Controller Design for Unknown Multivariable Systems Using Monotone Modelling Errors

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The problem of controller design for an unknown discrete or continuous multivariable system in the frequency domain based on open-loop step response data alone is considered. The approach is based upon the use of simple approximate plant models possessing the property that the resulting error in modelling the plant open-loop step response is both monotonic and sign definite as a function of time. In such circumstances the frequency domain properties of the approximating feedback system and the error involved in predicting steady state characteristics is sufficient data to predict the stability of the real feedback system. If integral controller elements are included, the results also provide sufficient conditions for exact tracking of step demands. Under certain circumstances, the technique has strong connections with the well-known inverse Nyquist array design method and can be regarded in part as a modification and generalization of that technique to cope with unknown interaction dynamics.

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1. Introduction

Frequency response methods[1-5] for the design of a forward path control element K for a m-input/m-output linear time-invariant plant G in the normal negative feedback configuration illustrated in Fig. 1(a) are now well-established, particularly in the unity-feedback case when the feedback element (measurement dynamics) F is the identity. Almost all of this work is based on the assumption that G is known exactly in the form of a state variable model or transfer function matrix representation or that the model is close enough to the real plant that the (assumed small) modelling errors are well within the tolerances implicit in the standard techniques of ensuring adequate gain and phase margins. The process model can then be used with confidence to perform design calculations such as simulation, transfer function matrix evaluation, calculation of poles and zeros and, on completion of the theoretical design exercise, to make confident predictions about the stability and performance of th real plant in the presence of the designed controller.

This paper is concerned with the problem of controller design when a plant model is not available in the sense that

(a) either the plant model is not known but open-loop plant <u>step</u> responses are available from plant tests, or

(b) the plant model is known but is so complex that design calculations other than simulation are not regarded as feasible (or necessary!) with available on-site computing facilities.

In both cases, the plant model is (from the designers viewpoint) 'unknown' and controller design must proceed using time-response data only or on some other basis. Two possible solution methodologies immediately suggest themselves i.e. identification[6] of a low order approximate model from off-line analysis of input/output data and the use of this model as a basis for control design studies or self-tuning control[7,8] using a control strategy based on an assumed low-order parametric system model and on-line identification of the required controller parameters. These are viable alternatives but they do require extensive off-line or on-line (resp.) computing facilities that may not be available to the design engineer at that place and time. Indeed, such facilities may be regarded as unnecessary. In such cases it is clearly necessary to attempt the design of the controller by some other technique. This topic is the concern of this paper.

The procedure followed has close structural connections with the methods of first-order multivariable control(9-12), being based upon the idea of (i) designing the controller K for an approximate model G_A of plant dynamics to ensure that the approximating feedback system shown in Fig. 1(b) has the required stability and transient performance characteristics and (i) providing easily checked conditions that ensure that the resultant controller stabilizes G in the real configuration of Fig. 1(a). The results here however, are distinct in that they apply to the class of stable process plant whereas others(9-12) are concerned with minimum-phase (possibly unstable) process plant Also, we do not necessarily assume here that the approximate plant model is of first-order form.

The underlying stability theory is described in section 2, relating the stability of Fig. 1(a) to that of Fig. 1(b) and the modelling error $G-G_A$. The concept of monotone modelling error is then introduced and it is demonstrated that the stability conditions can then be checked in practice without knowing the details of the dynamics of the plant G. The construction of appropriate plant approximations and control schemes is discussed in section 3. Although there are an infinity of possibilities, a number of obvious cases are described to illustrate the range and power of the ideas. A number of numerical case studies are described in Section 4.

Finally, we note that the field of non-adaptive 'unknown systems control' has only recently been identified as a theoretically feasible proposition[9-16] with contributions from Davison[13], Koivo[14], Porter[15], Astrom[16] and Owens[9-12]. The reader is referred to the references to find the details of the alternative techniques described there. The present paper is more in the spirit of the very recent publication by Carlucci and Vallauri[23] who have independently used similar techniques to assess the stability of feedback systems for a plant with known frequency response characteristics using a controller designed on the basis of a reduced order model. We do not assume any detailed knowledge of plant characteristics in this paper however.

2. Stability Theory for Unknown Systems

The foundations of any theory of controller design for unknown systems is inevitably based upon the construction of checkable conditions that ensure that the stability of the approximating feedback systems Fig. 1(b) implies the stability of the real feedback system Fig. 1(a). Sufficient conditions for this are derived in this section.

2.1 Stability Theory for Discrete Plant

Suppose that the plant G is derived from an underlying linear, time-invariant continuous time model with piecewise constant inputs and synchronous input actuation and output sampling of frequency h^{-1}. We will denote the strictly proper plant z-transfer function matrix by $G(z)$, and the proper forward path controller and feedback transfer function matrices by $K(z)$ and $F(z)$ resp. Suppose also that the plant approximate model G_A is described by the strictly proper z-transfer function matrix $G_A(z)$. Let $Q = GK$ and $Q_A = G_AK$ be the real and approximate (resp.) forward path systems (see Fig. 1).

The stability theorem proved later in the section relies upon the use of the well-known contraction mapping theorem[17-19] in a suitable

Banach space. The relevant space for the studies described below is the space $E^m(S)$ of m-vector valued functions of the complex variable s that are bounded and holomorphic in the open, connected region $S \triangleq \{z : 1 < |z| < R\}$ of the complex plane. The parameter R will be normally assumed to be large enough to make the interpretation of S as the 'unstable region' of the complex plane valid. The norm in $E^m(S)$ is taken to be[20]

$$||y|| \triangleq \sup_{s \in S} \max_{1 \leq i \leq m} |y_i(s)| \tag{1}$$

or, as analytic functions in a domain always achieve their maximum on the boundary of that domain,

$$||y|| = \sup_{s \in \partial S} \max_{1 \leq i \leq m} |y_i(s)| \tag{2}$$

where ∂S denotes the boundary of S. Clearly $\partial S = \{z : |z| = 1 \text{ or } |z| = R\}$.

Linear operators in $E^m(S)$ can be identified with mxm transfer function matrices whose elements are bounded and holomorphic in S. In the case when the elements of the transfer function matrix are rational functions of the complex variable z, it is trivially seen that the linear operator can be identified with a controllable, observable and stable discrete system. Conversely, every stable discrete system can be regarded as generating a linear operator in $E^m(S)$. If T is a linear operator in $E^m(S)$ then the operator norm induced by the vector norm (1) or (2) is[20]

$$||T|| \triangleq \sup_{s \in \partial S} \max_{1 \leq i \leq m} \sum_{j=1}^{m} |T_{ij}(z)| \tag{3}$$

The equations of the real feedback system of Fig. 1(a) are (neglecting the effect of initial conditions for simplicity),

$$y(z) = Q(z)r(z) - Q(z)\ F(z)\ y(z) \tag{4}$$

Adding $Q_A(z)F(z)y(z)$ to both sides of the equation and rearranging yields

$$\begin{aligned} y(z) = {} & (I_m + Q_A(z)F(z))^{-1}Q(z)r(z) \\ & + (I_m + Q_A(z)F(z))^{-1}(Q_A(z) - Q(z))Fy(z) \end{aligned} \tag{5}$$

Clearly, the system is input-output stable if, and only if, this equation has a unique solution $y \in E^m(S)$ for each demand $r \in E^m(S)$. Writing (5) in the operational form $y = Wy$, the following proposition follows trivially from the contraction mapping theorem by an argument similar to that used in (20).

Proposition 1: If the approximating feedback systems of Fig. 1(b) is stable and the transfer function matrix $(I_m + Q_A(z))^{-1}(Q_A(z) - Q(z))F$ has elements bounded and holomorphic in S, then the feedback system of Fig. 1(a) is stable if

(i) The state-variable model generating the composite system QF is both controllable and observable and

(ii) the contraction condition

$$\lambda \triangleq ||(I+Q_AF)^{-1}(Q_A - Q)F|| < 1 \tag{6}$$

is satisfied.

Proof: Writing $(I+Q_AF)^{-1}Q = (I+Q_AF)^{-1}((Q-Q_A)+Q_A)$ it is clear that W maps $E^m(S)$ into itself. Morever W is a contraction with contraction constant λ. Stability then follows from the contraction mapping theorem, the controllability and observability assumption guaranteeing the impossibility of hidden unstable modes.

Other versions of this lemma are easily derived. For example, writing the feedback equations in the form

$$\hat{y}(z) = F(z)Q(z)r(z) - F(z)Q(z)\hat{y}(z) \tag{7}$$

leads, after a similar procedure, to the result:

Proposition 2: If the approximating feedback system of Fig. 1(b) is stable and the transfer function matrix $(I_m + F(z)Q_A(z))^{-1}F(z)(Q_A(z)-Q(z))$ has elements bounded and holomorphic in S, then the feedback system of Fig. 1(a) is stable if FQ is both controllable and observable and the contraction constant

$$\lambda' \triangleq ||(I+FQ_A)^{-1} F(Q_A - Q)|| < 1 \tag{8}$$

Clearly proposition 1 and 2 are identical if $F = I_m$ or m=1, but, in general, they provide distinct sufficient stability conditions that can be checked if G, G_A, K and F are known. There is however, a technical problem that arises if the controller possesses an integral/summation component. In such cases, taking $m \geq 2$, the presence of G in the matrices $(I+Q_AF)^{-1}(Q_A - Q)F$ and $(I+FQ_A)^{-1}F(Q_A-Q)$ can mean that they will have a pole at the point $z = 1 \in \partial S$. The conditions of both Proposition 1 and 2 can therefore be violated in a situation of great practical interest. The way out of this problem is to write down the closed-loop system equations in terms of the plant input u,

$$u(z) = K(z)r(z) - K(z)F(z)G(z)\,u(z) \tag{9}$$

or, adding KFG_Au to both sides of the equation and rearranging yields

$$\begin{aligned} u(z) = {} & (I_m + K(z)F(z)G_A(z))^{-1}K(z)r(z) \\ & + (I_m + K(z)F(z)G_A(z))^{-1}K(z)F(z)(G_A(z)-G(z))u(z) \end{aligned} \tag{10}$$

and hence the lemma:

Lemma 1: If the approximating feedback system of Fig. 1(b) is stable and the transfer function matrix $G_A(z)-G(z)$ is bounded and holomorphic in S, then the feedback system of Fig. 1(a) is stable if the state variable model generating KFG is both controllable and observable and if

$$\lambda'' \triangleq ||(I+KFG_A)^{-1}KF(G_A-G)|| < 1 \tag{11}$$

Proof: The stability of Fig. 1(b) and the return-difference identity $|I+G_AKF| \equiv |I+KFG_A|$ indicate that both $(I+KFG_A)^{-1}K$ and $(I+KFG_A)^{-1}KF$ are bounded and holomorphic in S. Equation (10) therefore takes the form $u = W'u$ where W' maps $E^m(S)$ into itself and is a contraction with contraction constant λ''. Stability follows from the contraction mapping theorem, the controllability and observability assumptions guaranteeing the impossibility of hidden unstable modes.

