X}(u^2) = \{(3,0)^T,(0,2)^T\}.$$

(cf. fig.4.3).

This time, $H[\overline{X}(u^2)] \cap H[\underline{X}(u^2)] \neq \emptyset$, and so u^2 is an optimal solution of the problem and

$$\varepsilon(u^2)=\overline{f}^*(u^2) - \underline{f}^*(u^2) = 8\tfrac{1}{13} .$$

5. Conclusion .

In this paper we have considered the minimization of the extreme difference $\varepsilon(u)=\overline{f}^*(u)-\underline{g}^*(u)$, extending the conjugate strong duality theorem . It has been shown that when $H[\overline{X}(u)] \cap H[\underline{X}(u)] \neq \emptyset$, i.e. $0 \in H(u)$, then d=0. Thus u is the optimal solution for the extreme difference problem. If $H[\overline{X}(u)] \cap H[\underline{X}(u)] = \emptyset$, i.e. $0 \notin H(u)$, then $d \neq 0$, and d is the direction of steepest descent . Using this result , we have developed a procedure for minimizing the function ε, and applied it to an example of a function defined on a discrete subset of E_n.

6. References

(1). Fenchel, W , On Conjugate Convex Functions , Canadian J. Maths.1 pp 73-77, 1949.

(2). Fenchel, W, Convex Cones , Sets and Functions , Princeton Univ. 1953

(3). Rockafeller, R.T., Minimax Theorems and Conjugate Saddle Functions, Math. Scand. 14,pp 151-173,1964.

(4). Rockafeller, R.T., Extension of Fenchel's Duality Theorem for Convex Functions, Duke Maths. J.,33, pp 81-90, 1966.

(5). Bellman. R.,(Ed.) Mathematical Optimization Techniques , Univ. of California Press, Berkeley 1963.

(6). Whinston. A., A Dual Decomposition Algorithm for Quadratic Programming, Cahiers Centre Etudes Recherche Operational, 1964.

(7). Whinston. A., Some Applications of the Conjugate Function Theorem to Duality , in'Nonlinear Programming' , J.Obadie (Ed.) 1967.

(8). Karlin. S., Mathematical Methods and Theory in Games,Programming and Economics , Addison Wesley ,1959.

(9). Aviel. M., Nonlinear Programming Analysis and Methods , Prentice-Hall, 1976.

(10). Rockafeller. R.T., Convex Analysis, Princeton Univ. Press, 1970.

(11). Xiong. Y.L., The α-β Coordinate Method of Evaluating Flatness According to the Principle of Minimax, J. of H.I.T. 1980 (In Chinese).

(12). Jacobi. S.L.S., Kowalik. J.S. and Pizzo. J.T., Iterative Method for Nonlinear Optimization Problems , 1972.

Acknowledgment.

The authors would like to thank Prof. H. Nicholson for his advice and encouragement.

