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NORM BOUNDS FOR LINEAR SYSTEMS USING STEP RESPONSE DATA

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Research Report No. 159

September 1981

Not for publication

This work is supported by SERC under grant GR/B/23250

Abstract

Norms for operators play a fundamental role in the stability and approximation theory of linear and nonlinear systems. This note describes how certain induced operator norms can be bounded for continuous and discrete systems using plant step response data only.

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1. Continuous Systems

Any stable m-output/ℓ-input, linear, time-invariant system S(A,B,C,D) of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(o) = x_o \in R^n$$
$$y(t) = Cx(t) + Du(t) \qquad \ldots(1)$$

can be associated with a mapping L of the Banach space $C^{\ell}(0,\infty)$ of bounded continuous mappings of $(0,+\infty)$ into R^{ℓ} into $C^m(0,\infty)$. We can choose the norm in R^q as

$$||x||_q \triangleq \max_{1 \leq j \leq q} |x_j| \qquad \ldots(2)$$

where $x \in R^q$ is regarded as the column $x = (x_1, x_2, \ldots, x_q)^T$ and the norm in $C^q(0,\infty)$ as

$$||y|| \triangleq \sup_{t \geq 0} ||y(t)||_q \qquad \ldots(3)$$

The mapping L can be represented by the 'convolution'

$$(Lu)(t) = \int_0^t H(t')u(t-t')dt' + Du(t) \quad , \quad t \geq 0 \qquad \ldots(4)$$

where H(t) is the m$\times\ell$ impulse response matrix

$$H(t) \triangleq C\, e^{At} B \quad , \quad t \geq 0 \qquad \ldots(5)$$

Noting that the matrix norm $||\cdot||$ on m$\times\ell$ matrices induced by the vector norm (2) is just

$$||M|| \triangleq \max_{1\leq i\leq m} \sum_{j=1}^{\ell} |M_{ij}| \qquad \ldots(6)$$

then the stability assumption guarantees the existence of real numbers $M\geq 0$ and $\alpha>0$ such that, for each (i,j),

$$|H_{ij}(t)| \leq ||H(t)|| \leq M\,e^{-\alpha t} \quad \forall\ t\geq 0 \qquad \ldots(7)$$

(Note: although in what follows we are primarily interested in linear systems of the form of (1), we note that the results apply to any system with input/output map $u \to y = Lu$ with L of the form of (4) and (7) with H piecewise-continuous (say). Our results extend in this sense to differential-delay systems etc).

Our concern is the evaluation of the induced norm of L in $\mathcal{L}(C^{\ell}(0,\infty),\ C^{m}(0,\infty))$ defined by

$$||L|| = \sup_{||x||=1} \frac{||Lx||}{||x||} \qquad \ldots(8)$$

The approach taken will be to reduce the evaluation of $||L||$ to the evaluation of norms of single-input/single-output systems. To do this we will regard the system as decomposed in the form

$$L = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{pmatrix} \qquad \ldots(9)$$

where L_i is the map of $C^{\ell}(0,\infty)$ into $C^{1}(0,\infty)$ defined by $u \to y_i$ or, more explicitly,

$$(L_i u)(t) = \sum_{j=1}^{\ell} \left(\int_0^t H_{ij}(t')u_j(t-t')dt' + D_{ij}u_j(t)\right) \qquad \ldots(10)$$

Clearly each L_i is an ℓ-input/single-output stable system which can be decomposed in an obvious way into single-input/single-output stable systems by writing

$$L_i u = \sum_{j=1}^{\ell} L_{ij} u_j \qquad \forall\ u \qquad \ldots(11)$$

where L_{ij} maps $C^1(0,\infty)$ into itself and takes the obvious form

$$(L_{ij}u_j)(t) = \int_0^t H_{ij}(t')u_j(t-t')dt' + D_{ij}u_j(t) \qquad \ldots(12)$$

We can identify certain relations between the norms, namely,

Proposition 1:

$$||L|| = \max_i ||L_i|| \qquad \ldots(13)$$

Proof: Let q be the index such that $||L_q|| = \max_i ||L_i||$ and let $\varepsilon>0$ be arbitrary. There exists u such that $||u|| = 1$ and $y_q = L_q u$ satisfies $||y_q|| > ||L_q|| - \varepsilon$. Setting $y = Lu$ then $||L|| \geq ||y|| \geq ||y_q|| > ||L_q|| - \varepsilon$. As ε is arbitrary we get $||L_q|| = \max_i ||L_i|| \leqslant ||L||$. But $||y|| = \max_i ||y_i||$ $= \max ||L_i u|| \leq \max_i ||L_i||$ for any u of unit norm and the result follows.