Note that the stability of Fig. 1(b) ensures that $(I+KFG_A)^{-1}KF$ is bounded and holomorphic in S, even if K contains integral action! Note also that $G_A - G$ is stable if both G and G_A are stable but G can be unstable provided that G_A models the unstable part of G exactly.

2.2 Stability Theory for Continous Plant

If the plant G is described by a linear, time-invariant continuous model with strictly proper transfer function matrix G(s) and the controller and feedback element have proper transfer function matrices K(s) and F(s) respectively, then the results of section 2.1 apply directly in this case with S replaced by the 'Nyquist set' $S \triangleq \{s: \mathrm{Re}\,s > 0, |s| < R\}$ with boundary ∂S equal to the familiar Nyquist contour. All the results of section 2.1 remain valid and, in particular, lemma 1 still applies. For these reasons, we will tend to use notation suitable for the discrete case throughout the rest of the paper with the understanding that the equivalent results for the continuous case are obtained by replacing z by s and S by the Nyquist set.

2.3 Approximate Models with Monotonic Modelling Errors

The application of the above results to controller design for an unknown multivariable plant G is clearly frustrated by the need, in general, to know the details of an exact model of G to evaluate the contraction constant λ". In mathematical terms, noting that a sufficient condition for (11) to hold is that

$$\lambda''' \triangleq ||(I + KFG_A)^{-1}KF||.||(G_A-G)|| < 1 \tag{12}$$

it is clear that it is sufficient to have the value (or an upper bound) of the norm of the plant modelling error $G_A - G$. The problem considered below is the identification of practical situations when this norm can be evaluated easily using only <u>steady state data</u> obtained from plant open-loop step response tests.

The appropriate concept to invoke is a generalization[11] of the monotone property used by Astrom[16]:

Definition: An m-input/m-output, proper, discrete or continuous system will be said to be monotonic (resp. sign-definite) if, for each pair of indices (i,j), the response $y_i^{(j)}$ of the i^{th} system output from zero initial conditions to a unit step input in the j^{th} input is either monotonically increasing or monotonically decreasing (resp. takes only positive values or takes only negative values).

Some illustrative examples of these concepts in the scalar case are illustrated in Fig. 2. Note that, if a system is strictly proper then the property of monotonicity implies that of sign definiteness.

The important properties of such systems are given in the following lemma:

Lemma 2: (see Appendix 7) An m-input/m-output, strictly proper, discrete (resp. continuous), monotonic, sign-definite and stable system with transfer function matrix T has $E^m(S)$ norm

$$||T|| = ||T(1)||_m \quad (\text{resp.} ||T(o)||_m) \tag{13}$$

where $||.||_m = \max_{1\leq i\leq m} \sum_{j=1}^{m} |(.)_{ij}|$ is the matrix norm induced in $L(R^m)$ by the uniform norm $||.||_m = \max_{1\leq i\leq m} |(.)_i|$ in R^m.

The interpretation of this result is very important! Taking, initially, the discrete case, suppose that plant tests or simulations of a complex plant model of T are undertaken to evaluate the steady state response $t_i^{(j)}$ of the i^{th} output to a unit step in the j^{th} input. If (as should be self-evident by visual inspection) the system is stable, monotonic and sign-definite, then $||T|| = \max_{1\leq i\leq m} \sum_{j=1}^{m} |t_i^{(j)}|$. An identical result holds in the continuous case.

The main application of this notion is in the evaluation of $||G_A - G||$. The following results follows from the definition and lemma 2:

<u>Proposition 3</u>: If $y_i^{(j)}(k)$ (resp. $y_{Ai}^{(j)}(k)$) is the response from zero initial conditions of the i^{th} output of the <u>discrete</u> system G (resp. G_A) to a unit step in the j^{th} input, then $G_A - G$ is monotonic and sign-definite if, and only if, for each (i,j), the error response $\{y_{Ai}^{(j)}(k) - y_i^{(j)}(k)\}_{k\geq o}$ is either positive and monotonically increasing or negative and monotonically decreasing. In the case of $G_A - G$ monotonic, sign definite and stable, it has norm

$$||G_A - G|| = ||E||_m \tag{14}$$

where $E = G_A(1) - G(1)$ has elements E_{ij} that can be deduced from the time-response relation

$$E_{ij} \triangleq \lim_{k\to+\infty} \{y_{Ai}^{(j)}(k) - y_i^{(j)}(k)\} \tag{15}$$

<u>Proposition 4</u>: If $y_i^{(j)}(t)$ (resp. $y_{Ai}^{(j)}(t)$) is the response from zero initial conditions of the i^{th} output of the <u>continuous</u> system G(resp. G_A) to a unit step in the j^{th} input, then $G_A - G$ is monotonic and sign-definite if, and only if, for each pair (i,j) the error function $y_{Ai}^{(j)}(t) - y_i^{(j)}(t)$ is either positive and monotonically increasing or negative and monotonically decreasing. In the case of $G_A - G$ monotonic, sign-definite and stable, equation (14) holds with $E = G_A(o) - G(o)$, or, equivalently, the matrix with elements

$$E_{ij} \triangleq \lim_{t\to+\infty} \{y_{Ai}^{(j)}(t) - y_i^{(j)}(t)\} \tag{16}$$

In short, if the modelling error $G_A - G$ is monotonic and stable, we can evaluate $||G_A - G||$ by inspection of the steady state characteristics of the plant and approximate model only! The following theorem summarizes the above and is the main result obtained in this paper.

<u>Theorem 1</u>: Suppose that an unknown discrete (resp. continuous) linear plant is approximated by a discrete (resp. continuous) model G_A with the property that $G_A - G$ is monotonic, sign-definite and stable. If the controller K is designed to ensure that the approximating feedback system of Fig. 1(b) is stable, then K will also stabilize the real plant G in the real feedback configuration of Fig. 1(a) if KFG is controllable and observable and at least one of the following inequalities is satisfied

$$\lambda''''_{(1)} \triangleq \sup_{z\in\partial S} \max_{1\leq i\leq m} \sum_{j=1}^{m} \sum_{k=1}^{m} |((I_m+K(z)F(z)G_A(z))^{-1}K(z)F(z))_{ik}|\cdot|E_{kj}| < 1 \qquad (17)$$

$$\lambda''''_{(2)} \triangleq ||(I + KFG_A)^{-1}KF||\cdot||E||_m < 1 \qquad (18)$$

where E is the mxm 'steady state error matrix' defined in proposition 3 (resp. proposition 4).

<u>Proof</u>: We verify the conditions of lemma 1 will hold. By assumption Fig. 1(b) is stable, $G_A - G$ is stable and KFG is controllable and observable. Also, using (11),

$$\lambda'' = \sup_{z\in\partial S} \max_{1\leq i\leq m} \sum_{j=1}^{m} |((I_m + K(z)F(z)G_A(z))^{-1}K(z)F(z)(G_A(z)-G(z))_{ij}|$$

$$\leq \sup_{z\in\partial S} \max_{1\leq i\leq m} \sum_{j=1}^{m} \sum_{k=1}^{m} |((I_m+K(z)F(z)G_A(z))^{-1}K(z)F(z))_{ik}|\cdot|(G_A(z)-G(z))_{kj}| \qquad (19)$$

and hence, as G_A-G is stable, monotonic and sign-definite, $|(G_A(z)-G(z))_{kj}|$ is bounded by $|E_{k_j}|$ on ∂S and (19) reduces to $\lambda'' < \lambda''''_{(1)}$.

Finally

$$\lambda'' \leq \lambda''''_{(1)} = \sup_{z\in\partial S} \max_{1\leq i\leq m} \sum_{j=1}^{m} |((I_m+KFG_A)^{-1}KF)_{ij}| \sum_{k=1}^{m} |E_{jk}|$$

$$\leq \sup_{z\in\partial S} \max_{1\leq i\leq m} \sum_{j=1}^{m} |((I_m+KFG_A)^{-1}KF)_{ij}| \max_{1\leq k\leq m} \sum_{\ell=1}^{m} |E_{k\ell}|$$

$$= ||(I+KFG_A)^{-1}KF||.||E||_m = \lambda''_{(2)} \tag{20}$$

The theorem is hence proven as $\lambda'' < 1$ if (17) or (18) holds.

2.4 Checking the Stability Conditions:

The application of the result is best underlined by the following step by step design procedure:

Step 1: Obtain the plant responses to unit step inputs from zero inital conditions.

Step 2: Choose an approximate plant model G_A with the property that G_A-G is monotonic, sign-definite and stable.

Step 3: Design the controller K for G_A to obtain the required stability and performance characteristics from the approximating feedback system of Fig. 1(b).

Step 4: Check the contraction condition (17) or (18) by numerical or graphical means.

Step 5: Check that KFG is controllable and observable.

Steps 1 to 4 can easily be undertaken without a detailed knoledge of G. Step 5 does, however, require, in principle, some knowledge of G. The required knowledge is frequently structural rather than numerical and can hence often be deduced from physical considerations. For example, if F = I, m = 1 and the controller K is proportional plus integral and minimum phase it is only required to know that G is controllable and observable with no zero at the point z = 1 (in the discrete case) or s=o (in the continuous case). Equivalently G is controllable and observable with non-zero steady state response to step inputs.

