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ON THE STABILITY OF NONLINEARLY INTERCONNECTED LARGE SCALE SYSTEMS

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1. Introduction

There has been considerable interest in recent years in the input-output stability of large-scale systems, mainly in the case of nonlinear subsystems combined together at linear summing points (see, for example [1,6]). However, there has also been some literature on nonlinearly interconnected subsystems (e.g. [3]) where a geometric type of argument is employed.

In this paper, we shall use the graph theoretic techniques described in [6] , although for the case of nonlinearly interconnected subsystems, it will be seen that it is natural to use hypergraph theory (an account of which can be found in [4]). The interconnection hypergraph is defined and then decomposition theory is used to represent the hypergraph as a tree of strongly connected components. A minimal essential set is then found for each strongly connected component and a nonlinear system is derived which depends only on the outputs of the combining points corresponding to the vertices included in this minimal essential set.

In section 2, we shall describe our notation and in section 3 the system description will be developed. Section 4 is devoted to a study of the stability of an overall interconnected system and in sections 5,6,7 we describe the decomposition theory discussed above. Finally, conclusions are drawn in section 8.

2. Notation

In this paper, the notation which we use is fairly standard. $\mathbb{R}^n$ denotes n-dimensional Euclidean space and L will denote a general function space (which could be a Banach space or a Hilbert space, for example). L_e is the usual extended space of funtions all of whose truncations to a finite interval belong to L.

The main novelty of our notation is in the representation of a vector valued function. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a function defined on $\mathbb{R}^n$ with values in $\mathbb{R}^m$ and suppose that G is an nxn matrix and e is a column vector in $\mathbb{R}^n$.

Then, by G*e we shall mean the nxn matrix

$$(G_{ij}e_j)_{\substack{1\leq i\leq n\\ 1\leq j\leq n}} \qquad \text{(no summation!)}$$

and by f(G*e) we shall mean the vector-value

$$\begin{matrix} f_1(G_{11}e_1,\ldots,G_{1n}e_n) \\ \vdots \\ f_n(G_{n1}e_1,\ldots,G_{nn}e_n) \end{matrix} .$$

The nonlinear functions f will denote the operations of the nonlinear combining elements. Of course, if these reduce to summation points, then

$$f(G^*e) = Ge \text{ (ordinary matrix multiplication).}$$

3. System Description

The system we consider in this paper is of a very general type. It consists of a set of m linear or nonlinear subsystems connected together at nonlinear combining points. The subsystems are specified by input-output maps which may be stable or unstable. A typical subsystem is shown in fig. 1.

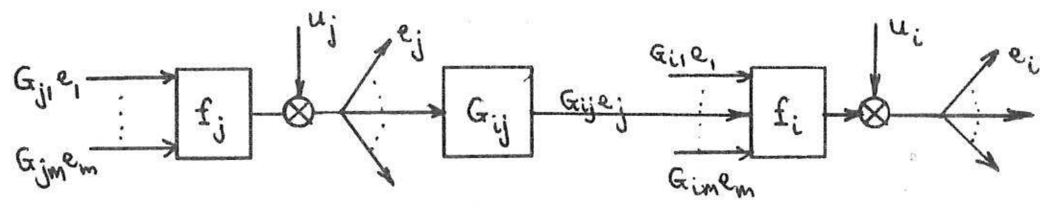

Fig. 1.

The nonlinear combining elements f_i, $1\leq i\leq m$, are maps

$$f_i : \bigoplus_{i=1}^{m} \{L_e\} \to L_e$$

in the general case, and in the case of memoryless combiners may be regarded as maps

$f_i : \mathbb{R}^m \to \mathbb{R}$.

We now introduce the following general assumption A1.

A1. Throughout this paper the operators G_{ij}, f_j will be assumed to be causal.

The equation for each nonlinear combining point may be written (comparing with fig. 1.)

