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ON THE COMPENSATION OF OPTIMAL SYSTEM ASYMPTOTIC ROOT-LOCI

by

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Abstract

This paper describes a procedure for manipulating the free parameters of the asymptotes of the optimal closed-loop poles of a time-invariant linear regulator as the weight of the input in the performance criterion approaches zero by suitable choice of weighting matrices Q and R.

1. Introduction

Given a stabilizable and detectable ℓ-input/m-output time-invariant linear system S(A,B,C) ofthe form

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(o) = x_o$$
$$y(t) = Cx(t) \qquad \ldots(1)$$

then the controller minimizing the performance index

$$J = \tfrac{1}{2}\int_o^{\infty} \{y^T(t)Qy(t) + p^{-1}u^T(t)Ru(t)\}dt$$

$$Q = Q^T > 0 \quad , \quad R = R^T > 0 \quad , \quad p > 0 \qquad \ldots(2)$$

is a constant state-feedback controller generated by solving a Riccati equation and[1] the resulting closed-loop system has poles equal to the left-half-plane solutions of the equation

$$|I_\ell + pG^T(-s)G(s)| = 0 \qquad \ldots(3)$$

where, if $Q^{\frac{1}{2}}$ and $R^{\frac{1}{2}}$ are the symmetric, positive-definite square-roots of Q and R respectively, we define

$$G(s) = Q^{\frac{1}{2}}C(sI_n - A)^{-1}B\,R^{-\frac{1}{2}} \qquad \ldots(4)$$

A well-known and accepted problem in optimal controller design is the choice of Q and R to produce acceptable design characteristics. One can regard this problem in terms of the intuitive notion of basing design on ensuring reasonable locations for the closed-loop poles but this is not a tractable proposition in general and hence, fairly recently[2-4], the notion of basing our choice of Q and R on the asymptotic behaviour of the unbounded roots of (3) as $p \to +\infty$ has been introduced. It has been recognized[1,5-8] that the analysis of the solutions of (3) as $p \to +\infty$ is simply a problem in asymptotic root-locus analysis and it is known[5-8] that the unbounded poles (the so-called 'infinite zeros') can be listed in the form

$$s_{j\ell r}(p) = p^{1/2k_j}\eta_{j\ell r} + \alpha_{jr} + \varepsilon_{j\ell r}(p)$$

$$\lim_{p\to\infty} \varepsilon_{j\ell r}(p) = 0$$

$$1 \le \ell \le k_j \quad , \quad 1 \le r \le d_j \quad , \quad 1 \le j \le q \qquad \ldots(5)$$

for suitable choices of integers q, k_j and d_j, $1 \le j \le q$, and complex scalars $\eta_{j\ell r}$, α_{jr}, $1 \le \ell \le k_j$, $1 \le r \le d_j$, $1 \le j \le q$. More precisely, it is now known that

(a) each 'pivot' α_{jr} is pure imaginary and equal to zero[8] for 'almost all' choices of Q and R,

(b) each 'asymptotic direction' $\eta_{j\ell r}$ takes the form $\lambda_{jr}\mu_{j\ell}$ where λ_{jr} is real and strictly positive and the $\mu_{j\ell}$, $1 \le \ell \le k_j$, are the distinct left-half-plane $2k_j^{th}$ roots of $(-1)^{k_j+1}$, and

(c) the integer 'orders' k_j are simply[9] the distinct integer structural invariants of the Morse group and the d_j's are their multiplicities.