The choice of model G_A is discussed in section 3 and the design of K for G_A can proceed by any available means (1-5). It is worthwhile, however, considering the means of checking step 4. In general there is no alternative to the numerical evaluation of $\lambda''''_{(1)}$ or $\lambda''''_{(2)}$ from the definitions and equation (3). There are however, two notable cases when this procedure can be considerably simplified:

Proposition 5: The conclusions of theorem 1 remain valid if also the system $(I+KFG_A)^{-1}KF$ is monotonic and sign-definite and (17) and (18) are replaced by

$$\lambda''''_{(1)} = \max_{1 \leq i \leq m} \sum_{j=1}^{m} \sum_{k=1}^{m} |H_{ik}| \cdot |E_{k_j}| \tag{21}$$

and

$$\lambda''''_{(2)} = ||H||_m \cdot ||E||_m \tag{22}$$

respectively. Here the mxm matrix H is just the steady state gain matrix.

$$H \triangleq \lim_{z \to 1} (I_m + K(z)F(z)G_A(z))^{-1} K(z)F(z) \tag{23}$$

(if the plant is discrete) or

$$H \triangleq \lim_{s \to o} (I_m + K(s)F(s)G_A(s))^{-1} K(s)F(s) \tag{24}$$

(if the plant is continuous) and is defined in terms of the known objects G_A, K and F only.

Proof: Lemma 2 applied to $(I+KFG_A)^{-1}KF$ in (18) implies (22) immediately. Also, if $(I+KFG_A)^{-1}KF$ is monotonic and sign-definite then so are its elements and hence the supremum in (17) is achieved at the d.c. frequency z=1 (in the discrete case) or s=o (in the continuous case). This proves (21) and ends the proof of the result.

The above result identifies a situation when a knowledge of the steady state characteristics of real and approximate plant is sufficient to guarantee stability. The following result has clear connections with the use of diagonal domonance ideas[1,2] and the standard use of Nyquist methods in classical control theory.

Proposition 6: The conclusions of theorem 1 remain valid if also the plant approximation G_A, the controller K and feedback element F are all scalar or diagonal multivariable (i.e. non-interacting) systems and (17) and (18) are replaced by the single condition.

$$\left| \frac{K_{kk}(z)F_{kk}(z)}{1+K_{kk}(z)F_{kk}(z)(G_A(z))_{kk}} \right| < \frac{1}{\sum_{j=1}^{m} |E_{kj}|} \triangleq R_k \quad , \forall z \in \partial S \quad , \quad 1 \leq k \leq m \tag{25}$$

In particular, if $(G_A)_{kk}$, K_{kk} and F_{kk} have no zeros on ∂S, (25) can be written as

$$|1 + F_{kk}^{-1}(z)K_{kk}^{-1}(z)(G_A(z))_{kk}^{-1}| > |(G_A(z))_{kk}^{-1}| \sum_{j=1}^{m} |E_{kj}| \triangleq d_k(z)$$

$$\forall \; z \in \partial S \quad , \quad 1 \leq k \leq m \tag{26}$$

(Note: the assumptions on the zero structure of G_A, K and F can be removed in the normal manner by indenting ∂S into the stable region of the complex plane).

Proof: Equation (25) is equivalent (under the diagonality assumption) to (17) and (18) as ∂S is compact. Under the stated conditions (26) is equivalent to (25). The result follows trivially.

A situation when this proposition can be applied is described in Section 3.1.4

A graphical interpretation of Proposition 6 indicates that the conditions (25) and (26) are easily checked in practice. More precisely, writing

$$\partial S = \partial_1 S \cup \partial_2 S \tag{27}$$

where $\partial_1 S$ is the familiar set (the positive imaginary axis)

$$\partial_1 S \triangleq \{s : s = \pm i\omega \text{ for some real frequency } \omega \geq 0\} \tag{28}$$

in the continuous case, and the unit circle

$$\partial_1 S \triangleq \{z : |z| = 1\} \tag{29}$$

in the discrete case. The set

$$\partial_2 S \triangleq \{s : \mathrm{Re}\,s > 0 \; , \; |s| = R\} \tag{30}$$

is the semi-circular component of the Nyquist contour in the continuous case and

$$\partial_2(S) = \{z : |z| = R\} \tag{31}$$

in the discrete case. Verification of (25) and (26) on ∂S boils down to verification on $\partial_1 S$ and $\partial_2 S$. In practice the set $\partial_1 S$ can be identified with the normal frequency response plots of the various systems whilst $\partial_2 S$ is present due to the mathematical technicalities. We make the following observations:

(a) The validity of (25) on $\partial_2 S$ as $R \to +\infty$ is most easily checked analytically be noting that, as KFG_A is strictly proper, it is sufficient that

$$\lim_{|z|\to\infty} |K_{kk}(z)F_{kk}(z)| < R_k \; , \; 1 \leq k \leq m \tag{32}$$

(b) The validity of (25) on $\partial_1 S$ is ensured if, for each index k in the range $1 \leq k \leq m$, the <u>Nyquist</u> plot of $K_{kk}F_{kk}/(1+K_{kk}F_{kk}(G_A)_{kk})$ $(z \in \partial S_1)$ lies entirely within the open disc of radius R_k and centre the origin of the complex plane. This notion is illustrated geometrically in Fig. 3(a).

(c) The validity of (26) on $\partial_1 S$ is ensured if, for each index k in the range $1 \leq k \leq m$, the <u>inverse</u> Nyquist locus $F_{kk}^{-1}(z)\ K_{kk}^{-1}(z)\ (G_A(z))_{kk}^{-1}$ $(z \in \partial_1 S)$ with superimposed 'confidence circles' at each frequency $z \in \partial_1 S$ of centre $F_{kk}^{-1}(z)K_{kk}^{-1}(z)(G_A(z))_{kk}^{-1}$ and radius $d_k(z)$ does not contain or touch the (-1,0) point of the complex plane. This notion is illustrated in Fig. 3(b), and has a clear similarity to the diagonal dominance concepts used in the inverse Nyquist array and method of dyadic expansion design techniques[1,2]. Note however, that the radii of the confidence circles are independent of controller gain and feedback element. This is the price that is paid for our lack of detailed knowledge of plant modelling error dynamics!

2.5 Robustness of the Final Design

Although the analysis, and consequent design procedure, discussed above is clearly capable of producing stable controllers for unknown plant (and hence, if integral action is included in K, of ensuring accurate tracking of step demands and total rejection of step disturbance signals) it is important to recognize that the procedure is robust in the sense that it will continue to stabilize the plant if, over a period of time, its dynamic characteristics change by less than a computable amount. More precisely, if the dynamic characteristics of the plant G change to produce the plant $\tilde{G}$, we can prove the following:

Proposition 7: If the conditions of theorem 1 hold, then the closed-loop system of Fig. 1(a) will retain its stability if $KF\tilde{G}$ is controllable and observable, $\tilde{G} - G$ is stable and

$$||\tilde{G} - G|| < \frac{1 - \mu}{||(I+KFG_A)^{-1}KF||} \qquad (33)$$

where $\mu < 1$ is any convenient upper bound for λ'' (e.g. $\lambda''''_{(1)}$ or $\lambda''''_{(2)}$)

Proof: For the configuration of Fig. 1(a) to remain stable with G replaced by $\tilde{G}$, it is sufficient (lemma 1) that $KF\tilde{G}$ is controllable and observable that $G_A-\tilde{G}$ is stable and $||(I+KFG_A)^{-1}KF(G_A-\tilde{G})|| < 1$. The first two conditions have been assumed and the second follows from the inequality,

$$\|(I+KFG_A)^{-1}KF(G_A-\tilde{G})\|$$
$$< \|(I+KFG_A)^{-1}KF(G_A-G)\| + \|(I+KFG_A)^{-1}KF(G-\tilde{G})\|$$
$$< \mu + \|(I+KFG_A)^{-1}KF\|.\|G-\tilde{G}\|$$
$$< 1 \qquad \text{(from (33))} \qquad (34)$$

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3. Construction of the Approximate PLant Model

The success of the techniques described in section 2 rely primarily upon our ability to construct an approximate plant model G_A with the properties:

(i) G_A is simple enough to make possible the design of K using available facilities,

(ii) the modelling error G_A-G is stable, and

(iii) G_A - G is also monotonic and sign-definite

It is acknowledge that there exists examples where the simplest solution to this composite problem is the (notionally unobtainable) solution $G_A = G$. In the following sections some example solutions to the problem are described together with conditions when the solutions exist. It will become clear that a solution certainly exists and can be easily computed if the unknown plant G is (a) stable and (b) non-oscillatory, but that, if one of these conditions is violated, the construction of G_A can become rather more complicated. Fortunately, the properties of stability and non-oscillation are commonly encountered properties of systems in the process industries.

3.1 Approximate Models for Discrete Plant

In principle there exists an infinite number of choices of G_A with the required properties that $G-G_A$ is stable, monotonic and sign-definite. In practice, however, G_A must be constructed using simple operations on transient data and should preferably be of a desired complexity i.e. of a simple form if 'back of envelope' calculations are to be used or of a more complex form if it is thought to be necessary and suitable computing facilities are available. This choice should be open to the designer! A number of useful possibilities are described below.

3.1.1 <u>The Case of G Known Exactly.</u>

If a model of the plant is known exactly (i.e. practice, to a reasonable accuracy) we can follow the technique of Vidyasagar[21] and express

$$G = G_+ + G_- \tag{35}$$

as the sum of a stable system G_- and an unstable system G_+ using, say, the standard technique of partial fraction expansions. A suitable choice of G_A could then be $G_A = G_+$, when $G - G_A = G_-$ is clearly stable. A check for monotonicity can then be undertaken by simulation or pole-zero analysis.