The norms of the L_i can be related to those of the L_{ij} by the obvious inequality,

$$||L_i|| \leq \sum_{j=1}^{\ell} ||L_{ij}|| \qquad \ldots(14)$$

and we can obtain $||L_{ij}||$ from the following result,

Proposition 2:

$$||L_{ij}|| = \int_0^\infty |H_{ij}(t)|dt + |D_{ij}| \qquad \ldots(15)$$

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and the obvious corollary follows that, using (13), (14) and (15),

$$||L|| \leq \max_{1 \leq i \leq m} \sum_{j=1}^{\ell} \left(\int_0 |H_{ij}(t)|dt + |D_{ij}| \right) \qquad \ldots(16)$$

<u>Proof of Proposition 2:</u> Let $\varepsilon > 0$ be arbitrary and note that the convergence of the integral

$$\int_0^\infty |H_{ij}(t)|dt \leq \int_0^\infty ||H(t)|| \, dt \leq M \int_0^\infty e^{-\alpha t} dt$$

$$\leq \frac{M}{\alpha} \qquad \ldots(17)$$

ensure that there exists $t_o \geq 0$ such that $\int_{t_o}^{\infty} |H_{ij}(t)|dt < \varepsilon/2$. Choosing the <u>piecewise</u> continuous input

$$u_j(t) = \begin{cases} \text{sgn } H_{ij}(t_o - t) & , \quad t < t_o \\ \text{sgn } D_{ij} & , \quad t \geq t_o \end{cases} \qquad \ldots(18)$$

then $||u|| = 1$, $y_i = L_{ij} u_j$ is bounded with

$$|y_i(t_o)| = \left| \int_0^{t_o} H_{ij}(t) u_j(t_o - t) dt + D_{ij} u(t_o) \right|$$

$$= \int_0^{t_o} |H_{ij}(t)|dt + |D_{ij}|$$

$$> \int_0^\infty |H_{ij}(t)|dt + |D_{ij}| - \varepsilon/2 \qquad \ldots(19)$$

It is trivial to verify that we can replace u_j by a continuous input $\tilde{u}_j$ satisfying $||\tilde{u}_j|| = 1$ and, denoting $\tilde{y}_i = L_{ij}\tilde{u}_j$, $|\tilde{y}_i(t_o) - y_i(t_o)| < \varepsilon/2$. Clearly,

$$\|L_{ij}\| \geq \|\tilde{y}_i\| \geq |\tilde{y}_i(t_o)| > |y_i(t_o)| - \varepsilon/2$$

$$> \int_0^\infty |H_{ij}(t)|dt + |D_{ij}| - \varepsilon \qquad \ldots(20)$$

and hence, as ε is arbitrary,

$$\|L_{ij}\| \geq \int_0^\infty |H_{ij}(t)|dt + |D_{ij}| \qquad \ldots(21)$$

The result follows as, for any input u_j, it is easily seen that

$$|y_i(t)| = |\int_0^t H_{ij}(t')u(t-t')dt' + D_{ij}u_j(t)|$$

$$\leq \{ \int_0^\infty |H_{ij}(t)|dt + |D_{ij}|\} \|u_j\| \qquad \ldots(22)$$

hence reversing the inequality in (21).

It is however possible to strengthen (16) to equality using the result:

<u>Proposition 3</u>:

$$\|L_i\| = \sum_{j=1}^{\ell} \|L_{ij}\| \quad , \quad 1\leq i\leq m \qquad \ldots(23)$$

<u>Proof</u>: We need to show that the inequality in (14) can be reversed. Let $\varepsilon>0$ be arbitrary and choose $t_o\geq 0$ so that, $1\leq j\leq \ell$,

$$\int_{t_o}^\infty |H_{ij}(t)|dt < \varepsilon/2\ell \qquad \ldots(24)$$

If u_j is given by (18) for $1\leq j\leq \ell$, then, writing $y_i = \sum_j L_{ij}u_j$,

$$|y_i(t_o)| = |\sum_{j=1}^{\ell} (\int_0^{t_o} H_{ij}(t)u_j(t_o-t)dt + D_{ij}u_j(t_o))|$$

$$= \sum_{j=1}^{\ell} (\int_0^{t_o} |H_{ij}(t)|dt + |D_{ij}|)$$

$$> \sum_{j=1}^{\ell} (\int_0^{\infty}|H_{ij}(t)|dt + |D_{ij}|) - \varepsilon/2 \qquad \ldots(25)$$

As in the proof of proposition 2, replace u by $\tilde{u}$ such that $||\tilde{u}|| = 1$ and $|\tilde{y}_i(t_o)-y_i(t_o)| < \varepsilon/2$. A similar argument to (20) yields

$$||L_i|| > \sum_{j=1}^{\ell} (\int_0^{\infty}|H_{ij}(t)|dt + |D_{ij}|) - \varepsilon \qquad \ldots(26)$$

and the result follows from (15) and the fact that ε is arbitrary.

In fact we have proved the following theorem relating the norm of L to the impulse response matrix H(t):

Theorem 1:

$$||L|| = \max_{1\leq i\leq m} ||L_i|| = \max_{1\leq i\leq m} \sum_{j=1}^{\ell} ||L_{ij}||$$

$$= \max_{1\leq i\leq m} \sum_{j=1}^{\ell} (\int_0^{\infty}|H_{ij}(t)|dt + |D_{ij}|)$$

$$\leq \int_0^{\infty} ||H(t)|| \, dt + ||D|| \qquad \ldots(27)$$

Proof: The equalities follow from (13), (23), (15) and the inequality from

$$\max_{1\leq i\leq m} \sum_{j=1}^{\ell} (\int_0^{\infty}|H_{ij}(t)|dt + |D_{ij}|)$$

$$\leq \int_0^{\infty} \max_i \sum_j |H_{ij}(t)|dt + \max_i \sum_j |D_{ij}| \qquad \ldots(28)$$

and application of the definition (6).