(3.1a) $$e_i = u_i + f_i(G_{i1}e_1, \dots, G_{ij}e_j, \dots, G_{im}e_m)$$

or

(3.1b) $$(I-F)e = u$$

where

$$e \overset{\Delta}{=} \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \qquad u \overset{\Delta}{=} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

and $F: \bigoplus_{i=1}^{m} L_e \to \bigoplus_{i=1}^{m} L_e$ is the (nonlinear) map defined by

$$F_i(e) = f_i(G_{i1}e_1, \dots, G_{im}e_m).$$

Note that L_e is a Frechet space (cf. [10]) as the inductive limit of the Banach spaces $L(0,T)$ for $T>0$. There are various ways of studying the existance theory for the equation (3.1), but the assumptions on F mainly fall into three categories:

(i) F is a contraction for each space $L(T_1,T_2)$, ([5]).

(ii) F is monotone. ([7])

(iii) F is completely continuous (i.e. maps bounded sets to compact sets).([8]).

In each of these cases (with certain additional technical assumptions) one can obtain a unique causal in verse $(I-F)^{-1}$ defined on $\bigoplus_{i=1}^{m} L_e$.

However, in the practical situation, it is unlikely that any of these conditions will hold for an arbitrary system of the form shown in Fig. 1.

We shall therefore assume in the sequel that I-F has a unique causal inverse defined on the whole of $\bigoplus_{i=1}^{m} L_e$ and rely on physical intuition or the nature of a particular system to guarantee such an inverse exists. Such an assumption is justified to a certain extent by the fact that we are interested mainly in stability and if a solution does not exist then to ask for the system to be stable is somewhat meaningless. Therefore we make the assumption

A2. $(I-F)^{-1}$ exists and is causal on $\bigoplus_{i=1}^{m} \{L_e\}$.

It will be convenient in the following to replace the system in Fig.1. by one in which the nonlinearity f_i does not depend explicity on the input $G_{ii}e_i$. This can be done easily at the expense of increasing the order of the system. We merely introduce a new summing point for each i as shown in Fig. 2.

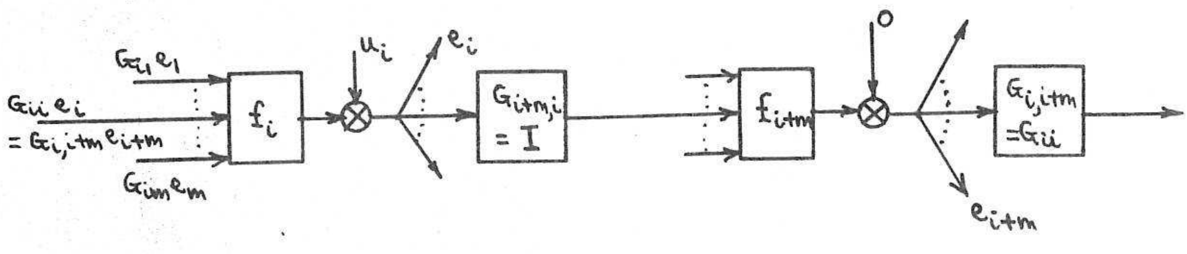

Fig. 2.

In this system $G_{i+m,i}$ is the identity operator and

$$f_{i+m}(G_{i+m,1}e_1, \ldots, G_{i+m,2m}\hat{e}_{2m}) = G_{i+m,i}e_i = e_{i+m} = e_i .$$

Note that the G matrix now becomes

$$G = \left[\begin{array}{c|c} (G_{ij})_{\substack{1\le i\le m \\ 1\le j\le m}} & \begin{matrix} G_{11} & & 0 \\ & \ddots & \\ 0 & & G_{mm} \end{matrix} \\ \hline \begin{matrix} I & & 0 \\ & \ddots & \\ 0 & & I \end{matrix} & 0 \end{array}\right]$$

(we shall use the same letter G for the 'augmented system'; this should not cause any confusion). We note that the diagonal of the new G matrix is ignored, and we have shown that the original system can be replaced by one in which the output of a nonlinear combiner is not fed directly back into its input. This will enable us to apply the theory of [3] and also it will simplify the graph theory structure of the system since this will have no self-loops (see section 6).