To illustrate the technique, consider the unstable discrete plant described by the second-order transfer function $G(z) = 1/z(z-1)$. Using partial fraction expansions, express G in its stable and unstable components

$$G(z) = \frac{1}{(z-1)} - \frac{1}{z} \tag{36}$$

and choose the unstable approximate model $G_A(z) = 1/(z-1)$ when the (deadbeat) modelling error $G - G_A = - 1/z$ is clearly monotonic and sign-definite with $||G-G_A|| = 1$. Choosing proportional control and unity feedback for simplicity, the approximating feedback system of Fig. 1(a) is clearly stable if $|K-1| < 1$ (i.e. $K^{-1} > \frac{1}{2}$) and KFG is trivially controllable and observable. The conditions of Theorem 1 will then be satisfied if $\lambda''_{(2)} < 1$ when the stability of G in the presence of K is guaranteed. It is verified by graphical and analytic methods that

$$\lambda''''_{(2)} = \sup_{\substack{|z|=1\\|z|=R}} \left| \frac{K}{1+{}^{K}/_{(z-1)}} \right| = \max\left\{ |K| , \frac{1}{|K^{-1}-\tfrac{1}{2}|} \right\} < 1 \qquad (37)$$

(and hence that the given controller can be <u>guaranteed</u> to stabilize the real plant) if $0 < K < {}^{2}/_{3}$. This predicted range should be compared with the actual range of gains $0 < K < 1$ that ensure stability. Clearly the range predicted on the basis of the approximate model is pessimistic but not unduly so. In fact, bearing in mind the large modelling error $G - G_A$, the results are highly successful for prediction of stability characteristics. The transient responses are compared in Fig. 4 for the choice of $K = {}^{1}/_{3}$ and compare quite favourably.

3.1.2 Plants with Asymptotically Monotone Step Responses

If the plant model is known, then the technique of section 3.1.1 can always be applied. It is however, impossible to apply if G is not known or is too complex to handle. In such cases we can work with transient step response data obtained from plant test or complex model simulations if the plant has the following property (see section 3.1.3 for a relaxation of this property):

<u>Definition</u>: An m-input/m-output, proper, continuous or discrete system is said to be asymptotically monotone if there exists a time $t^* > 0$ such that, for each pair of indices (i,j), the response $y_i^{(j)}$ of the i^{th} system output from zero initial conditions to a unit step demand in the j^{th} input is either monotonically increasing or monotonically decreasing. (Note: in the discrete case, attention is restricted to the response at the sample times only).

Some illustrative examples of this concept in the scalar, continuous case are illustrated in Fig. 5 where we see that asymptotically monotone systems are simply systems whose dominant modes are nonoscillatory. This is frequently the case in many process industries and is easily identified from the transient data.

We now state, without proof, the following simple result relating the continuous and discrete cases:

Proposition 8: An m-input/m-output asymptotically monotone continuous system, under synchronous sampling and input actuation, gives rise to an asymptotically monotone discrete system.

The importance of the notion of asymptotic monotonicity in relation to the work of section 2 lies in the following fundamental result:

Proposition 9: To each m-input/m-output, proper, stable and asymptotically monotone discrete plant G, there exists an infinity of approximate models G_A with the property that $G - G_A$ is stable, monotonic and sign-definite.

Proof: We prove the existence of one G_A, the existence of an infinity following trivially by inspection of the constructive argument used. Let $y_i^{(j)}(k)$, $k \geq 0$ be the i^{th} output response sequence obtained from the plant from zero initial conditions to a unit step input in channel j and let $t^* \leq k^* h$ (k^* integer) be as defined in the definition. Using the technique outlined in ref. 22, set

$$(G_A(z))_{ij} = \sum_{k=1}^{k^*} z^{-k}\{y_i^{(j)}(k) - y_i^{(j)}(k-1)\} \tag{38}$$

to be the $(i,j)^{th}$ element of $G_A(z)$ i.e. the approximate model G_A is deadbeat and has identical step responses to G in the (discrete) time interval $o \leq k \leq k^*$ and is constant for $k > k^*$. Noting that the step responses of G-G_A are obtained simply by subtracting the step responses of G_A from those of G, the monotonicity and sign-definiteness of $G - G_A$ is trivially verified by the graphical argument illustrated in Fig. 6.

It is clear therefore that the construction of G_A is a simple task if G is asymptotically monotone but, if $t^* >> h$ (i.e. $k^* >> 1$), the model may have to be of high order. In many process control applications, however, it is anticipated that this will not be a problem if the plant step responses are monotonic (and hence $t^* = 0$) and relatively slow sampling is used. The dimensionality problem does not occur in this case, nor does it in any case where t^* is 'small' and the sampling rate is not too fast.

3.1.3 Asymptotic Monotonicity under Minor Loop Feedback

It is possible to partially extend the above notions to systems that oscillate in a straightforward manner provided that we can find a constant (say) feedback K_o such that $G' \triangleq (I + GK_o)^{-1}G$ is asymptotically monotone. The simple control scheme of Fig. 1(a) can then be extended to that shown in Fig. 7(a) where the 'plant' G' seen by K is asymptotically monotone. An approximation G_A to G' can be constructed as in section 3.1.2 and the analysis of section 2 applied with G replaced by G'.

Although conceptually appealing, it is generally true that the removal of oscillation by constant minor loop feedback requires the use of positive feedback e.g. for the case of the oscillating (continuous) system $G(s) = 1/(s^2 + 2s + 2)$, the removal of oscillation requires that $K_o < -1$ and stability requires $K_o > -2$. This positive feedback loop may be removed by noting from Fig. 7(a) with $r = o$ that $y = -GKy - GK_oy = -G(K + K_o)y$ and hence that the stability of this configuration is identical to that of the configuration shown in Fig. 7(b).

3.1.4 Multivariable Systems with Monotonic Iteraction Effects

In the multivariable case, although a suitable approximate model could be constructed using the techniques of section 3.2., the controller design on the basis of G_A can still be a complex problem due to the presence of interaction terms modelling the interaction effects of G. In such cases, it is natural to search for conditions when a diagonal/non-interacting approximation can be constructed. This problem is answered easily as follows:

Proposition 10: If the m-input/m-output, strictly proper, stable and asymptotically monotone discrete plant G has monotonic interaction effects $y_i^{(j)}$ $(j \neq i)$, then there exists an infinity of non-interacting approximate models G_A with the property that $G - G_A$ is stable, monotonic and sign-definite.

Proof: Simply set $(G_A(z))_{ij}$ as in equation (38) if $i = j$ and equal to zero if $i \neq j$.

A particularly interesting and easily applied graphical stability theorem bearing a close resemblance to the well-known inverse Nyquist array[1,2] results and directly applicable in these situations has already been given in Proposition 5 and the following discussion. The applicability of the result in any given case depends upon the steady-state magnitudes of the interaction effects E_{kj}, $k \neq j$. If interaction effects are large then the use of a diagonal model G_A ensures that the radii R_k, $1 \leq k \leq m$ (equation (25)) are small or, equivalently, the confidence circles have large radius. The theoretical conditions (25) or (26) then clearly require low loop gains to guarantee stability and hence the design may be too conservative. Conversely, if interaction effects are relatively small, the radii R_k are large and the confidence circles small, leading to few problems in satisfying (25) or (26).

3.1.5 First Order Plant Models

One particularly exciting possibility implicit in the form of approximation technique discussed in this paper is that it may be possible to construct an approximate model G_A with the required properties plus the bonus that the control design for G_A can be undertaken in an analytical/'pencil and paper' manner. One obvious candidate here is a first-order multivariable model[10] of the form

$$G_A(z) = ((z-1)B_o + B_1)^{-1}, \quad |B_o| \neq 0 \tag{39}$$

where B_o and B_1 are suitably chosen constant mxm matrices. Taking $B_o = B_1$, it may even be possible to consider the use of the pure-delay/deadbeat model

$$G_A(z) = z^{-1}B_o^{-1} \tag{40}$$

in a similar manner to that of Astrom in ref. (16). The simplicity of the resulting stability criteria can be illustrated by considering the case of the design of integral controllers K as follows:

Proposition 11: Let the discrete plant G be m-input/m-output, strictly proper, stable and monotonic and $F = I_m$ and consider the deadbeat approximate model given in equation (40) with

$$B_o^{-1} = \begin{pmatrix} y_1^{(1)}(1) & \cdots & y_1^{(m)}(1) \\ \vdots & & \vdots \\ y_m^{(1)}(1) & \cdots & y_m^{(m)}(1) \end{pmatrix} \tag{41}$$

nonsingular (see Fig. 8) and the integral controller $K(z) = \alpha B_o z/(z-1)$

where α is a real scalar gain. The controller K will then stabilize the unknown plant G in the configuration of Fig. 1(a) if KFG is both controllable and observable and the following inequalities hold

$$|1 - \alpha| < 1 \tag{42}$$

$$\frac{|\alpha| \, . \, ||B_o||_m \, . ||E||_m}{1 - |1-\alpha|} < 1 \tag{43}$$

Proof: We simply check the conditions of theorem 1 are valid. Certainly KFG is controllable and observable by assumption and it is easily verified that (42) is necessary if K is to stabilize the approximate model G_A given in (40). Condition (43) is equivalent to condition (18) as $(I + KFG_A)^{-1}KF = (z\alpha/(z-1+\alpha))B_o$ and hence $||(I + KFG_A)^{-1}KF|| = (|\alpha|/(1-|1-\alpha|)) \, ||B_o||_m$. This completes the proof of the proposition.

As a final caution, we note that the above result implies the 'hidden' design condition of slow sampling. To illustrate this point, note that if sampling is fast then B_o^{-1} is 'small' and hence B_o is 'large'. Examination of (42) and (43) then indicates that α must be small and positive and hence (43) reduces to $||B_o||_m||E||_m < 1$ which may not be satisfied! In such conditions the theory is indicating that the approximate model is 'too crude' to be the basis for confident closed-loop predictions for the real plant. A better model could however, be constructed using the technique of section 3.1.2 or a slower sampling rate implemented.

3.2 Approximate Models for Continuous Plant

In many ways the choice of approximate model G_A for a continuous unknown plant G follows can be based upon similar considerations to those of discrete plant. It appears to be the case, however, that the construction of such G_A is not quite so straightforward as in the discrete case. These similarities and problems are discussed below.

It is trivially verified that the general procedure outlined in Section 3.1.1 in the case of G known exactly carries through to the continuous case with no change. Also the concept of asymptotic monotonicity of G introduced in Sections 3.1.2 and 3.1.3 carries through. In particular, the existence result of Proposition 9 extends with no change other than in the proof where the deadbeat model construction (which has no continuous counterpart) must be replaced by a trivial but indirect argument based upon the observation that an asymptotically monotone continuous plant must have (at least) one non-oscillatory mode. The main problem here is that, although a discrete deadbeat model with the required monotonicity properties can be easily fitted to plant data, using the construction of (39) it is not obviously so easy to fit a model to continuous data. In such cases it may be necessary to resort to trial and error!

