ROBUST STABILITY OF MULTIVARIABLE FEEDBACK SYSTEMS WITH RESPECT TO LINEAR AND NONLINEAR FEEDBACK PERTURBATIONS

by

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Abstract

Abstract quantitative measures of the robustness of multivariable feedback systems with respect to feedback perturbations are presented. The approach provides a means of considering a class of unstable plant perturbations and gives some indication of the effect of the choice of controller structure on robustness.

In recent papers [1]-[4] the problem of providing quantitative measures of the robustness of the multivariable feedback system illustrated in Fig.1(a) with respect to stable perturbations ΔG of the forward path system G. Attention has been focussed on the effect of additive perturbations $G \to G+\Delta G$ and multiplicative perturbations $G \to G(I+\Delta G)$ of the forward path system G. One disappointing aspect of these studies is that only stable perturbations are allowed whereas it is intuitively obvious that the feedback system can be stable in the presence of many (but not all) unstable perturbations. In some applications, for example in the nuclear industry [5], small variations in plant parameters over a long period of time can convert an open-loop stable plant into an unstable plant. Clearly the control system must be robust enough to cope with such unstable perturbations. Some topological aspects of unstable plants have been introduced in [6]-[7] but computable quantitative measures are not provided.

This paper provides a robustness analysis with respect to a class of perturbations corresponding to (possibly) unstable linear perturbations to the plant G in the modified linear configuration shown in Fig.1(b) and simultaneous nonlinear perturbations to feedback F. The 'forward path controller' K is separated from the plant G as it will be assumed that the control system characteristics are known to high accuracy.

An abstract functional analytic approach is taken. The plant G is regarded as a linear map of an input vector space U into an output vector space Y, K a linear map of Y into U and F a linear map of Y into itself. The spaces U and Y are not assumed to be topological spaces but we suppose that there exists suitable linear vector sub-spaces $U_o \subset U$ and $Y_o \subset Y$ endowed with chosen norm topologies with respect to which both U_o and Y_o are Banach (sub)spaces. A system G (say) is stable (in the bounded-input/bounded-output sense) if inputs $u \in U_o$ are mapped into outputs $y \in Y_o$ or, more formally, the restriction $G|U_o$ of G to U_o has range in Y_o. In the case of a linear system this is equivalent (with suitable continuity assumptions) to the requirement that $G|U_o$ is bounded. No causality structure is assumed.

Consider initially the linear system of Fig.1(b) defined by the relations

$$y = G\,u \qquad\qquad u = K\,e$$

$$e = r - Fy \qquad\qquad \ldots(1)$$

It is trivially verified that these relations define a linear map $r \mapsto y$ on Y written

$$y = L_c r \qquad\qquad \ldots(2)$$

We will suppose that this closed-loop system is stable in the sense that $L_c|Y_o$ maps Y_o into itself. Consider now a perturbation ΔG of the plant G of such a form that $G+\Delta G$ has a representation

$$y = G(u-v)$$
$$v = Hy \qquad \ldots(3)$$

corresponding to a feedback perturbation of G as illustrated in Fig.2. Assume also that the linear map H is well-defined and stable in the sense that it is defined on the range of G and maps $R(G)\cap Y_o$ into U_o.

Remark 1: It is not assumed that G, K or F are stable, nor that the perturbed plant $G+\Delta G$ has any particular stability characteristics. In fact G can be stable and $G+\Delta G$ unstable (or vice versa) as is illustrated by taking $U = Y$ = space of rational functions of the complex variable s and $U_o = Y_o$ = Banach subspace of functions bounded and analytic in the 'Nyquist set' $\{s : \mathrm{Re}\, s>0,\ |s|<R\} \triangleq \Omega$ with R 'large' and $\|\cdot\| = \sup_{s\in\partial\Omega}|\cdot(s)|$. If G is represented by multiplication by the transfer function $2/(s+1)$ then G is clearly bounded on U_o but choosing H to be the (negative) identity $v(s) = -y(s)$, it is trivially seen that $G+\Delta G$ has transfer function $2/(s-1)$ and hence is unstable.

Remark 2: Using U, Y, U_o and Y_o as in remark 1 with G represented by multiplication by a transfer function $G(s) = n(s)/d(s)$ of rank (the degree $\partial(d)$ of $d - \partial(n)$) equal to $k\geq 1$, then the class of admissible H can be written in the form $n'(s)/d'(s)$ with rank $\leq 1-k$ or, equivalently, as the sum of a polynomial in s of degree $\leq k-1$ and a strictly proper transfer function with poles outside the closure of the Nyquist set. The set of all possible plants $G+\Delta G$ generated by such perturbations is hence represented by the set of all minimum-phase, stable or unstable transfer functions of rank k with zero polynomial nd' and characteristic polynomial dd'+nn' where d' is Hurwitz and $\partial(n')-\partial(d') \leq k-1$.