4. Stability of an interconnected system

Before discussing the graph theory decomposition of the system we shall first consider the stability of an overall interconnected system. Suppose that, in the original system (Fig. 1.), the maps

$$f_i : \bigoplus_{i=1}^{m} L_e \to L_e$$

are Frechet differentiable when restricted to L_T (the Banach space of truncations of elements of L_e to the interval $[0,T]$); i.e.

$$f_i : \bigoplus_{i=1}^{m} L_T \to L_T$$

is Frechet differentiable. (Note that if $P_T:L_e \to L_T$ is the projection, and $I_T:L_T \subset L_e$ is the inclusion, then we are denoting $P_T f_i I_T$ by the same symbol f_i;

this should not cause confusion. Also, $f_i\,(\bigoplus_{i=1}^{m} L_T) \subseteq L_T$ by causality.)

Consider the equestion

$$Ie_i - f_i\,(G_{i1}e_1, \ldots., G_{ii}e_i, \ldots., G_{in}e_n) = u_i.$$

If we assume that G_{ii} is also Frechet differentiable as a map from L_T to L_T, then if

$$I - \mathfrak{F} f_i \mathfrak{F} G_{ii} \quad (\mathfrak{F} = \text{Frechet derivative})$$

is invertible at each point of the domain of f_i, then, by the implicit function theorem, we can solve uniquely for e_i in terms of the e_j's for $j \neq i$, i.e. $\exists$ a function,

$$g_i^T : \bigoplus_{i=1}^{m-1} L_T \to L_T \tag{4.1}$$

such that

$$e_i = g_i^T\,(G_{i1}e_1, \ldots., \widehat{G_{ii}\,e_i}, \ldots., G_{in}e_n\,)$$

where $\hat{}$ denotes that this argument is omitted. Now, because each f_i and G_{ij} is causal, if $T_1 < T_2$ it follows easily that

$$g_i^{T_1} = g_i^{T_2}\Big|_{\left\{\bigoplus_{i=1}^{m-1} L_{T_1}\right\}}$$

($= g_i^{T_2}$ restricted to the subspace $\bigoplus_{i=1}^{m-1} L_{T_1}$), and hence by the definition of inductive limits, the set of maps $\{g_i^T\}$ $T \geq 0$ uniquely define a map,

$$g_i : \bigoplus_{i=1}^{m-1} L_e \to L_e$$

which is causal by (4.1).

Following [6] , we introduce

<u>Definition 4.1</u> The <u>overall system</u> S is stable if $\forall$ $i,j=1,\ldots,m$, the maps $u \mapsto G_{ij}\left[(I-F)^{-1}u\right]_j$ are stable.

We then have

<u>Lemma 4.2</u> Suppose that

(a) Each G_{ij} is stable for $i \neq j$.

(b) For each unstable G_{ii}, we assume that g_i exists and is a stable mapping; i.e. $g_i : \bigoplus_{i=1}^{m-1} \{L\} \to L$.

(c) The mapping $(I-F)^{-1}$ is stable .

Then the overall system is stable. □

The proof of this result is simple and generalizes Theorem I of [6]. Its usefulness is limited, however, by the strong assumption that for an unstable G_{ii}, we have Frechet differentiability of f_i and G_{ii} together with (for example) a condition of the form

$$\|\mathfrak{J}f_i(e_1)\| \, \|\mathfrak{J}G_{ii}(e_2)\| < 1.$$

for all $e_1, e_2 \in L_T$. (This latter condition will guarantee the existence of $(I-\mathfrak{J}f_i\mathfrak{J}G_{ii})^{-1}$.)

In systems of the type considered in [2] , where the combiner is defined by digital logic we do not have such differentiability, but we are usually able to derive a norm condition on f_i of the form

(4.2) $$|f_i(e_1,\ldots,e_m)| \leq \tilde{f}_i(\|e_1\|,\ldots,\|e_m\|)$$

for some function $\tilde{f}:\mathbb{R}^{n+} \to \mathbb{R}^+$ and for $e_1,\ldots,e_m \in L_T$.