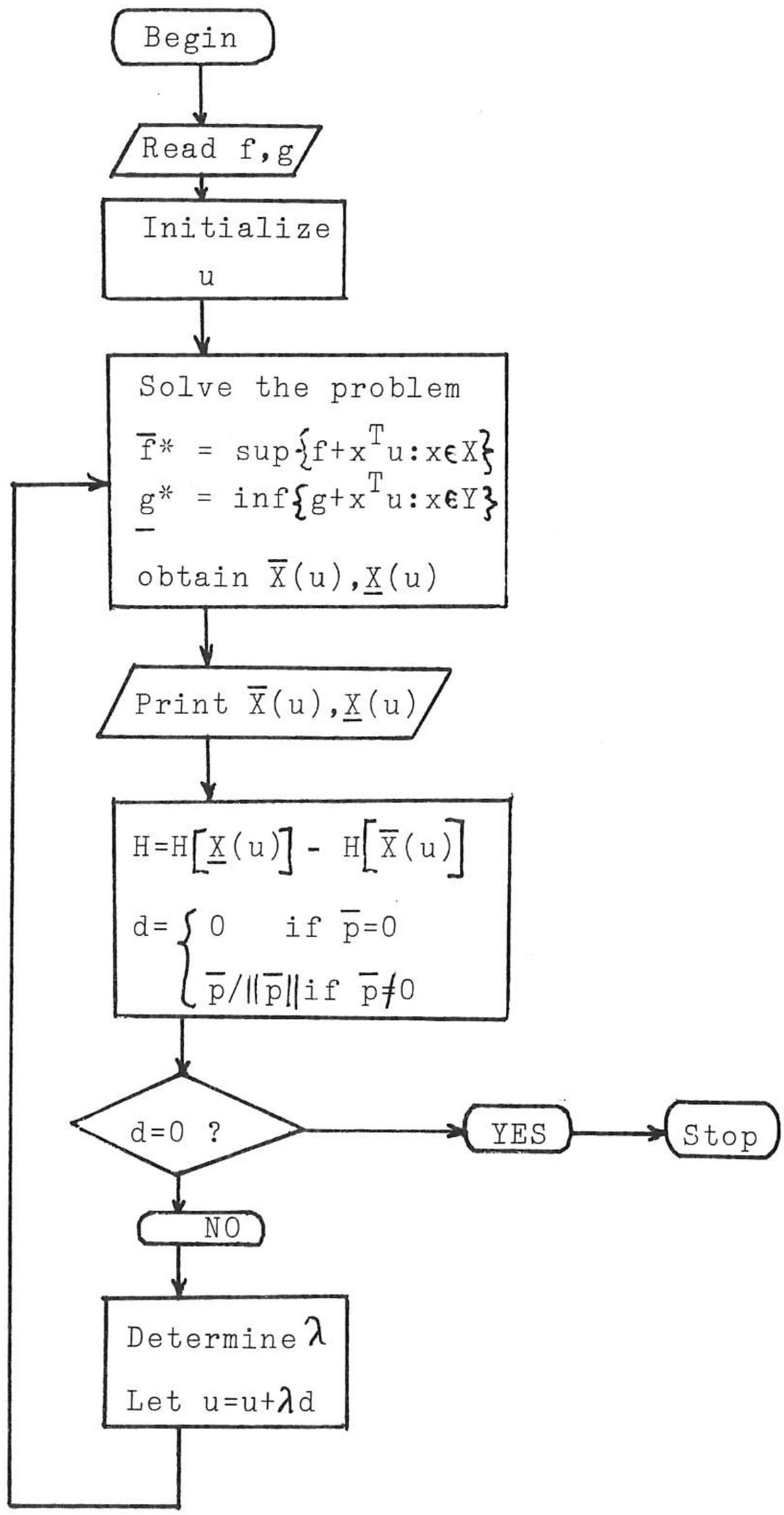
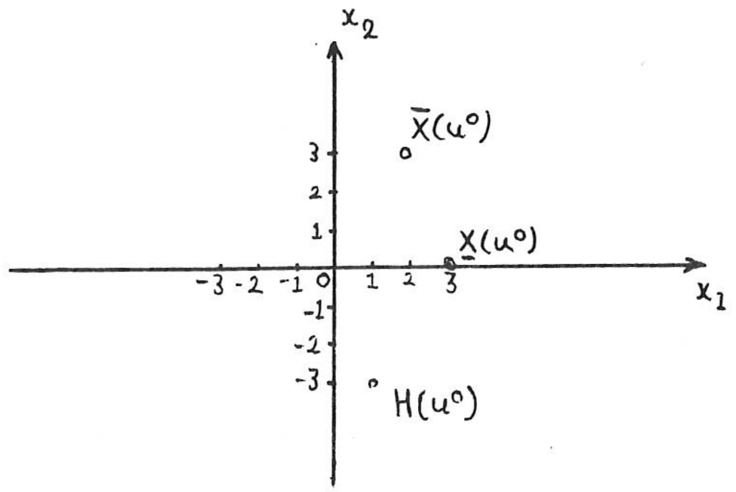
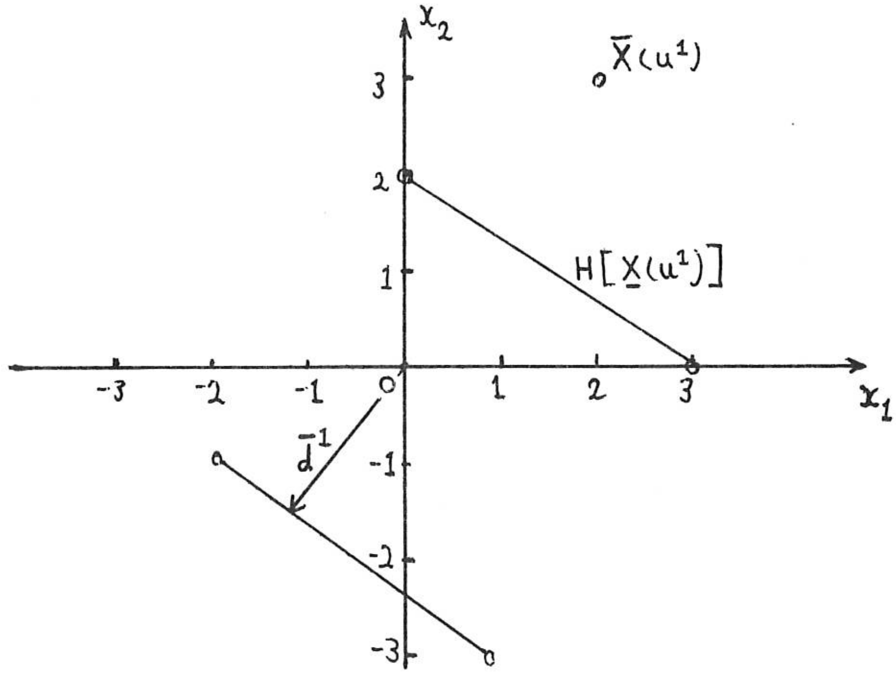