There is clearly a lot of structure in the asymptotic behaviour. In fact there is so much structure that many of the parameters are not available to the systems designer as design parameters. More precisely, if one poses the intuitive design objective(2-4) of ensuring that the optimal closed-loop poles move to infinity at a desired rate and at specified angles (Butterworth patterns) to the real axis, it is clear that the $\{k_j\}$ are the dominant plant characteristic governing this behaviour. Unfortunately(9), the values of these integers cannot be influenced by choice of Q and R hence reducing our room for manoeuvre. We do however have some control over the rate (if not the angle) of approach to infinity as the positive-real parameters λ_{jr} are clearly related to the approximate radii of the Butterworth patterns. It is the purpose of this note to point out a systematic method for choosing Q and/or R to produce arbitrary specified values of the parameters λ_{jr}, $1 \leq r \leq d_j$, $1 \leq j \leq q$. This formal algorithmic problem parallels that discussed in ref (10) for non-optimal systems but, in the optimal case, the pivot has a much reduced importance as (see comment (a) above) it is 'almost always' zero. It will be ignored here.

A duality between Q and R is outlined in section 2 and used to reduce the discussion to that of developing procedures for the choice of R alone. The basic results in the form of an algorithm are given in section 3. Unfortunately, it does not seem to be possible at this stage to answer the question 'what is a good choice of λ_{jr}?'. This is a problem shared by other fields such as non-optimal root-loci(10) and, in another form, pole-allocation(11) and will only be resolved by experience in practice with algorithms of the type described below and elsewhere(2-4).

2. The Duality between Q and R

A duality between Q and R is revealed by using a standard determinental identity (see ref (12), p7) to write

$$|I_\ell + pG^T(-s)G(s)| \equiv |I_m + pH^T(-s)H(s)| \qquad \ldots(6)$$

where

$$H(s) = G^T(-s) = R^{-\frac{1}{2}}B^T(-sI_n-A)^{-1}C^TQ^{\frac{1}{2}} \qquad \ldots(7)$$

Suppose now that a technique is available for choosing the matrix R for the system S(A,B,C) to produce the specified asymptotic parameters. Such a technique can then be converted into a procedure for choosing Q by noting that

$$H(s) = \tilde{Q}^{\frac{1}{2}}(-B)^T(sI_n+A^T)^{-1}C^T\tilde{R}^{-\frac{1}{2}} \qquad \ldots(8)$$

(where $\tilde{Q} = R^{-1}$ and $\tilde{R} = Q^{-1}$) is the transfer function matrix generated by the optimal control problem for $S(-A^T, C^T, -B^T)$ with performance index (2) with Q replaced by $\tilde{Q}$ and R replaced by $\tilde{R}$. A choice of $\tilde{R}$ for this problem to produce the specified asymptotic properties will generate a suitable choice of Q for S(A,B,C) by setting $Q = \tilde{R}^{-1}$. For this reason, the remainder of the paper will only consider manipulation of the 'R matrix'.

3. Choice of Q

Suppose that $m \geq \ell$ and that S(A,B,C) is left-invertible and consider the problem of modifying the R matrix to change the revealed asymptotic parameters λ_{jr}, $1 \leq r \leq d_j$, $1 \leq j \leq q$, into specified new values $\tilde{\lambda}_{jr}$, $1 \leq r \leq d_j$, $1 \leq j \leq q$. The results represent a generalization of recent work (see eg. ref (4)) from the case of $m = \ell$ and $|CB| \neq 0$.

The following lemma is fundamental:

Lemma 1: Equation (3) remains valid if G(s) is replaced by

$$\tilde{G}(s) = Q^{\frac{1}{2}} C(sI_n - A)^{-1} BV \qquad \ldots(9)$$

where V is any matrix such that $VV^T = R^{-1}$.

Proof: It is easily verified that $VV^T = R^{-1}$ if, and only if, $V = R^{-\frac{1}{2}}U$ for some orthogonal matrix U. The result then follows from the identity $|I_\ell + pG^T(-s)G(s)| \equiv |I_\ell + p\tilde{G}^T(-s)\tilde{G}(s)|$.