We shall now use the augmented system, which is of order 2m if the original system is of order m. However, for simplicity, we shall continue to use m to denote the order of the augmented system . Note that, in the augmented system, both f_i and $\tilde{f}_i$ are independent (explicitly) of e_i.

It will be assumed that all the G_{ij} are stable and we shall denote their gains by γ_{ij}. Introduce the functions

$$F_i(x_1,\ldots,x_m) = x_i - f_i(\gamma_{i1}x_1,\ldots,\hat{\;}_i,\ldots,\gamma_{im}x_m) - \nu_i$$

(where the ν_i represent the norms of the inputs u_i). Then by (a trivial extension) of [3, proposition 3.1], the sets

$$\mathfrak{M}_i = \{x \in \mathbb{R}^m : F_i(x) = 0\}$$

are (m-1)-dimensional submanifolds of $\mathbb{R}^m$. We also define

$$P_i^+ = \{x \in \mathbb{R}^m : F_i(X) \geq 0\}$$

$$P_i^- = \{x \in \mathbb{R}^m : F_i(x) \leq 0\}$$

and let

$$\omega^* = \sup\{ \left(\bigcap_{i=1}^{m} P_i^-\right) \cap \mathbb{R}^{+m} \} .$$

It is shown in [3] that $\omega^* = q(\nu_1,\ldots,\nu_m)$ for some function $q : \mathbb{R}^m \to \mathbb{R}^+$. Then, we have

Theorem 4.3 [3]. If $\omega^* < \infty$ and $q : \bigoplus_{i=1}^{m} L \to L$, then the overall system is stable. □

The above two results can be used in the case of small-scale highly interconnected systems. In the case of large-scale systems which consists of such small-scale systems loosely connected together, it has been found that a graph-theoretic decomposition technique is sometimes useful (see [2], [6]). We shall consider this approach in section 6, but first we shall introduce the notions of graph and hypergraph theory that we shall need.

5. Graphs and Hypergraphs

The theory which we shall outline in this section can be found mainly in [4]. However the notion of directed hypergraph appears to be new.

Definition 5.1. A directed graph $G=(X,E)$ is a pair of sets X and E where $E \subseteq X \times X$, the artesian product of X with itself. The set X is called the vertex set and E the edge set of G, and if $e \in E$ then $e = (x_1,x_2)$ can be regarded as a directed line segment from x_1 to x_2.

Let $|S|$ denote the order i.e. the number of elements of a set S. Then

Definition 5.2 An undirected graph $G' = (X',E')$ is a pair of sets X',E' where

(5.1) $$E' \subseteq \{S \in \mathcal{P}(X') : |S| \leq 2\}.$$

($\mathcal{P}(X')$ is the power set of X').

The notion of undirected graph can be generalized to that of a hypergraph by removing the condition $|S| \leq 2$ in (5.1). Thus

Definition 5.3 A hypergraph $G'_H=(X'_H,E'_H)$ is a pair of (finite) sets such that

$$E'_H \subseteq \mathcal{P}(X'_H), \quad e_i \neq \emptyset \text{ for each } e_i \in E'_H,$$

$$\bigcup_{E'_H} e_i = X'.$$

We can now generalize the concept of directed graph to that of directed hypergraph by regarding an edge e of a hypergraph as being partitioned into disjoint subsets e_1, e_2, which can be thought of as generalized vertices, and then a 'directed hyperedge' is an arrow joining e_1 to e_2. Thus, more formally,

Definition 5.4 A directed hypergraph $G_H = (X_H, E_H)$ is a pair of (finite) sets such that

$$E_H \subseteq \mathcal{P}(X_H) \times \mathcal{P}(X_H) .$$

If $e \in E_H$ is a hyperedge of G_H, then $e = (X_1, X_2)$ for some $X_1, X_2 \in \mathcal{P}(X_H)$ and X_1 is called the initial vertex and X_2 the final vertex of e, denoted respectively e_I, e_F.