In the multivariable case, the comments made in Section 3.1.4 still hold in that, if a diagonal model will lead to acceptable results, the resulting simplicity in controller design is very desirable. If G has monotone interaction effects, then it is easily verified that Proposition 10 carries through to the continuous case. This notion again leads to a form of 'inverse Nyquist array' where the confidence circles provide some measure of the confidence that can be placed upon the approximate model that neglects the unknown interaction dynamics.

Finally, the essential principle introduced in section 3.1.5 of using approximate models for which analytical construction of controllers is possible clearly applies to the continuous case and the continuous first order model[2,9,12] is the obvious equivalent to (39). There is however no equivalent to the deadbeat model (40) and hence Proposition 11 has no continuous counterpart. It is clear therefore that the construction of models G_A for a continuous plant G requires a little more thought. This is a topic for future study.

4. Illustrative Examples

The material of section 3 is a convincing justification of the fact that there are many ways of approximate modelling in an effective but simple manner. The examples described in this section are designed to underline this fact, to indicate the degree of flexibility available to the design engineer and also to point out that a number of problems can occur.

4.1 Controller Design for an Unknown Discrete Scalar System

Consider the stable, non-minimum phase, continuous time system with transfer function

$$G(s) = \frac{(2-s)}{(s+1)(s+2)} \tag{44}$$

That is assumed to be unknow for the purposes of this study. Assume however, that we have access to the unit step response given in Fig. 9(a). Note immediately that the system is asymptotically monotone with $t^* = \log_e 2 = 0.693$. For simplicity, we will take a sampling interval $h = t^*$ and, for comparative purposes, note that

$$G(z) = \frac{(3/_8)}{(z-\frac{1}{2})\ (z-\frac{1}{4})} \tag{45}$$

is the transfer function of the resulting discrete scalar system. Consider now the results obtained from the choice of a number of approximate models.

4.1.1 Design Based on a First Order Model

The very special choice of sample interval means that we can choose a first order deadbeat model based on the choice of $k^* = 1$ in equation (38). This approximate model is the trivial system $G_A(z) = 0$ and, using Proposition 3, it is seen that $G_A - G$ is stable and both monotonic and sign-definite with the norm

$$||G_A - G|| = ||E||_1 = 1 \tag{46}$$

computed from equation (15) and the steady state data deduced from Fig. 9 (a).

Restricting our attention to the choice of proportional, unity feedback control (i.e. $F(z) \equiv 1$), we see that <u>any</u> gain K will stabilize the trivial system $G_A(z) \equiv 0$. Applying theorem one, such a controller will also stabilize the unknown plant G provided that KFG = KG is controllable and observable and $\lambda''_{(2)} < 1$. The first of these conditions reduces to the requirement that G is both controllable and observable whilst the second reduces to

$$\sup_{\substack{|z|=1\\|z|=R}} \left| \frac{K(z)\ F(z)}{1+K(z)F(z)G_A(z)} \right| \cdot ||E||_1 = |K| < 1 \tag{47}$$

which provides an initial estimate of the range of gains that will ensure closed-loop stability. This is clearly rather a pessimistic estimate when compared with the real range of gains $-1 < K < 7/3$ that assure stability of the real plant. These predictions can be improved however by increasing the order of the approximate model. This is discussed below.

4.1.2 Design Based on a Second Order Deadbeat Model

Choosing $k^* = 2$ in equation (38) leads to the approximate, second-order deadbeat model

$$G_A(z) = \frac{3/8}{z^2} \tag{48}$$

whose step response is given, together with the real plants in Fig. 9(b). Clearly G_A-G is stable and both monotonic and sign-definite with norm (Proposition 3)

$$||G_A - G|| = ||E||_1 = 5/8 \tag{49}$$

Using proportional, unity feedback control the controllability and observability condition on KFG required by theorem 1 for K to stabilize G again reduces to the need for G to be both controllable and observable. The requirement that K stabilizes G_A reduces to

$$\frac{3}{8} \; |K| < 1 \tag{50}$$

and the contraction condition $\lambda''''_{(2)} < 1$ is simply

$$\sup_{\substack{|z|=1 \\ |z|=R}} \left| \frac{K z^2}{z^2 + \frac{3}{8} K} \right| \frac{5}{8} < 1 \tag{51}$$

or, equivalently, bearing (50) in mind,

$$\frac{5}{8} \quad \max \{ |K| , \frac{|K|}{1 - \frac{3}{8}|K|} \} = \frac{\frac{5}{8} |K|}{1 - \frac{3}{8}|K|} < 1 \tag{52}$$

Rearranging leads to $|K| < 1$ and hence, combining with (50), we conclude that the proportional controller K designed on the basis of the second order deadbeat model will also stabilize the unknown real plant if $|K| < 1$. Comparing this result with equation (47) indicates that there is no extra benefit to be obtained by using this higher order model. A somewhat surprising result? Suppose, however, that we accept this guaranteed gain band as a reasonable basis for control synthesis and attempt to augment the controller by an integral term,

$$K(z) = K_1 + \frac{K_2 z}{(z-1)} = K_1 + K_2 + \frac{K_2}{z-1} \tag{53}$$

The required condition that $\lambda''''_{(2)} < 1$ cannot however be satisfied as, if $K_2 \neq 0$,

$$\sup_{\substack{|z|=1 \\ |z|=R}} \left| \frac{K(z)F(z)}{1+K(z)F(z)G_A(z)} \right| \; ||E||_1$$

$$\geq \lim_{z \to 1} \left| \frac{K(z)F(z)}{1+K(z)F(z)G_A(z)} \right| \quad \frac{5}{8}$$

$$= \frac{5}{3} > 1 \tag{54}$$

The theory cannot therefore be used on the basis for design using the model of equation (48) if integral action is required. Perhaps a higher order model is required?

4.1.3 Design Based on Higher Order Models

The next stage of model complexity could be envisaged by showing $k^* = 3$ in equation (38) to yield the third-order, deadbeat model

$$G_A(z) = \frac{\frac{3}{8}\,(z + \tfrac{3}{4})}{z^3} \tag{55}$$

whose step response, together with that of the real plant, is given in Fig. 9(c). Again $G_A - G$ is stable, monotonic and sign-definite and

$$||G_A - G|| = ||E||_1 = \frac{11}{32} \tag{56}$$

For illustrative purposes however we will retain the second order nature of the approximate model but remove its deadbeat characteristics. More precisely, we will attempt a second order 'delay-lag' model of the form

$$G_A(z) = \frac{\frac{3}{8}}{z(z-\lambda)} \tag{57}$$

where $-1 < \lambda < 1$. Such a model certainly produces a step response matching that of the plant up to the second (k=2) sample and, by suitable choice of λ, it may be possible to produce a stable, monotonic and sign-definite modelling error $G_A - G$. This must clearly be undertaken by trial and error in general but, for simplicity, we will cheat a little by choosing $\lambda = \frac{1}{2}$ and verifying that it is a reasonable choice by using (45) and (57) to compute

$$G_A(z) - G(z) = \frac{(-3/32)}{z(z-\frac{1}{4})(z-\frac{1}{2})} \tag{58}$$

which is clearly stable, monotonic and sign-definite with

$$||E||_1 = \tfrac{1}{4} \tag{59}$$

The plant and approximate model step responses are shown in Fig. 9(d).

The first step in the design of a proportional, unity feedback controller for the unknown plant G is to design K for G_A. To this end, the inverse Nyquist plot of G_A is given in Fig. 10(a) in the half-circle $\{z : z = e^{i\theta}, -\pi \leq \theta \leq 0\}$. Bearing in mind the open-loop stability of G_A, it is seen that K stabilizes G_A in the gain range

$$-\frac{4}{3} < K < \frac{8}{3} \tag{60}$$

To ensure that K also stabilizes G, it is sufficient (Theorem 1) that G (and hence KFG) is both controllable and observable and that the contraction constant $\lambda''''_{(2)} < 1$. Taking the positive gain range $0 < K < 8/3$, we see that

$$\lambda''''_{(2)} = \sup_{\substack{|z|=1 \\ |z|=R}} \left| \frac{G_A^{-1}(z)}{1 + K^{-1} G_A^{-1}(z)} \right| \frac{1}{4}$$

$$= \frac{1}{4} \max\left\{ \frac{\frac{8}{3}}{K^{-1} \frac{8}{3} - 1} , K \right\} < 1 \tag{61}$$

if $0 < K < \frac{8}{5}$. Applying theorem 1, we deduce that the proportional controller stabilizing G_A in the range defined by (60) will also stabilize G in the range

$$0 < K < \frac{8}{5} \tag{62}$$

This prediction should be compared with the real range of (positive) gains $0 < K < \frac{7}{3}$. Although still a pessimistic estimate, note that the increased effort devoted to obtaining an improved approximate model has paid dividends in that the predictions are considerably better than those obtained in sections 4.1.1 and 4.1.2.

Choosing K = 1 for simplicity, the stability predictions are verified again in this case by the inverse Nyquist plot of $G_A K$ with (Fig. 10(b)) superimposed confidence circles. Note that the (-1,0) point does not lie in or on any circle, that $|K| . ||E||_1 = \frac{1}{4} < 1$ and hence that the contraction condition is clearly satisfied. The closed-loop unit step responses of the real and approximating feedback systems are given in Fig. 11 for comparative purposes.

Finally, if integral action is required, this approximate model can be used as the basis of design, as, if K_2 is small, the two-term controller will leave the inverse Nyquist plot of Fig. 10(b) essentially unaffected except in the vicinity of the d.c. (z=1) point which will be bent towards the origin. As the confidence circles have radii of the order of 1/3 in this region, note that the (-1,0) will still lie outside the confidence circles indicating that the contraction condition will still be satisfied and hence stability retained. A detailed discussion of this possibility is omitted.