Given the impulse response data $H(t)$ and D, (27) enables us to compute or bound $||L||$. In the more practical situation where the system step response matrix

$$Y(t) \triangleq \int_0^t H(t')dt' + D \qquad \ldots(29)$$

is available from plant tests or model simulations ($Y_{ij}(t)$ being the response of y_i from zero initial conditions to a unit step in u_j) then H and D can, in principle be computed. The main result of this section indicates however that it is possible to avoid the evaluation of integrals in (27) and, in cases where the system is non-oscillatory, to compute $||L_{ij}||$, and hence $||L||$, from a finite number of carefully selected data points of Y_{ij}.

Theorem 2:

For each pair of indices (i,j), there is a finite or infinite sequence of times $0 = t_{ij0} < t_{ij1} < t_{ij2} < \ldots$ such that $[0,\infty) = \bigcup_k [t_{k-1}, t_k)$ and

$$||L_{ij}|| = \sum_{k\geq 1} |Y_{ij}(t_{ijk}) - Y_{ij}(t_{ijk-1})| + |Y_{ij}(t_{ijo})| \qquad \ldots(30)$$

Moreover, the sequence $\{t_{ijk}\}$ is any sequence with any one of the following (equivalent) properties,

(a) $H_{ij}(t)$ is either ≥ 0 or ≤ 0 in every interval $t_{ijk} \leq t \leq t_{ijk+1}$

(Note: the sign of H_{ij} may vary from interval to interval)

(b) $Y_{ij}(t)$ is monotonic on every interval $t_{ijk} \leq t \leq t_{ijk+1}$.

Proof: Simply use (15) and note that (a) yields the identity

$$||L_{ij}|| = \sum_{k\geq 1} \int_{t_{ijk-1}}^{t_{ijk}} |H_{ij}(t)|dt + |D_{ij}|$$

$$= \sum_{k\geq 1} \left| \int_{t_{ijk-1}}^{t_{ijk}} H_{ij}(t)dt \right| + |D_{ij}| \qquad \ldots(31)$$

and the result follows as

$$Y_{ij}(o) = Y_{ij}(t_{ijo}) = D_{ij}, \text{ and}$$

$$\int_{t_{ijk-1}}^{t_{ijk}} H_{ij}(t)dt = \int_0^{t_{ijk}} H_{ij}(t)dt - \int_0^{t_{ijk-1}} H_{ij}(t)dt$$

$$= Y_{ij}(t_{ijk}) - Y_{ij}(t_{ijk-1}) \qquad \ldots(32)$$

The equivalence of (a) and (b) is a trivial consequence of the definition (29).

This result has a clear advantage over Proposition 2 in that only a countable (and, in many cases, finite) amount of step response data is needed to evaluate $||L_{ij}||$ and hence $||L||$ exactly. The actual construction used is illustrated in Fig.1 using both impulse and step response properties (a) and (b). It is immediately clear that the t_{ijk} can be taken, by inspection, to be the crossover points of the impulse response $H_{ij}(t)$ or the stationary points of the step response. It is also clear that only a finite number of t_{ijk} are needed if the system is non-oscillatory (or, more generally, asymptotically monotone) but that an infinite number of t_{ijk} are required if the system has dominant oscillatory