We shall now define a graph G_R called the representative graph of the directed hypergraph G_H as follows; the vertices V_R of G_R are in a one-to-one correspondence with the edges of G_H, and so we can write $V_R = E_H$ and there is an edge e_R of G_R joining the vertices v^1_R, $v^2_R \in V_R$ if and only if

$$(v^1_R)_F \cap (v^2_R)_I \neq \emptyset.$$

It will be convenient to use the representative graph of G_H for a system since we can then use the standard decomposition theory for the system graph [see, 6].

6. Decomposition Theory for the Overall System

In this section we shall generalize the ideas developed in [6] to the present case of nonlinearly connected systems. In order to generalize slightly our results, we shall suppose that each of the nonlinear combining elements f_i is a sum of several nonlinear functions. Thus

$$f_i(x_1, \ldots, x_m) = \sum_{j=1}^{n_i} f_{ij}(x_1, \ldots, x_m) . \tag{5.1}$$

However, in a large-scale system each f_{ij} will not necessarily be a function of all the inputs $x_1,\dots,x_m$. If f_{ij} depends explicitly only on the inputs x_ℓ where $\ell \in L \subseteq \{1,\dots,m\}$, we shall write

$$f_{ij}(x_1,\dots,x_m) = f_{ij}(x_L) \ .$$

Also, in the equation (6.1), if f_{ij_1} and f_{ij_2} depend explicitly on the same variables x_ℓ we shall assume that their sum has been represented by a single function in the summation on the right hand side. (This will avoid multiple hyperedges in the interconnection hypergraph). We are now in a position to define the interconnection (directed) hypergraph of the system, as follows. The vertices of the hypergraph are in a one-to-one correspondence with the nonlinear combining points of the system, of which there are m. Hence the vertices can be taken to be the set $\bar{m} = \{1,\dots,m\}$ of the first m natural numbers. The directed hyperedges are subsets of $\mathcal{P}(\bar{m}) \times \mathcal{P}(\bar{m})$ of the form

$$(L\ ,\{i\}) \quad , \qquad 1 \le i \le m$$

where L is such that there exists, in the expression (6.1), a nonlinear combining function f_{ij} which depends explicitly on the inputs x_ℓ for $\ell \in L$ (and only those inputs).

We shall now need the concept of cycle in a directed hypergraph, which is a sequence $(e_1,\dots,e_n,e_{n+1}=e_1)$ of hyperedges of the hypergraph such that

$$(e_i)_F \cap (e_{i+1})_I \neq \emptyset \ .$$

Lemma 6.1 A directed hypergraph G_H has a cycle $(e_1,\dots,e_n,e_1)$ iff the representative graph G_R of G_H has a cycle.

Proof If $(e_1,\dots,e_n,e_1)$ is a cycle in G_H, then $e_1,\dots,e_n$ are vertices of G_R and since $(e_i)_F \cap (e_{i+1})_I \neq \emptyset$, the definition of G_R implies that there is an edge in G_R from e_i to e_{i+1}. The reverse argument is equally simple. □

We can now use the decomposition theory described in[6]. The following definitions are required.

Definition 6.2 A diagraph (i.e. directed graph) is connected if, for any pair of vertices v_1, v_2, there is a chian of directed edges joining either v_1 to v_2 or v_2 to v_1. It is strongly connected if there are chains of directed edges joining both v_1 to v_2 and v_2 to v_1, (i.e. every vertex lies on a cycle). A strongly connected component (SCC) is a maximal strongly connected subgraph.

It should be noted that we can define similar concepts for directed hypergraphs, and by lemma 6.1 there is a one-to-one correspondence between the SCC's of a directed hypergraph and its representative graph. Using the results of [6], we can decompose a connected representative graph G_R of a connected hypergraph G_H as follows:

(a) Find all the SCC's $C_1, \ldots, C_\mu$ of G_R

(b) Define a new graph G_{RC} on μ vertices $(\omega_1, \ldots, \omega_\mu)$ (corresponding to the components $C_1, \ldots, C_\mu$ of G_R) such that $(\omega_i, \omega_j) \in E(G_{RC})$ (the edge set of G_{RC}) iff $\exists$ vertices v_i, v_j of G_R in C_i, C_j respectively, such that (v_i, v_j) or $(v_j, v_i) \in E(G_R)$. Clearly G_{RC} is a tree .