4.2 Controller Design for an Unknown Multivariable System

Consider the stable, two-input/two-output system with transfer function matrix

$$G(s) = \frac{8/3}{(s+2)(s+4)} \begin{pmatrix} s+3 & 1 \\ 1 & s+3 \end{pmatrix} \tag{63}$$

that is assumed to be unknown for the purposes of this example. Assume however that we have access to a set of responses from zero initial conditions to unit steps in the individual inputs 1 and 2. The responses to a unit step in the first input are given in Fig. 12, the responses to a unit step in the second input being obtained (by symmetry) by interchanging the labels y_1 and y_2. Note immediately that the system has monotone interaction effects of approximately 30% in magnitude relative to the diagonal terms. This makes it possible to consider a diagonal approximate model. Suppose, for simplicity, that some effort is made to model the diagonal terms and hence to obtain the approximate model

$$G_A(s) = \frac{\frac{8}{3}(s+3)}{(s+2)(s+4)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \triangleq g(s)\ I_2 \tag{64}$$

Note that $G_A - G$ is stable, monotonic and sign-definite with

$E = 1/3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and hence that

$$||G_A - G|| = ||E||_2 = 1/3 \qquad (65)$$

Restricting our attention to the case of proportional, unity feedback control with $K(s) = k\ I_2$ where the gains are identical in each loop. It is easily verified that K stabilizes G_A if $k > 0$, and hence that, if G (and hence FKG) is controllable and observable and $\lambda''_{(2)} < 1$, the controller K will also stabilize the unknown plant G. The condition $\lambda''_2 < 1$ can be written as

$$\sup_{\substack{s=i\omega \\ |s|=R}} \left| \frac{k}{1+kg(s)} \right| \quad 1/3 < 1 \qquad (66)$$

This relationship could be analysed graphically in a number of ways but, for simplicity, note that $Re\{g(s)\} > 0$ on the D-contour and hence that (66) reduces to the range of gains

$$-\ 3/4 < k < 3 \qquad (67)$$

that guarantees the stability of the closed-loop system for the real unknown plant. This prediction should be compared with the actual stability range $k > -\ 3/4$. There is clearly some pessimism involved in totally neglecting the unknown interaction dynamics but this is only to be expected if no attempt is made to model them.

Finally, choosing $k = 2$ within the guaranteed stability band, the closed-loop responses of the real and approximating feedback systems are given in Fig. 13 for comparative purposes. The inclusion of integral action $K(s) = k(1 + \frac{1}{Ts)})$ could be considered to remove the steady state errors but this is omitted here. We will note however, that the chosen approximate model is good enought for this purpose as, examining the inverse Nyquist plot of gk given in Fig. 14 with superimposed confidence circles and noting that integral action bends the locus towards the origin in the vicinity of $s = o$ but leaves the radii of the confidence circles unaffected, it is clear that the (-1,0) point will not enter any confidence circle if the reset time T chosen is long enough. The contraction condition will therefore still be satisfied and hence stability retained provided that the controllability and observability of KFG has not been violated.

5. Conclusions

The paper has considered the general problem of identifying a large class of practical situations where controller design can be undertaken for a dynamically complex continuous or discrete, scalar or multivariable process using only simple (and, in some cases, pencil and paper) operations on transient data obtained from simulations or plant tests. A detailed knowledge of the open-loop plant model is not required provided that it is stable and the analysis is particularly simple if the plant has no oscillatory properties in the open-loop. In many ways the analysis can be regarded as a rigorous justification of the well-known intuitive fact that it is not necessary to model a system too accurately if the model is to be used for feedback control design. The primary contribution of this paper in this sense is to quantify what we mean by 'sufficient' accuracy in terms of the transient error between plant and its approximate model and to express the criteria in terms that can easily be checked graphically for both scalar and multivariable systems.

The procedures described are based upon the use of a simple (and possibly highly accurate) model of the plant dynamics deduced, say, from transient data. The degree of flexibility available to the designer is considerable. In particular, and within certain well-defined constraints, he can choose the order and dynamic complexity of the model to be used. In general the design of the control system based on the approximate model can proceed using well-known frequency domain design techniques[(1-5)]. Although one might envisage that this step will be undertaken with the aid of a CAD facility, the flexibility in choice of model order may reduce the design to a pencil and paper study. In general, however, a higher order model will tend to produce less conservative results.

Finally, the key to the simplicity of the techniques developed here is the construction of approximate models with monotone modelling error. Although this is shown to be easily achieved for nonoscillating discrete plant, a few problems can arise when consideration is given to oscillating or continuous plant. This is not thought to be a major problem as many industrial processes do not have open-loop oscillatory tendencies and the use of discrete plant models is increasing as the use of digital control elements increases. It is however, under consideration and will be the subject of a future report.

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7. Appendix

To prove lemma 2, consider first the discrete case and expand $T(z)$ in the form

$$T(z) = \sum_{k=1}^{\infty} z^{-k} T^{(k)} \qquad (68)$$

Stability ensures that this series is absolutely convergent for $|z| \geqslant 1$. It is easily verified that T is both monotonic and sign-definite if, for each pair of indices (i,j), the sequence $\{T_{ij}^{(k)}\}_{k\geqslant 1}$ consists of entirely positive or entirely negative terms. In this case, we see that, for $|z| \geqslant 1$,

$$|T_{ij}(z)| \leqslant \sum_{k=1}^{\infty} |T_{ij}^{(k)}| = |\sum_{k=1}^{\infty} T_{ij}^{(k)}| = |T_{ij}(1)| \qquad (69)$$

and hence that

$$||T|| = \max_{1\leqslant i\leqslant m} \sup_{z\varepsilon\partial D} \sum_{j=1}^{m} |T_{ij}(z)|$$

$$= \max_{1\leqslant i\leqslant m} \sum_{j=1}^{m} |T_{ij}(1)| = ||T(1)||_m \qquad (70)$$

In the continuous case, we can write

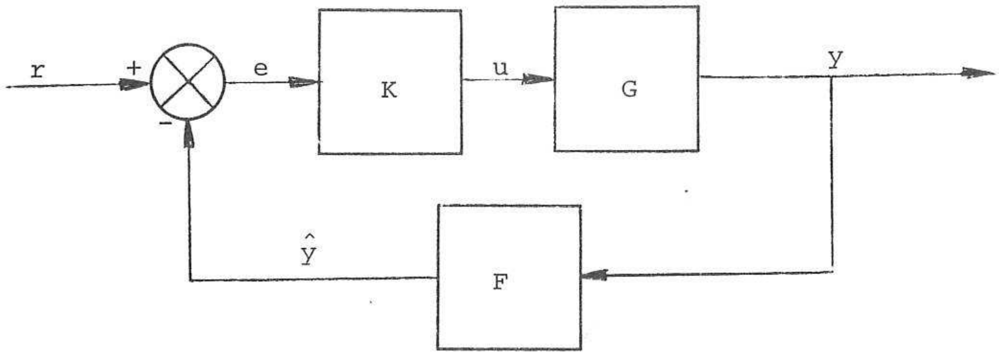
$$T(s) = \int_o^{\infty} H(t)e^{-st}\, dt \qquad (71)$$

where $H(t)$ is the plant impulse response matrix. Stability ensures that the integral exists for $\text{Re } s \geqslant o$. Again we note that T is monotonic and sign-definite if, and only if, for each pair of indices (i,j), the function $H_{ij}(t)$ is entirely positive or negative for $t \geqslant o$. Under these conditions, taking $\text{Re } s \geqslant o$,

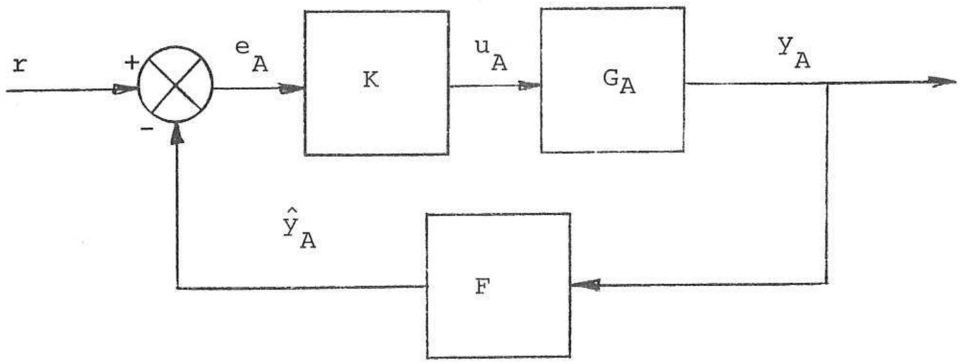
$$|T_{ij}(s)| \leqslant \int_o^{\infty} |H_{ij}(t)|dt$$

$$= |\int_o^{\infty} H_{ij}(t)dt| = |T_{ij}(o)| \qquad (72)$$

and the result follows using a similar argument to (70).



(a)



(b)

Fig. 1 Real (a) and approximating (b) feedback schemes.