We also need the following construction:

Lemma 2[8]: There exists an orthogonal transformation T_1 and a unimodular polynomial matrix of the form

$$M(s) = \begin{pmatrix} I_{d_1} & O(s^{-1}) & \cdots & O(s^{-1}) \\ 0 & I_{d_2} & & \vdots \\ \vdots & & & O(s^{-1}) \\ 0 & \cdots & 0 & I_{d_q} \end{pmatrix} \qquad \ldots(10)$$

(where the notation $O(s^k)$ is used to denote a function with the property that $\lim_{|s|\to\infty} s^{-(k+1)} O(s^k) = 0$) such that

$$M^T(-s)T_1^T G^T(-s)G(s)T_1 M(s) = \text{block diag } \{Q_j(s)\}_{1\le j\le q} + O(s^{-(2k_q+2)}) \qquad \ldots(11)$$

where the $d_j \times d_j$ transfer function matrices $Q_j(s)$ have uniform rank[12,13] $2k_j$ and take the form $N_j^T(-s)N_j(s)$ for some $m \times d_j$ left-invertible transfer function matrices $N_j(s)$, $1\le j\le q$.

In fact, applying known techniques[12,13], the characterization of equation (5) follows quite simply[8]. In particular, the following result is easily proved:

Lemma 3: The real, strictly positive numbers $\lambda_{jr}^{2k_j}$, $1 \leq r \leq d_j$, are the eigenvalues of the real, symmetric positive-definite matrix

$$Q_j^{(2k_j)} \triangleq \lim_{|s| \to \infty} s^{2k_j} Q_j(s)(-1)^{k_j} \qquad \ldots(12)$$

Consider now the real constant nonsingular matrix

$$L = \text{block diag}\,\{L_j\}_{1 \leq j \leq q} \qquad \ldots(13)$$

where the nonsingular matrices L_j have dimensions $d_j x d_j$, $1 \leq j \leq q$. Multiplying equation (11) from the left and right by L^T and L respectively yields

$$\tilde{M}^T(-s)T_1^{\ T}(T_1L^TT_1^{\ T}G^T(-s)G(s)T_1LT_1^{\ T})T_1\tilde{M}(s)$$

$$= \text{block diag}\,\{L_j^{\ T}Q_j(s)L_j\}_{1 \leq j \leq q} + O(s^{-(2k_q+2)}) \qquad \ldots(14)$$

where $\tilde{M}(s)$ (defined by $M(s)L \equiv L\tilde{M}(s)$) has the same structure as M(s). In fact, we obtain the following main result of this paper:

Theorem: If $\tilde{G}(s) \triangleq G(s)T_1LT_1^{\ T}$, then the left-half-plane solutions of the relation

$$|I_\ell + p\tilde{G}^T(-s)\tilde{G}(s)| = 0 \qquad \ldots(15)$$

are the closed-loop poles of S(A,B,C) with state feedback controller minimizing the performance criterion of equation (2) with R replaced by R_o where

$$R_o^{-1} \triangleq R^{-\frac{1}{2}} T_1 L L^T T_1^T R^{-\frac{1}{2}} \qquad \ldots(16)$$

Moreover, the unbounded solutions of (15) have the form of (5) but where, in particular, the parameters λ_{jr}, $1\le r\le d_j$, $1\le j\le q$, are replaced by the real, strictly positive parameters $\tilde{\lambda}_{jr}$, $1\le r\le d_j$, $1\le j\le q$. The real, strictly positive numbers $\tilde{\lambda}_{jr}^{2k_j}$, $1\le r\le d_j$, are the eigenvalues of

$$\tilde{Q}_j^{(2k_j)} \triangleq L_j^T Q_j^{(2k_j)} L_j \quad , \qquad \ldots(17)$$

$1\le j\le q$.