We now prove a simple result providing a quantitative measure of robustness with respect to the defined class of linear feedback perturbations:

Theorem 1: If the feedback system of Fig.1(b) is stable, then the system is also stable in the presence of linear feedback perturbations of plant dynamics if

(i) the control mapping K has an inverse, and

(ii) $L_c K^{-1} H \mid Y_o$ has range in Y_o and is bounded in the sense that

$$\lambda \triangleq \|L_c K^{-1} H\| < 1 \qquad \ldots(4)$$

Moreover, under these conditions, the responses y and y' of the real and perturbed systems satisfy the relation

$$\|y'-y\| \leq \frac{\lambda}{1-\lambda} \|y\| \qquad \ldots(5)$$

(Note: (5) provides an upper bound on the response error y'-y introduced by the perturbation).

Remark 3: Conditions (i) and (ii) (in practical terms) imply that K must have a stable inverse or, in transfer function terms, that K is minimum-phase. This is clearly no restriction in practice.

Proof of Theorem 1: The closed-loop dynamics of the perturbed system can be written as

$$y' = G(u-v) \quad , \quad v = Hy'$$
$$u = K\,e \quad , \quad e = r-Fy' \qquad \ldots(6)$$

or, after a little manipulation,

$$y' = L_c(r - K^{-1}Hy) \qquad \ldots(7)$$

The result is now a trivial consequence of the Banach contraction theorem [8].

Corollary 1: With the above assumptions, the feedback system of Fig.1(b) is stable in the presence of any bounded feedback perturbation satisfying

$$||H|| < 1/\,||L_c K^{-1}|| \qquad \ldots(8)$$

Proof: Follows trivially from (4), writing $||L_c K^{-1} H|| \leq ||L_c K^{-1}|| \cdot ||H||$.

These results have a structural similarity to others (see, for example, [1]) but has the advantage of (a) allowing unstable plant perturbations and (b) providing an intuitive insight into controller effects on robustness. More precisely, if we interpret K as an abstract controller 'gain', the presence of the inverse K^{-1} in (4) and (8) suggests that high gains tend to increase the robustness of the system. This is, of course, rather simplistic as L_c depends upon K also, but it does suggest the existence of the possibility (see [9], [10] for specific examples of this idea in practice). To illustrate this point, consider the example of remark 1 and let K be represented by the proportional gain $g>0$. It is seen that $L_c = (1+GK)^{-1}GK = 2g/s+1+2g$

with $\|L_c K^{-1}\| = 2/(1+2g)$ and hence, using (8), the closed-loop system is stable in the presence of feedback perturbations of norm $\|H\| < g + \frac{1}{2}$ which increases monotonically with gain. In fact, if $g>\frac{1}{2}$, this perturbation set generates unstable plant perturbations as is seen by choosing H to be the negative identity.

An interesting corollary to the theorem relates the work to the inverse Nyquist array design technique [12], [13] and provides a direct robustness interpretation of that method:

Corollary 2: Let G, K and F be mxm rational transfer function matrices with the diagonal structure $F(s) = K(s) = I_m$ and $G(s) = \text{diag}\{g_j(s)\}_{1\leq j\leq m}$. If the feedback configuration of Fig.1(b) is stable, then it is also stable in the presence of stable feedback perturbations H(s) satisfying

$$\sup_{s\in\partial\Omega} \max_{1\leq j\leq m} \sum_{\ell=1}^{m} \left| \frac{1}{1+g_j^{-1}(s)} H_{j\ell}(s) \right| < 1 \qquad \ldots(9)$$

Proof: Follows from the theorem using the spaces described in [13].

Remark 4: We recover the classical diagonal dominance condition from (9) by writing $H = (G+\Delta G)^{-1} - G^{-1}$ whenever they exist and are analytic in Ω and choosing the diagonal terms of H to be identically zero. Equation (9) then reduces to the requirement that $I+(G+\Delta G)^{-1}$ be diagonally dominant on $\partial\Omega$.

Finally, although robustness with respect to linear perturbations of the plant is an extremely important practical notion, it is equally important to examine whether or not this robustness is maintained if nonlinear effects are introduced. The problem considered below is the robustness of the configuration of Fig.1(b) when the plant is subjected to linear feedback perturbations of the type discussed above and the linear feedback F is simultaneously replaced by F+N where N is a memoryless map of Y into itself of the form

$$N = N_1 + N_2 \qquad \ldots(10)$$

where N_1 and N_2 map Y_o into itself, N_1 has finite incremental gain k_1 such that

$$||N_1 y - N_1 z|| \leq k_1 ||y-z|| \qquad \forall\ y, z \in Y_o \qquad \ldots(11)$$