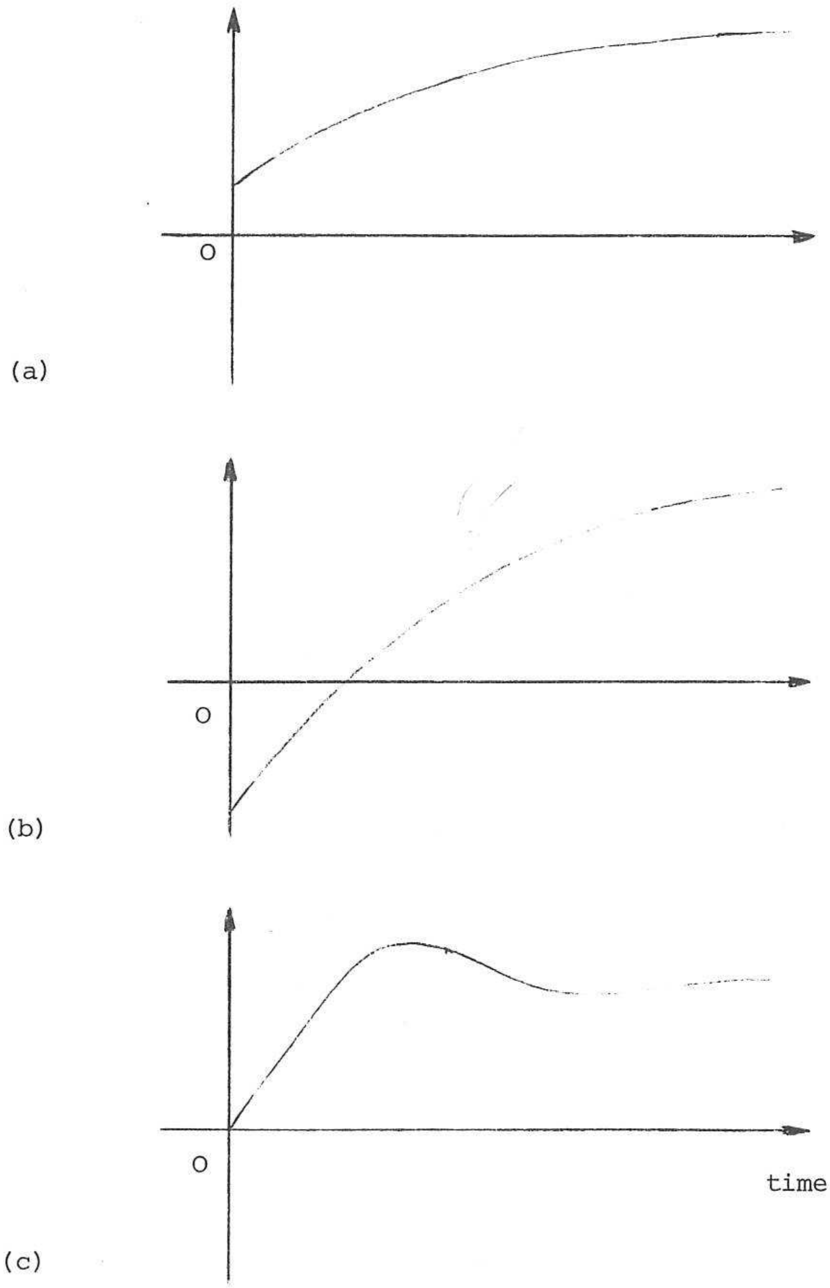


Fig. 2. (a) Monotonic and sign-definite

(b) Monotonic but not sign-definite

(c) Sign-definite but not monotonic

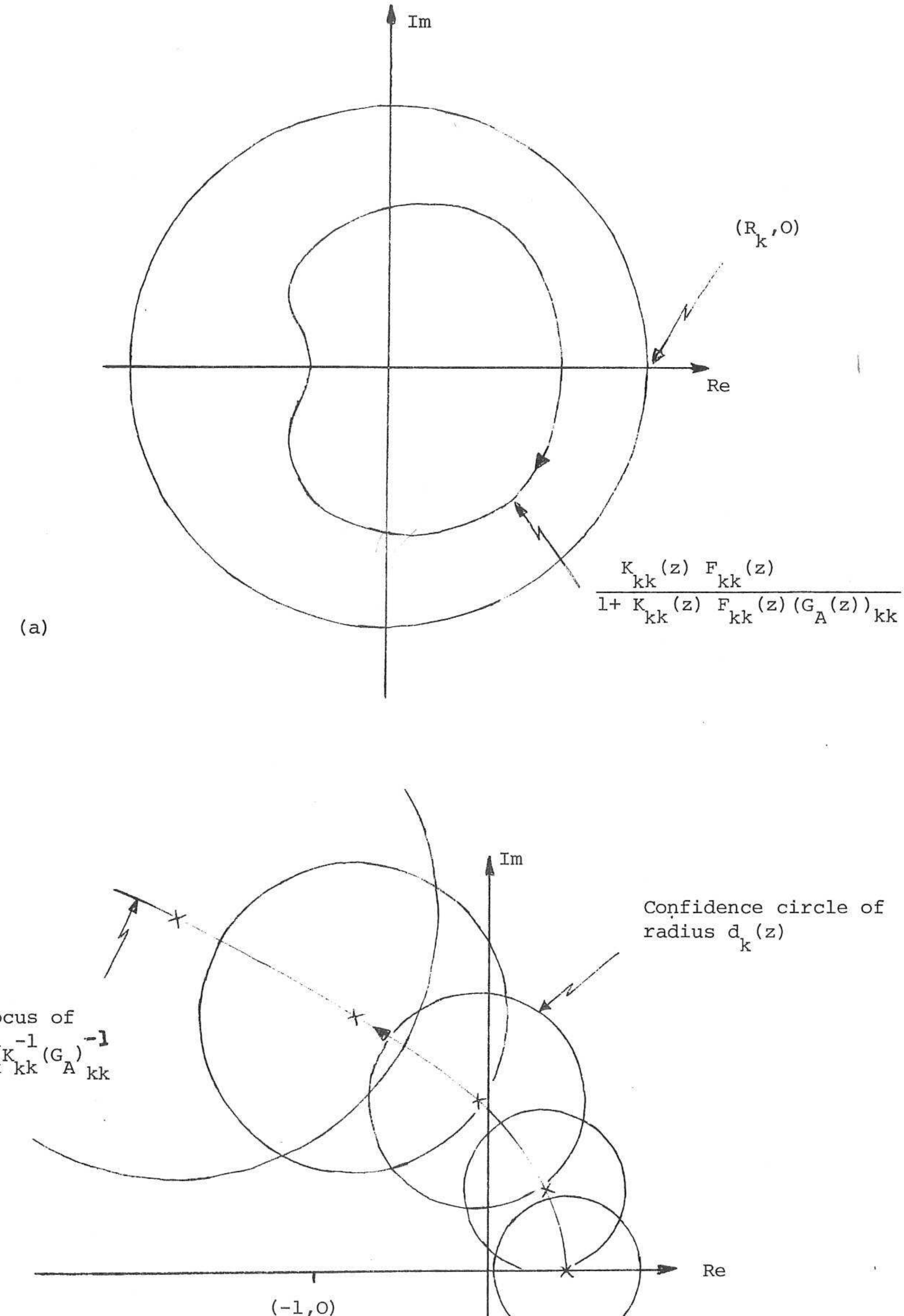


Fig. 3. Graphical representations of the contraction condition

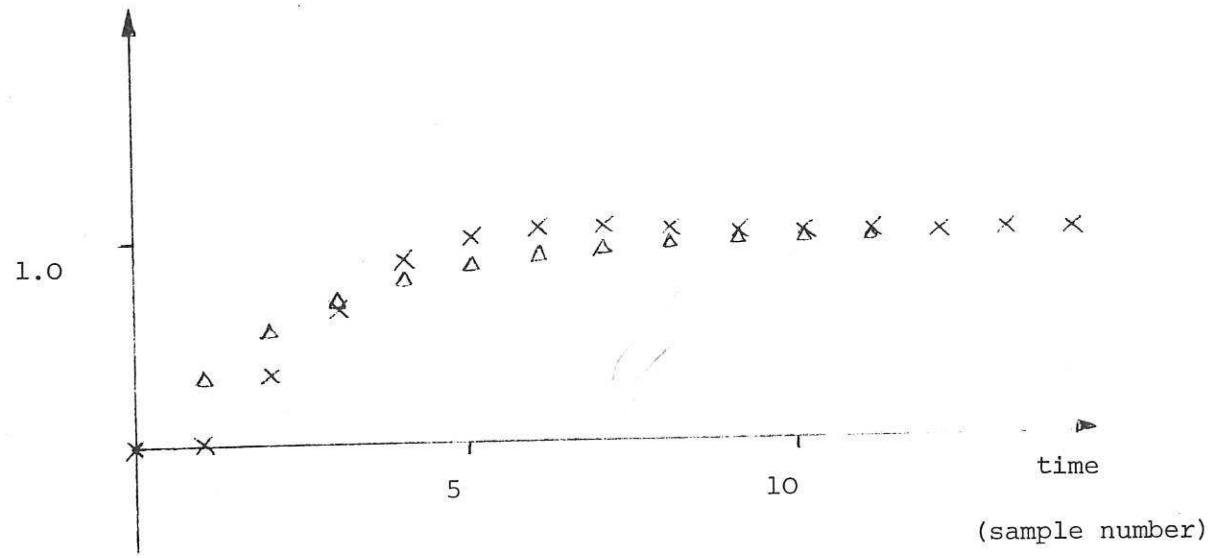
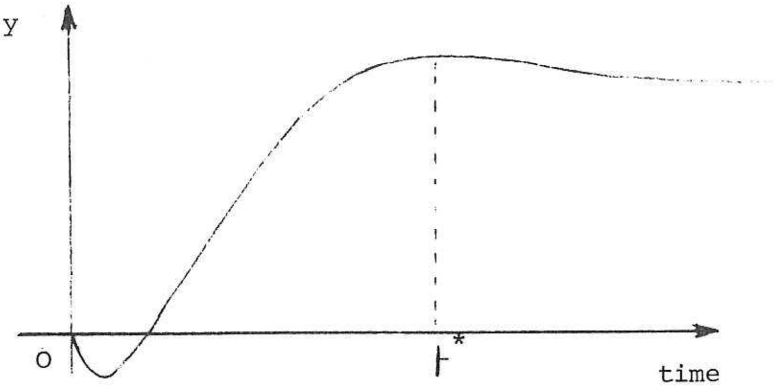


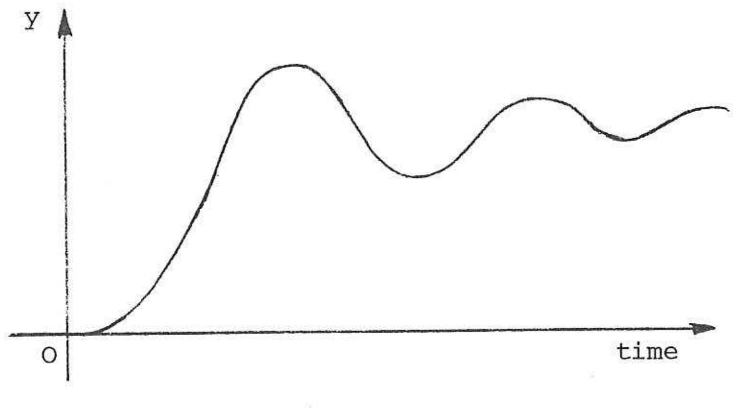
Fig. 4 Unit step responses of the real and approximating feedback systems

x = real systems response y

Δ = approximate system response y_A



(a)



(b)

Fig. 5 Asymptotically monotone (a) and non-asymptotically monotone (b) scalar continuous system responses