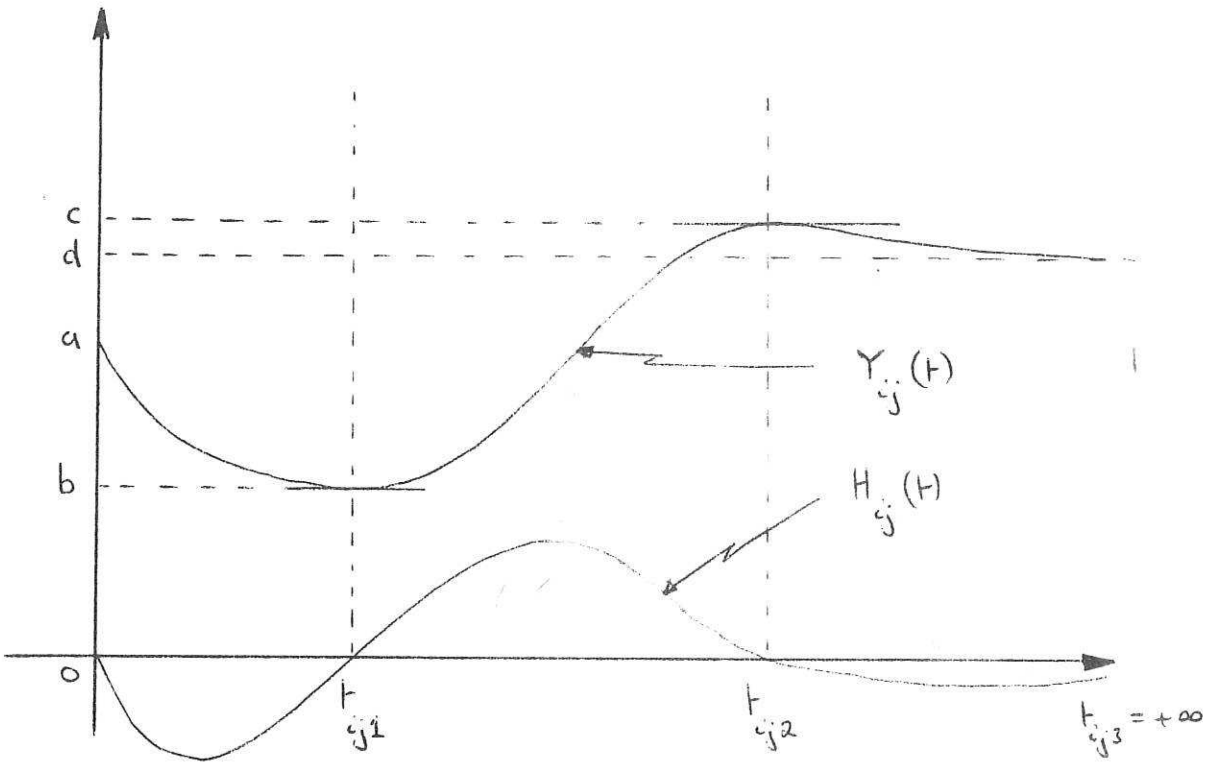

Fig.1. Illustrating the calculation $\|L_{ij}\| = a+(a-b)+(c-b)+(c-d)$

modes. The problem of summing (30) in this case is clearly insoluble as only a finite amount of data is normally available. The following corollary indicates that an arbitrary accuracy is possible if long finite data records are available.

Corollary 2.1.

Given any $\varepsilon>0$, there exists a time T such that knowledge of $Y_{ij}(t)$ on any interval $[0,\hat{T}]$ with $\hat{T}\geq T$ ensures that the numerical estimate

$$E_{ij} \triangleq |Y_{ij}(\hat{T}) - Y_{ij}(t_{ijk^*})| + \sum_{k=1}^{k^*} |Y_{ij}(t_{ijk}) - Y_{ij}(t_{ijk-1})| + |Y_{ij}(t_{ijo})| \qquad \ldots(33)$$

(where k* is the largest index such that $t_{ijk} < \hat{T}$) satisfies the accuracy relation

$$||L_{ij}|| - \varepsilon < E_{ij} \leq ||L_{ij}|| \qquad \ldots(34)$$

Proof: The proof is virtually identical to that of theorem 2 applied to the approximation

$$||L_{ij}|| \geq |D_{ij}| + \int_0^{\hat{T}} |H_{ij}(t)|\,dt \triangleq E_{ij} \qquad \ldots(35)$$

to $||L_{ij}||$. Clearly, if $\hat{T}$ is large enough, the convergence of the integral as $\hat{T}\to+\infty$ guarantees arbitrary accuracy.

Finally, we note the following bound on $||L||$.

Corollary 2.2:

By replacing $\{t_{ijk}\}$ by an ordering of $\bigcup_{ij}\{t_{ijk}\}_{k\geq 1}$, we can take $t_{ijk} = t_k$ (independent of i,j) for all (i,j) and k. Under this construction, $t_0 = 0$ and

$$||L|| \leq \sum_{k\geq 1} ||Y(t_k) - Y(t_{k-1})|| + ||Y(t_o)|| \qquad \ldots(36)$$

Proof: Simply note that, using the above,

$$||L|| = \max_{1\leq i\leq m} \sum_{j=1}^{\ell} \Big(\sum_{k\geq 1} |Y_{ij}(t_k) - Y_{ij}(t_{k-1})| + |Y_{ij}(t_o)|\Big)$$

$$\leq \sum_{k\geq 1} \max_{1\leq i\leq m} \sum_{j=1}^{\ell} |Y_{ij}(t_k) - Y_{ij}(t_{k-1})| + \max_{1\leq i\leq m} \sum_j |Y_{ij}(t_o)| \qquad \ldots(37)$$

which is simply (36).

The bound (36) would appear however to be only of theoretical interest.

2. Discrete Systems

The results described above carry over with little change to the analysis of a stable, ℓ-input/m-output, linear, time-invariant discrete system

$$x_{k+1} = \Phi x_k + \Delta u_k$$
$$y_k = Cx_k + Du_k \qquad \ldots(38)$$

and the 'induced' linear mapping L defined by the 'convolution',

$$(Lu)_k = \sum_{j=1}^{k} H_j u_{k-j} + Du_k \qquad \ldots(39)$$

mapping ℓ_∞^{ℓ} into ℓ_∞^{m}. Here H_j is the 'Markov parameter'

$$H_j = C\Phi^{j-1}\Delta \quad , \quad j\geq 1 \qquad \ldots(40)$$

and ℓ_∞^{q} is the natural Banach space associated with sequences $y = \{y_o, y_1, \ldots\}$ in R^q with norm

$$\|y\| = \sup_{k\geq o} \|y_k\|_q < +\infty \qquad \ldots(41)$$

The stability assumption is equivalent to the existence of real numbers $M\geq 0$ and $0\leq\lambda<1$ satisfying

$$\|H_j\| \leq M\lambda^j \quad , \quad j\geq 1 \qquad \ldots(42)$$

We note that the results depend only upon the convolution and stability structure imposed on the problem and hence can be applied to a larger class of system than (38).

Comparing (4) and (39) we see that we need only replace integration by summation and the impulse response H(t) by the impulse sequence $\{H_1, H_2, \ldots\}$. As such the proofs of the following results are virtually identical to those of their equivalents in section 1 without the complications introduced by continuity etc. They are hence omitted.