Proof: The first part of the result follows from the definition of G and $\tilde{G}$, bearing in mind lemma 1. Equation (14) then implies that $\tilde{G}$ satisfies lemma 2 with M and Q_j, $1\le j\le q$, replaced by $\tilde{M}$ and $\tilde{Q}_j = L_j^T Q_j L$, $1\le j\le q$. Standard results[12,13] then indicate that the general characterization of equation (5) remains valid with (lemma 3) $\lambda_{jr}^{2k_j}$, $1\le r\le d_j$, replaced by the eigenvalues of $\lim_{s\to\infty} s^{2k_j} L_j^T Q_j(s) L_j (-1)^{k_j} = L_j^T Q_j^{(2k_j)} L_j$.

The theorem provides an explicit method for manipulation of the asymptotic structure of the optimal root-locus by systematic manipulation of the parameters λ_{jr} describing the radii (when multiplied by $p^{1/2k_j}$) of the Butterworth patterns. For example, suppose that a given choice of Q and R yield infinite zeros with, in particular, parameters λ_{jr}, $1\le r\le d_j$, $1\le j\le q$. These can be obtained by application of known numerical algorithms[8,12,13] to compute (amongst other things) the matrix T_1 and the Markov parameters $Q_j^{(2k_j)}$, $1\le j\le q$, and subsequent application of lemma 3. Suppose that it is desired that the parameters $\{\lambda_{jr}\}$ be replaced by $\tilde{\lambda}_{jr}$, $1\le r\le d_j$, $1\le j\le q$. Write,

$$Q_j^{(2k_j)} = U_j \operatorname{diag}\{\lambda_{jr}^{2k_j}\}_{1\le r\le d_j} U_j^T \quad , \quad 1\le j\le q \qquad \ldots(18)$$

where U_j is the orthogonal eigenvector matrix of $Q_j^{(2k_j)}$ and set

$$L_j = U_j \operatorname{diag} \{\tilde{\lambda}_{jr}^{k_j}/\lambda_{jr}^{k_j}\}_{1\leq r\leq d_j} W_j^T , \quad 1\leq j\leq q \qquad \ldots(19)$$

where W_j is an orthogonal matrix, $1\leq j\leq q$. It is trivially verified that

$$\tilde{Q}_j^{(2k_j)} = W_j \operatorname{diag} \{\tilde{\lambda}_{jr}^{2k_j}\}_{1\leq r\leq d_j} W_j^T , \quad 1\leq j\leq q \qquad \ldots(20)$$

and hence, by the theorem, that the desired objective has been achieved.

Finally, we note that the objective of manipulating the available parameters of the asymptotic optimal root-locus is easily and systematically achieved using the above approach and that the arbitrary matrices W_j can be set equal to identities with no loss in generality as, using (16),

$$R_o^{-1} = R^{-\frac{1}{2}} T_1 \text{block diag} \{U_j \operatorname{diag}\{(\frac{\tilde{\lambda}_{jr}}{\lambda_{jr}})^{2k_j}\}_{1\leq r\leq d_j} U_j^T\}_{1\leq j\leq q} T_1^T R^{-\frac{1}{2}} \qquad \ldots(21)$$

which is independent of $W_1, W_2, \ldots, W_q$.

4. Conclusions

It has been shown that the notion[(2-4)] of using the asymptotic structure of the optimal root-locus for a system $S(A,B,C)$ as the basis of choosing Q and R matrices in the performance index has a simple, elegant and quite general solution if it is regarded as the formal problem of choice of Q and R to produce specified values for the free parameters of the system asymptotes. In this sense the results are a complete generalization of previous work[(2-4)] in this area with the exception that no attempt has been made to manipulate the closed-loop asymptotic eigenvectors. The contribution represents also a generalization of the compensation ideas for non-optimal systems [10] to the optimal case but, at this stage, suffers from the same problem that bedevills other areas such as pole-assignment [11], namely, 'what should be regarded as good choices of asymptotic parameters/poles?'. As in pole-assignment, it appears that this problem will not be resolved at the theoretical level. Only application of the ideas in practice can yield the required insights. It is hoped that the algorithms described in this paper make the application procedures a straightforward computational (if not conceptual) matter.

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