and $N_1 0 = 0$ and N_2 is bounded in the sense that there exists a scalar $q \geq 0$ such that $N_2 Y \subset Y_o$ and

$$||N_2 y|| \leq \frac{q}{2} \qquad \forall\ y \in Y \qquad \ldots(12)$$

Theorem 2: If the feedback system of Fig.1(b) is stable, then the system is also stable in the presence of simultaneous linear feedback perturbations of the plant and nonlinear perturbations of the feedback dynamics if conditions (i) and (ii) of Theorem 1 are satisfied and also

$$\mu \triangleq (1-\lambda)^{-1} ||L_c|| k_1 < 1 \qquad \ldots(13)$$

Moreover, under these conditions, the responses y and y' of the real and perturbed feedback systems are related by the inequality

$$||y'-y|| \leq \left\{\frac{\mu}{1-\mu} + \lambda\right\}\frac{1}{(1-\lambda)} ||y|| + ||L_c|| \frac{q}{2} \frac{1}{(1-\lambda)(1-\mu)} \qquad \ldots(14)$$

<u>Proof:</u> The feedback relations of the perturbed system are

$$y' = G(u-v) \quad , \quad v = Hy' \quad , \quad u = Ke$$

$$e = r - Fy' - Ny' \qquad \ldots(15)$$

which can be written as

$$y' = L_c\,(r - K^{-1}Hy' - Ny') \qquad \ldots(16)$$

or, bearing in mind the fact that (4) implies that $(I+L_cK^{-1}H)^{-1}$ exists in Y_o,

$$y' = (I+L_cK^{-1}H)^{-1}L_c(r-Ny') \qquad \ldots(17)$$

This equation has a unique solution $y' \in Y_o$ for each choice of $r \in Y_o$ if $(I+L_cK^{-1}H)^{-1}L_cN$ is a contraction. Taking initially the case of $N_2 = 0$ (ie $q = 0$), this is clearly the case with contraction constant μ as

$$\|(I+L_cK^{-1}H)^{-1}L_c(Ny-Nz)\|$$

$$\leq \|(I+L_cK^{-1}H)^{-1}\| \cdot \|L_c\|\, k_1\, \|y-z\|$$

$$\leq \frac{\|L_c\|\, k_1}{1-\lambda}\, \|y-z\| \qquad \ldots(18)$$

using (4). We can in fact verify (14) in this case by choosing the first iterate $y_o' = 0$ in the successive approximation sequence and note that $y_1' = (I+L_cK^{-1}H)^{-1}L_cr = (I+L_cK^{-1}H)^{-1}y$ to give

$$\|y' - (I+L_cK^{-1}H)^{-1}y\| \leq \frac{\mu}{1-\mu}\, \|(I+L_cK^{-1}H)^{-1}y\| \qquad \ldots(19)$$

Using the relationship $\|(I+L_cK^{-1}H)^{-1}\| \leq (1-\lambda)^{-1}$ and the triangle inequality then yields

$$\|y'-y\| \leq \|y' - (I+L_cK^{-1}H)^{-1}y\| + \|(I+L_cK^{-1}H)^{-1}L_cK^{-1}Hy\|$$

$$\leq \frac{\mu}{(1-\mu)} \frac{1}{(1-\lambda)} ||y|| + \frac{\lambda}{(1-\lambda)} ||y||$$

$$= (\frac{\mu}{1-\mu} + \lambda) \frac{1}{(1-\lambda)} ||y|| \qquad \ldots(20)$$

The case of $N_2 \neq 0$ is treated by noting that it is equivalent to the situation when $r \in Y_o$ is replaced by $r - N_2 y' \in Y_o$. Stability is therefore unaffected by N_2 and (20) holds with y replaced by $y - L_c N_2 y'$ ie we obtain

$$||y'-y|| \leq ||y'-(y-L_cN_2y')|| + ||L_cN_2y'||$$

$$\leq (\frac{\mu}{1-\mu} + \lambda) \frac{1}{1-\lambda} ||y-L_cN_2y'|| + ||L_c|| \frac{q}{2}$$

$$\leq (\frac{\mu}{1-\mu} + \lambda) \frac{1}{1-\lambda} \{ ||y|| + ||L_c|| \frac{q}{2} \} + ||L_c|| \frac{q}{2} \qquad \ldots(21)$$

which is simply (14). This completes the proof of the theorem.

Remark 5: If, as is usually the case in nonlinear studies, a causality structure exists, the result holds with all vectors and operators replaced by their truncated counterparts. The proof is trivial.

In conclusion, a partial answer to the problem of robustness of feedback systems to unstable plant perturbations has been provided by considering stable linear feedback perturbations to the plant. The results are a generalization of those underlying known design techniques [14] and also enable the effect of nonlinear feedback perturbations on both stability and response characteristics to be assessed.

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