We do however need to identify L with single-output systems using the decomposition (9) with L_i mapping ℓ_∞^{ℓ} into ℓ_∞^{1} defined by

$$(L_i u)_k = \sum_{j=1}^{\ell} \left(\sum_{r=1}^{k} (H_r)_{ij} (u_{k-r})_j + D_{ij} (u_k)_j \right) \qquad \ldots(43)$$

and decomposing L_i into single-input/single-output system as in (11) with

$$(L_{ij} u_j)_k = \sum_{r=1}^{k} (H_r)_{ij} (u_{k-r})_j + D_{ij} (u_k)_j \qquad \ldots(44)$$

where L_j maps ℓ_∞^{1} into itself. In more detail we get.

Proposition 4: (c.f. proposition 1)

$$||L|| = \max_i ||L_i|| \qquad \ldots(45)$$

Proposition 5: (c.f. proposition 2)

$$||L_{ij}|| = \sum_{k=1}^{\infty} |(H_k)_{ij}| + |D_{ij}| \qquad \ldots(46)$$

Proposition 6: (c.f. proposition 3)

$$||L_i|| = \sum_{j=1}^{\ell} ||L_{ij}|| \qquad \ldots(47)$$

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and hence

Theorem 3: (c.f. theorem 1)

$$||L|| = \max_{1 \le i \le m} ||L_i|| = \max_{1 \le i \le m} \sum_{j=1}^{\ell} ||L_{ij}||.$$

$$= \max_{1 \le i \le m} \sum_{j=1}^{\ell} \left(\sum_{k=1}^{\infty} |(H_k)_{ij}| + |D_{ij}| \right)$$

$$\leq \sum_{k=1}^{\infty} \|H_k\| + \|D\| \qquad \ldots(48)$$

Introducing the step response sequence,

$$Y_k = \sum_{j=1}^{k} H_j + D \quad , \qquad k \geq 0 \qquad \ldots(49)$$

we obtain the following equivalent of theorem 2:

Theorem 4:

For each pair of indices (i,j), there exists a finite or infinite sequence of sampling instants $0 = k_{ijo} < k_{ij1} < \ldots\ldots$ such that

$$\|L_{ij}\| = |(Y_o)_{ij}| + \sum_{r \geq 1} |(Y_{k_{ijr}})_{ij} - (Y_{k_{ijr-1}})_{ij}| \qquad \ldots(50)$$

Moreover, the sequence $\{k_{ijr}\}$ is any sequence with one of the following equivalent properties

(a) $(H_k)_{ij}$ is either ≥ 0 or ≤ 0 in every integer interval

$k_{ijr} < k \leq k_{ijr+1}$.

(b) $(Y_k)_{ij}$ is monotonic on every interval $k_{ijr} \leq k \leq k_{ijr+1}$.

The graphical construction is shown in Fig.2.

The use of finite data sequences is described by:

Corollary 4.1:

Given any $\varepsilon > 0$, there exists a time k' such that knowledge of $(Y_k)_{ij}$, $k \geq 0$, on any interval $k \leq \hat{k}$ with $\hat{k} \geq k'$ ensures that the numerical estimate