Fig. 3.1 Block Diagram of the Main Program for the Extreme Difference Problem.



x_2
$\bar{X}(u^0)$
$\underline{X}(u^0)$
$H(u^0)$
x_1
-3 -2 -1 0 1 2 3
3 2 1 -1 -2 -3

fig. 4.1



x_2
$\bar{X}(u^1)$
$H[\underline{X}(u^1)]$
$\bar{d}^1$
x_1
-3 -2 -1 0 1 2 3
3 2 1 -1 -2 -3

fig. 4.2.

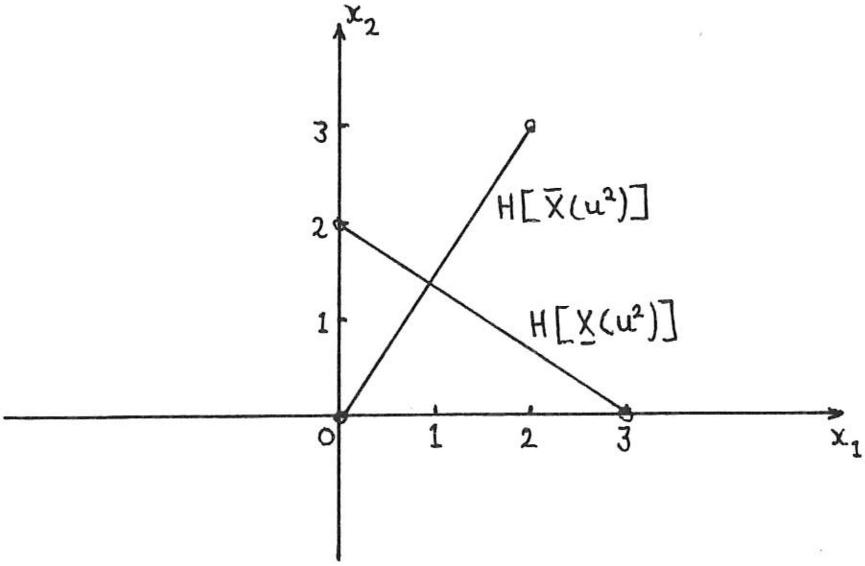


fig. 4.3.