(c) Relabel the vertices of G_{RC} such that its adjacency matrix is lower triangular. If we continue to denote the new vertices by $(\omega_1, \ldots, \omega_\mu)$, then we finally relabel the vertices of the SCC's $C_1, \ldots, C_\mu$ such that those of C_i have lower numbers than those of C_{i+1}, $1 \leq i \leq \mu-1$.

The adjacency matrix of G_R is then of the form

$$A_{G_R} = \begin{bmatrix} A_{11} & 0 & \cdot\cdot & 0 \\ A_{21} & A_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ A_{\mu 1} & \cdots & & A_{\mu\mu} \end{bmatrix} .$$

Denote the corresponding SCC's of G_H by $C_1^H,\ldots,C_\mu^H$.

Before proceeding with the decomposition of the general system we note that, following [6], the augmented feedback system of fig. 2 may be represented as a nonlinear feedback system as shown in fig. 3 where $G_{.j}$ is the j^{th} column of G,

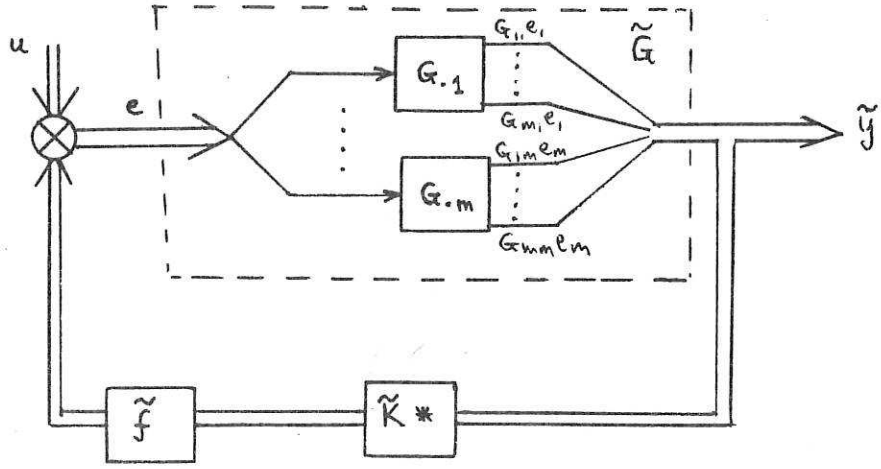

Fig. 3.

and
$$\tilde{G} = \text{diag}\,(G_{.1},\ldots,G_{.m}),$$
$$\tilde{K} = [I_1,\ldots,I_m] \in \mathbb{R}^{m\times m^2}$$

where I_i is the mxm identity matrix with the i^{th} diagonal element 1 replaced by 0 (since in the augmented system e_i does not feed directly back to the input of i^{th} loop), and

$$\tilde{f}_i(\tilde{k}_i) = f_i(k_i,k_{i+m},\ldots,k_{i+m(m-1)})$$

where $\tilde{k}_i$ is the i^{th} row of $\tilde{K}*\tilde{y}$.

Returning to the augmented nonlinear system we have seen that its representatice graph G_R can be renumbered so that its adjacency matrix is block-triangular. Refering this back to the original hypergraph it is easy to see that the equations (3.1) become

$$(6.2)\quad \begin{bmatrix} e_1^c \\ \vdots \\ e_\mu^c \end{bmatrix} = \begin{bmatrix} f_1^c \\ \vdots \\ f_\mu^c \end{bmatrix} \left(\begin{bmatrix} G_{11}^c & 0 & \cdots & 0 \\ G_{21}^c & G_{22}^c & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ G_{\mu 1}^c & G_{\mu 2}^c & & G_{\mu\mu}^c \end{bmatrix} * \begin{bmatrix} e_1^c \\ \vdots \\ e_\mu^c \end{bmatrix} \right) + \begin{bmatrix} u_1^c \\ \vdots \\ u_\mu^c \end{bmatrix}$$