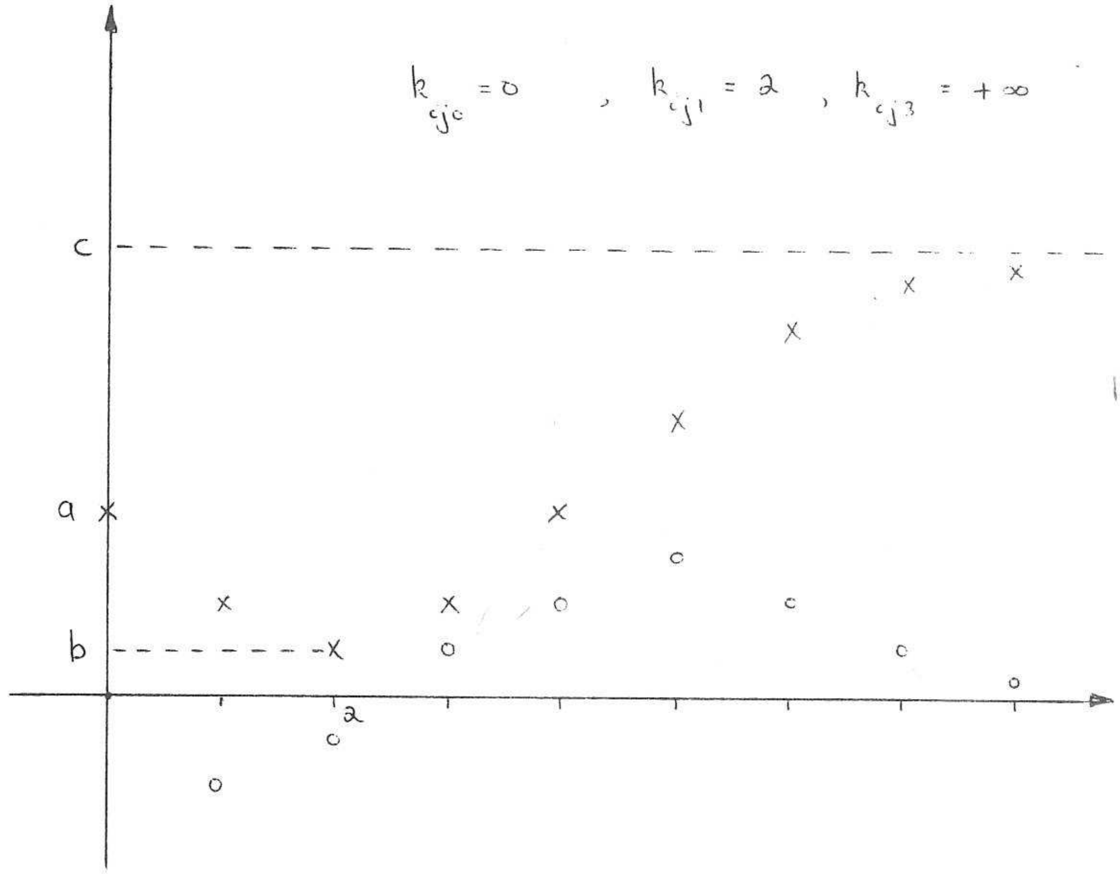

Fig.2. Illustrating the calculation $\|L_{ij}\| = a+(a-b)+(c-b)$

$$E_{ij} \triangleq |(Y_o)_{ij}| + \sum_{r=1}^{r^*} |(Y_{k_{ijr}})_{ij} - (Y_{k_{ijr-1}})_{ij}| + |(Y_{\hat{k}})_{ij} - (Y_{k_{ijr^*}})_{ij}| \qquad \ldots(51)$$

(where r^* is the largest index such that $k_{ijr^*} < \hat{k}$) satisfies the accuracy relation

$$\|L_{ij}\| - \varepsilon < E_{ij} \leq \|L_{ij}\| \qquad \ldots(52)$$

The following bound on $\|L\|$ can also be obtained:

Corollary 4.2:

By replacing $\{k_{ijr}\}$ by an ordering of $\bigcup_{ij}\{k_{ijr}\}_{r\geq 1}$, we can take $k_{ijr} = k_r$ (independent of (i,j)) for all)i,j) and r. Under this construction

$$||L|| \leq \sum_{r\geq 1} ||Y_{k_r} - Y_{k_{r-1}}|| + ||Y_o|| \qquad \ldots(53)$$

Finally, the discrete system theory can be used to answer the practical problem of estimating the norm of a continuous system L of the form of (1) if the system step response is available only at (synchronous) sample times 0,h,2h,3h,... Clearly this data only allows the calculation of the norm **of** the system regarded as a discrete system L_d of the form of (38) with sample interval h. We can however prove the following result indicating that a good estimate is obtained at fast sampling rates.

Proposition 7:

$$||L|| = \lim_{h\to 0+} ||L_d|| \qquad \ldots(54)$$

Proof: The graphical construction for both $||L||$ and $||L_d||$ indicates that

$$||L_d|| \leq ||L|| \qquad h > 0 \qquad \ldots(55)$$

and that $||L_d||$ increases monotonically as h decreases. Clearly therefore

$$\lim_{h\to 0+} ||L_d|| \leq ||L|| \qquad \ldots(56)$$

To show that equality must hold, simply note that as $h\to 0+$, the estimate of $||L_d||$ obtained from a finite data record Y_k, $0\leq k\leq t_o/h$, is arbitrarily close to the estimate of $||L||$ obtained from $Y(t)$ on the interval $0 \leq t \leq t_o$. But (Proposition 2.1), suitable choice of t_o enables the estimate of $||L||$ to be arbitrarily accurate and hence that $||L_d||$ is close to $||L||$ under fast sampling conditions. Equality therefore holds in (56).

Finally note that the result can also be interpreted as stating that small errors in estimating the t_{ijk} will only lead to small errors in estimating $||L||$.

3. Some Illustrative Examples

3.1. Systems with Monotonic, Sign-definite Step Responses

If the continuous system (12) is monotonic and sign-definite (ie either $H_{ij}(t) \geq 0 \quad t \geq 0$ and $D_{ij} \geq 0$ or $H_{ij}(t) \leq 0 \quad t \geq 0$ and $D_{ij} \leq 0$) then $Y_{ij}(t)$ increases or decreases monotonically from 0 at $t = 0-$. Clearly $t_{ijo} = 0$ and $t_{ij1} = +\infty$ and

$$||L_{ij}|| = |Y_{ij}(\infty) - Y_{ij}(o)| + |Y_{ij}(o)|$$

$$= |Y_{ij}(\infty)| \qquad \ldots(57)$$

Clearly, the steady state value of the step response is sufficient data to calculate the norm in this case.

The equivalent result for the discrete system (44) is that

$$||L_{ij}|| = |(Y_\infty)_{ij}| \qquad \ldots(58)$$

if either $(H_k)_{ij} \geq 0 \quad k \geq 1$ and $D_{ij} \geq 0$ or $(H_k)_{ij} \leq 0 \quad k \geq 1$ and $D_{ij} \leq 0$. In this case, $k_{ijo} = 0$ and $k_{ij1} = +\infty$.

3.2. Norms for First Order Systems

The stable first-order models

$$\dot{y}(t) + \lambda y(t) = gu(t) \quad , \quad \lambda>0 \qquad \dots(59)$$

and

$$y_{k+1} + \lambda y_k = gu_k \quad , \quad |\lambda|<1 \qquad \dots(60)$$

have norms g/λ and $g/(1+\lambda)$ respectively as they have monotone step responses (see section 3.1).

3.3. Norms for Second-order Systems

Consider the second order system

$$\ddot{y} + 2\xi\omega_o\dot{y} + \omega_o{}^2 y = \omega_o{}^2 u \qquad \dots(61)$$

If the damping ratio $\xi \geq 1$, the step response is monotonic and hence (section 3.1) the system has norm $||L|| = 1$. If however $0<\xi<1$, the situation changes completely but the norm can still be evaluated in closed-form. More precisely, the system response from zero initial conditions to a unit step input is simply obtained from the transfer function

$$g(s) = \frac{\lambda\bar{\lambda}}{(s-\lambda)s-\bar{\lambda})} \qquad \dots(62)$$

where $\lambda = -\xi\omega_o - j\omega_o\surd(1-\xi^2)$ to be

$$Y(t) = 1 + \frac{\bar{\lambda}e^{-\lambda t}}{(\lambda-\bar{\lambda})} + \frac{\lambda e^{-\bar{\lambda}t}}{(\bar{\lambda}-\lambda)} \qquad \dots(63)$$

Solving the equation $\dot{Y} = 0$ yields the time sequence

$$t_k = \frac{2k\pi j}{\bar{\lambda}-\lambda}$$

$$= \frac{k\pi}{\omega_o\surd 1-\xi^2} \qquad k = 0,1,2,\dots \qquad \dots(64)$$

From (63) we see that

$$Y(t_k) = 1 + 2\mathrm{Re}\left(\frac{\bar{\lambda}}{\lambda-\bar{\lambda}} e^{-\lambda t} \right)$$

$$= 1 + 2Ae^{-\xi\omega_o t_k}\cos(\omega_o\sqrt{1-\xi^2}\,t_k+\phi) \qquad \ldots(65)$$

where A and ϕ are deduced from the polar decomposition

$$\frac{\bar{\lambda}}{\lambda-\bar{\lambda}} = Ae^{j\phi} \qquad \ldots(66)$$

and hence, using (64), that

$$Y(t_k) = 1 + 2Ae^{-\xi\omega_o t_k}\cos(k\pi+\phi)$$

$$= 1 + (-1)^k 2Ae^{-\xi\omega_o t_k}\cos\phi \qquad \ldots(67)$$

Clearly, $Y(t_o) = 0$, and

$$Y(t_k) - Y(t_{k-1})$$

$$= (-1)^k 2A\, e^{-\xi(1-\xi^2)^{-\frac{1}{2}}(k-1)\pi} \times$$

$$(e^{-\xi(1-\xi^2)^{-\frac{1}{2}}\pi} + 1)\cos\phi \qquad \ldots(68)$$

so that

$$||L|| = (1 + e^{-\xi(1-\xi^2)^{-\frac{1}{2}}\pi})\,|2A\cos\phi|\,\times$$

$$\sum_{k=1}^{\infty} e^{-\xi(1-\xi^2)^{-\frac{1}{2}}(k-1)\pi}$$

$$= \frac{(1 + e^{-\xi(1-\xi^2)^{-\frac{1}{2}}\pi})}{(1 - e^{-\xi(1-\xi^2)^{-\frac{1}{2}}\pi})} \qquad \ldots(69)$$

where we have used the observation that the sum is a geometric series and, from (67) and the fact that $Y(t_o) = 0$, that $1+2A\cos\phi = 0$.

Finally, note that

$$\lim_{\xi\to 1-} ||L|| = 1 \quad , \quad \lim_{\xi\to 0+} ||L|| = +\infty \qquad \ldots(70)$$

indicating that $||L||$ is a continuous function of ξ on $(0,+\infty)$ and unbounded in the vicinity of $\xi = 0+$. $||L||$ is independent of ω_o as it is independent of scaling of time.

4. Frequency and Time-domain Norms

Application of functional analytic techniques in stability theory can use 'frequency' or 'time'-domain based analysis. We see below that bounds for time-domain norms can also be bounds for frequency domain norms.

4.1. Continuous Systems

The continuous system (1), after Laplace transforming and rearrangement, can be written in the form

$$y(s) = G(s)u(s) \qquad \ldots(71)$$

where $y(s)$ (resp. $u(s)$) is the Laplace transform of the $m\times 1$ (resp. $(\ell\times 1)$) output (resp. input) vector and G is the $m\times\ell$ transfer function matrix

$$G(s) = C(sI_n-A)^{-1}B + D \qquad \ldots(72)$$

As the system is stable it can be regarded as a map of $E_\ell(S)$ into $E_m(S)$ where $E_q(S)$ denotes the Banach space of $q\times 1$ vector-valued functions of the complex variable, bounded and analytic in the 'Nyquist region'

$$S \triangleq \{s : \text{Re}s > 0 \quad , \quad |s| < R\} \qquad \ldots(73)$$

where R is 'large' in the standard sense. If the boundary of S (ie the standard Nyquist D-contour) is denoted ∂S, the norm in $E_q(S)$ is just

$$||y|| \triangleq \sup_{s \in \partial S} ||y(s)||_q \qquad \dots(74)$$

and the induced operator norm is

$$||G|| \triangleq \sup_{s \in \partial S} ||G(s)|| \qquad \dots(75)$$

Noting that

$$G(s) = \int_0^\infty e^{-st} H(t)dt \qquad \dots(76)$$

is defined for Re s > $-\alpha$, we see that the norm of G and L are related through H. More precisely,