where, for $\alpha \geq \beta$, $(\alpha, \beta = 1,\ldots,\mu)$,

$$V_\alpha^c = \bigcup_{e \in C_\alpha^H} (e_I \cup e_F)$$

$$u_\alpha^c = \text{vector } (u_i)_{i \in V_\alpha^c} \quad \text{(of dimension } m_\alpha)$$

$$e_\alpha^c = \text{vector } (e_i)_{i \in V_\alpha^c}$$

$$G_\alpha^c = \text{matrix } [G_{ij}]_{i \in V_\alpha^c,\, j \in V_\beta^c}$$

f_α^c = vector $(f_i)_{i \in V_\alpha^c}$ of nonlinear operators at the vertices of V_α^c.

It is evident from (6.2) that our original system may now be decomposed into μ subsystems. At the 'top' of the hierarchy is the subsystem with system matrix G_{11}^c, i.e.

$$(6.3)\qquad e_1^c = f_1^c\,(G_{11}^c e_1^c) + u_1^c$$

and, by construction, this system is strongly connected. If we can derive stability conditions C_1 for this system, then, under these conditions

$$u_1^c \in \bigoplus_{i=1}^{m_1} \{L\} \Rightarrow e_1^c \in \bigoplus_{i=1}^{m_1} \{L\}$$

The next subsystem in the hierarchy is

$$(6.4)\qquad e_2^c = f_2^c(G_{21}^c e_1^c\,,\ G_{22} e_2^c) + u_2^c$$

Assume that all the off-diagonal blocks $G_{\alpha\beta}$, $\alpha > \beta$ have finite gain and replace the subsystem (6.4) by the system

$$(6.5) \qquad e_2^c = \hat{f}_2^c\,(v_2\,,\,G_{22}e_2^c)$$

where v_2 is the $2m_2$-vector $(G_{21}^c e_1^c\,, u_2^c)$ and

$$\hat{f}_2^c(v_2, G_{22}e_2^c) = f_2^c(G_{21}^c e_1^c\,, G_{22}e_2^c) + u_2^c\ .$$

Now the system (6.5) is a strongly interconnected system of the type (6.3), except that now, not only are the feedback elements nonlinearly connected, but the input v_2 also enters the system nonlinearly. Also, if we have conditions C_2 for the stability of the system (6.5) (i.e. such that $v_2 \in \bigoplus_{i=1}^{2m_1} \{L\} \Rightarrow e_2^c \quad \bigoplus_{i=1}^{m_1} \{L\}$) then under the joint conditions $C_1 \cup C_2$

we have that

$$(u_1^c\,,\,u_2^c) \in \bigoplus_{i=1}^{m_1+m_2} \{L\} \Rightarrow (e_1^c\,,\,e_2^c) \in \bigoplus_{i=1}^{m_1+m_2} \{L\}\ .$$

Continuing in this manner, it is clear that, if we can obtain stability conditions on a general strongly connected nonlinear system whose inputs enter nonlinearly, then the subsystems of (6.2) defined by the operators f_i^c can be considered seperately, and, extending the above notation in an obvious manner, we have, under the condition $C_1 \cup \ldots \cup C_\mu$, the stability condition

$$(u_1^c, \ldots, u_\mu^c) \in \bigoplus_{i=1}^{m} \{L\} \Rightarrow (e_1^c, \ldots, e_\mu^c) \in \bigoplus_{i=1}^{m} L \quad (m = \sum_{i=1}^{\mu} m_i)$$

for the overall system.

We have seen, therefore, that the stability of the overall system, which was reduced to one of the form (6.2) can be characterized in terms of the stability of strongly interconnected systems of the form

$$(6.6) \qquad e = f(G^*e,\ u)$$

where f is a general nonlinear function and u is the system input. We could use the results of section 4 (which can easily be generalized to such systems) to answer stability questions for the system (6.6) or we can decompose the hypergraph of this strongly interconnected system into its spanning tree and its complement. This will now be considered in section 7.