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Proposition 8:

For the linear system (1)

$$||G|| \le ||L|| \qquad \dots(77)$$

Proof: If the supremum in (75) is achieved for a point $|s| = R$ on the compact set ∂S, then, letting $R \to \infty$, $||G|| = ||D||$ which is clearly less than or equal to $||L||$ (set t = 0 in (4)). If however the supremum is achieved at $s = i\omega$ on the imaginary axis and q is the index such that

$$||G|| = ||G(i\omega)|| = \sum_{j=1}^{\ell} |G_{qj}(i\omega)| \qquad \dots(78)$$

let $u(t) = \mathrm{Re}\, u_o e^{i\omega t}$. Clearly the resultant response is just

$$y(t) = \mathrm{Re}\ G(i\omega) u_o e^{i\omega t} + y_o(t) \qquad \dots(79)$$

where, for suitable choice of $M' \ge 0$

$$||y_o(t)||_m \le M' e^{-\alpha t} \qquad \dots(80)$$

Let t_o be such that $\|y_o(t)\|_m < \varepsilon$ for $t \geq t_o$ where $\varepsilon > 0$ is arbitrary and consider $y_q(t)$ for $t \geq t_o$ ie

$$|y_q(t) - \sum_{j=1}^{\ell} \mathrm{Re}G_{qj}(i\omega)u_{oj}e^{i\omega t}| < \varepsilon \qquad \ldots(81)$$

Choosing $u_{oj} = e^{-i\phi_j}$ where ϕ_j is the phase of $G_{qj}(i\omega)$ yields, via (78)

$$|y_q(t) - \sum_{j=1}^{\ell} |G_{qj}(i\omega)| \cos \omega t|$$

$$= |y_q(t) - \|G\| \cos \omega t\| < \varepsilon \qquad \ldots(82)$$

Noting that $\|u\| = 1$, we conclude that

$$\|L\| \geq \|y\| \geq \sup_{t \geq t_o} |y_q(t)|$$

$$> \sup_{t \geq t_o} \|G\| |\cos\omega t| - \varepsilon = \|G\| - \varepsilon \qquad \ldots(83)$$

The result follows as ε is arbitrary.

There are a number of obvious corollaries obtained by substituting for $\|L\|$ from theorem 1. These are omitted. We will however state one interesting case.

Proposition 9:

If the linear system (1) is monotonic and sign-definite

$$\|G\| = \|L\| = \|Y(\infty)\| \qquad \ldots(84)$$

Proof: The identity $\|L\| = \|Y(\infty)\|$ follows from the discussion in section 3.1 and $\|G\| = \|L\|$ as $\|G\| \leq \|L\|$ from proposition 8 and

$$G(o) = \int_0^{\infty} H(t)dt = Y(\infty) \qquad \ldots(85)$$

by definition.

4.2. Discrete Systems

After taking z-transforms, the discrete system (38) can be written as

$$y(z) = G(z)u(z) \qquad \ldots(86)$$

where G(z) is the z-transfer function matrix

$$G(z) = C(zI_n - \Phi)^{-1}\Delta + D \qquad \ldots(87)$$

The stability assumption then implies that the system G maps $E_\ell(S)$ into $E_m(S)$ with S defined to be

$$S \triangleq \{z : 1 < |z| < R\} \qquad \ldots(88)$$

with compact boundary,

$$\partial S \triangleq \{z : |z| = 1 \text{ or } |z| = R\} \qquad \ldots(89)$$

where R is taken to be 'large'. The discrete equivalent to proposition 8 is easily proved:

Proposition 10:

For the discrete system (38),

$$||G|| \leq ||L|| \qquad \ldots(90)$$

Proof: If the supremum in (74) is achieved at a point z such that $|z| = R$ then, as $R \to +\infty$, $||G|| = ||D|| \leq ||L||$ by (39) (set k = 0). If however the supremum is achieved at $z = \beta$ with $|\beta| = 1$ and q is the index such that

$$||G|| = ||G(\beta)|| = \sum_{j=1}^{\ell} |G_{qj}(\beta)| \qquad \ldots(91)$$

set $u_k = \mathrm{Re}\,\hat{u}_o\beta^k$. The resultant response is

$$y_k = \mathrm{Re}\,G(\beta)u_o\beta^k + y_o(k) \qquad \ldots(92)$$

where, for suitable choice of $M' \geq 0$,

$$||y_o(k)||_m \leq M'\lambda^k \quad , \quad k \geq 0 \qquad \ldots(93)$$

The remainder of the proof follows the pattern set out in proposition 8 and is omitted.

The discrete equivalent of proposition 9 is

Proposition 11:

If the discrete system (38) is monotonic and sign-definite

$$||G|| = ||L|| = ||Y_\infty|| \qquad \ldots(94)$$

Proof: Note that

$$G(z) = \sum_{j=1}^{\infty} z^{-j} H_j + D \qquad \ldots(95)$$

converges for $|z|>\lambda$ and hence that $||G(o)|| = ||Y_\infty||$. But the discussion of section 3.1 indicates that $||L|| = ||Y_\infty||$ ie $||G|| \geq ||G(o)|| = ||L||$. The result follows by combination with (90).