7. Decomposition of a SCS

In this section we shall consider a strongly connected system of the form (6.6) and derive a stability condition for this system by decomposing its corresponding hypergraph G_H into two parts. The decomposition of G_H is obtained as follows: we successively remove hyperedges from the graph which do not disconnect the hypergraph. This process is continued until no further hyperedge with this property can be found. We will then be left with a hypertree and a collection of hyperedges which have been removed from G_H. Let V_H' be the vertices of the hypertree and put $V_H'' = V_H \setminus V_H'$ where V_H is the vertex set of G_H. Therefore, if we let

f_1 = vector of nonlinear combining functions of dimension m_1, each element corresponding to a vertex in V_H' $(m_1 = |V_H'|)$

f_2 = vector of nonlinear functions corresponding to vertices in V_H'' (of order $m_2 = |V_H''|$).

$$G_{11}^T = (G_{ij})_{i,j \in V_H'}$$

$$G_{12}^T = (G_{ij})_{i \in V_H'',\, j \in V_H'}$$

$$G_{21}^T = (G_{ij})_{i \in V_H',\, j \in V_H''}$$

$$G_{22}^T = (G_{ij})_{i,j \in V_H''}$$

$$e_1 = (e_i)_{i \in V_H'}, \quad e_2 = (e_i)_{i \in V_H''}$$

$$u_1 = (u_i)_{i \in V_H'}, \quad u_2 = (u_i)_{i \in V_H''}$$

The equations of the system can therefore be written in the form

$$e_1 = f_1(G_{11}^T * e_1 \, , \, G_{12}^T * e_2 \, , \, u_i)$$

$$e_2 = f_2(G_{21}^T * e_1 \, , \, G_{22}^T * e_2 \, , \, u_2)$$

However, because V_H' is the vertex set of a tree and we are considering the augmented system, we can arrange that G_{11}^T is strictly lower triangular (i.e. has zeros above and on the main diagonal). Hence e_{11} is just a function of u_1 and e_2, i.e.

$$e_{11} = f_{11}(G_{12}^T * e_2, u_1). \quad (e_1 = (e_{11}, \ldots, e_{1m_1}))$$

(By a slight abuse of notation we shall use the same letter f_1 for this function.) Therefore,

$$e_{12} = f_{12}(f_{11}(G_{12}^T * e_2, u_1), (G_{11}^T)_{22} e_{12} \, , \, G_{12}^T * e_2, u_1) \, .$$

Continuing in this way, we can express e_1 as a new function g_1 of e_2 and u_1. Thus,

(7.1) $$e_1 = g_1(e_2, u_1)$$

(g_1, of course, also depends on the subsystems G). Therefore, we have an equation which just depends on e_2, u_1, u_2; namely,

(7.2) $$e_2 = f_2(G_{21}^T * g_1(e_2, u_1), \, G_{22}^T * e_2, u_2).$$

It is clear from this development that if we choose the edges which are removed from the original hypergraph to belong to a minimal essential set (i.e. a set with the least number of elements which will reduce G_H to a tree), then the dimension of e_2 in (7.2) will be a minimum. The stability problem for a general strongly connected system can therefore be considerably reduced and is equivalent to solving that for equation (7.2), to which can be applied the methods of section 4.

8. Conclusions

In this paper we have generalized the theory of [6] to take account of nonlinear combining elements. An example of such a system can be found in [2] , where a discussion of elementary graph theory is presented. The natural setting for the present situation is that of hypergraph theory (c.f. [4]) which is used here to give a structural model of the system in topological terms. This model can then be decomposed using first the strongly connected component decomposition and then, for each such subsystem, a minimal essential set is found. The latter procedure was used, in effect, in [2] although there the full significance of the procedure was not emphasized. The subsystem thus obtained can be approached with the theory such as in [3] , and stability conditions for the overall system can be developed.

9. References

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