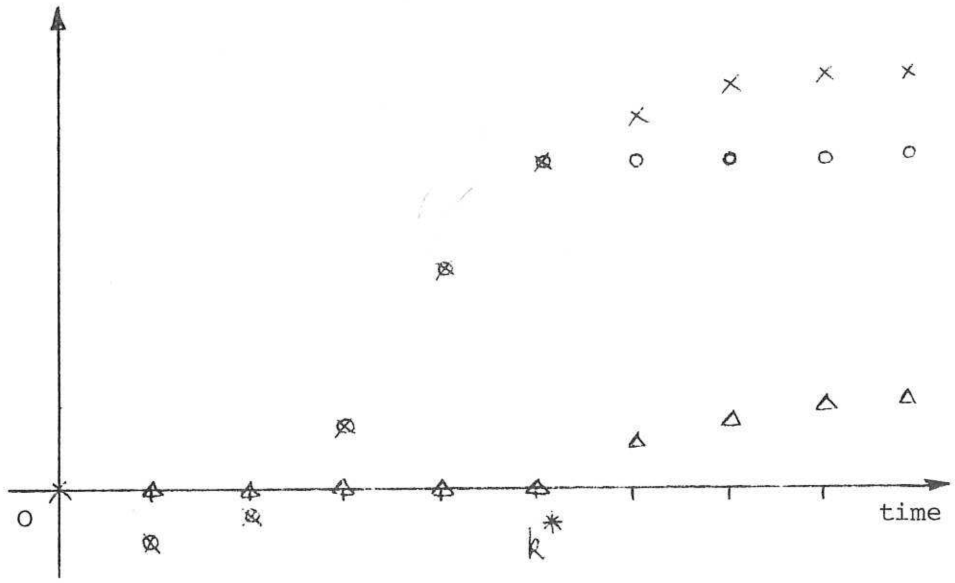
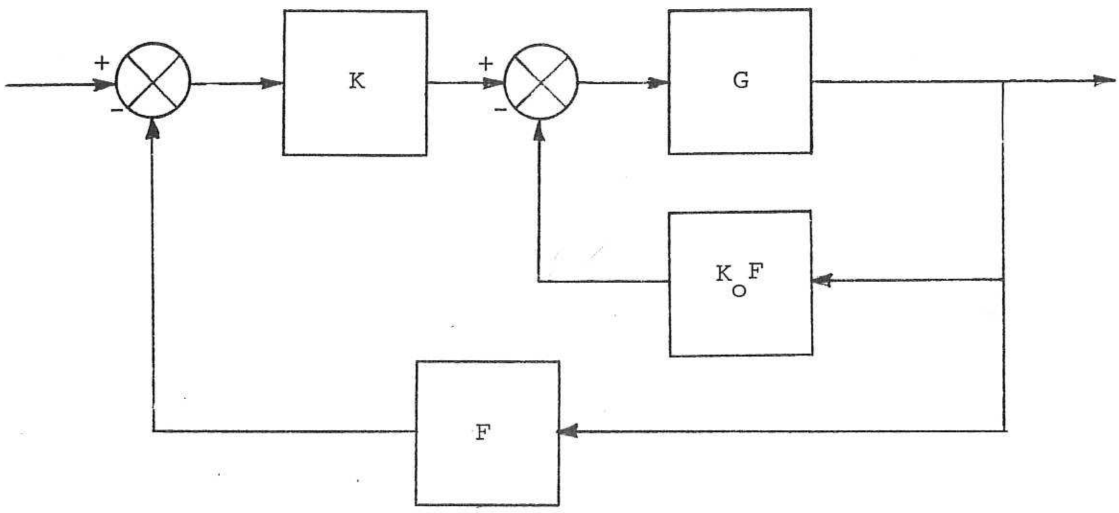
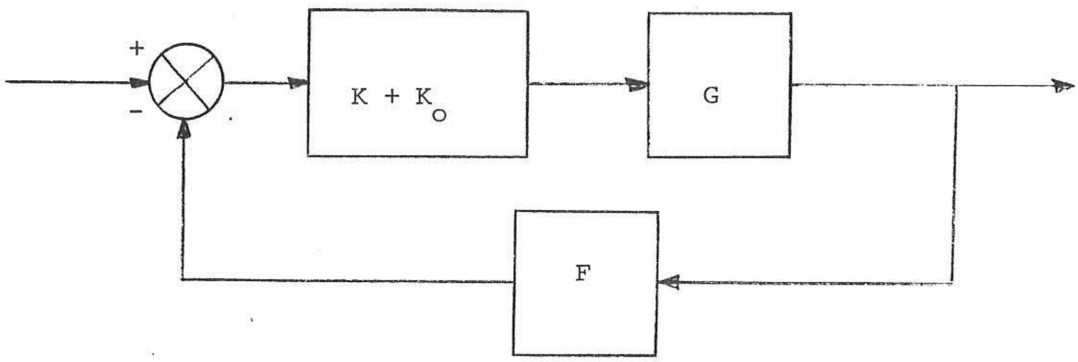


Fig. 6 Construction of model with monotone error
X = step response of G
O = step response of G_A
Δ = step response of $G - G_A$



(a)



(b)

Fig. 7. Minor loop feedback (a) and alternative (b) feedback configurations

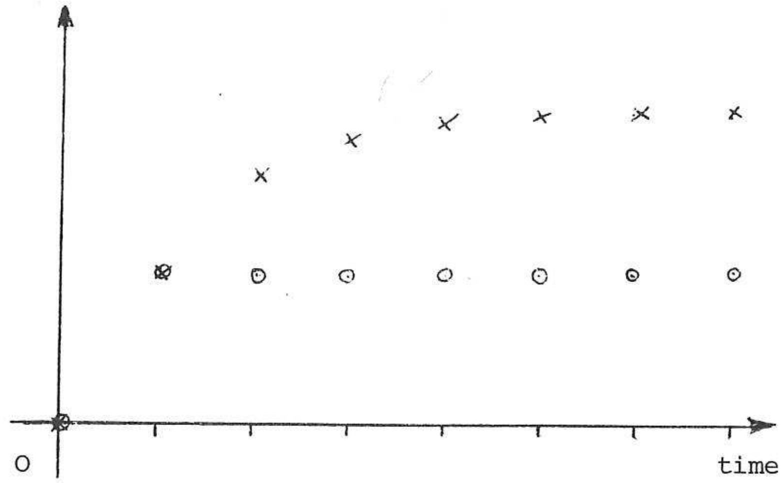
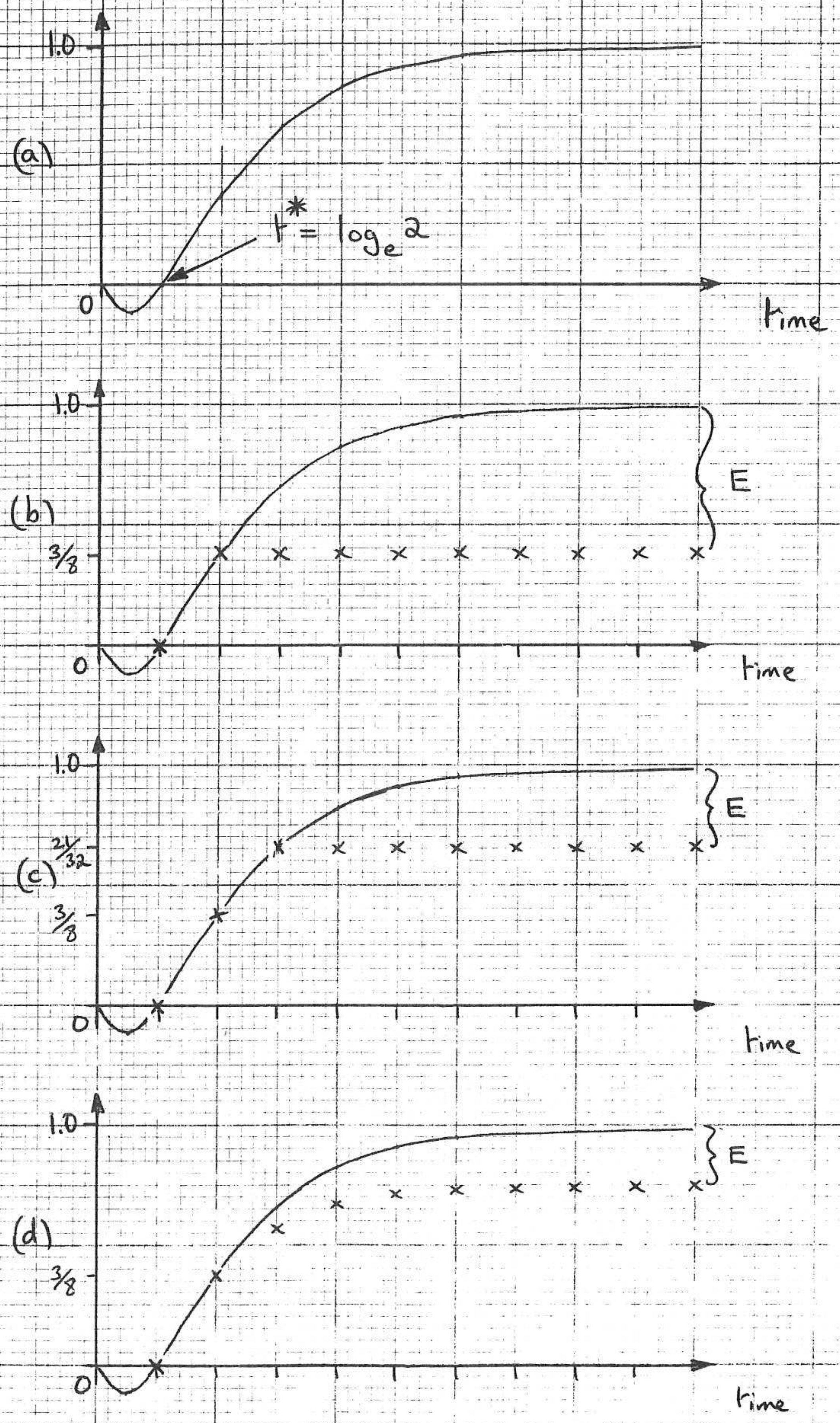


Fig. 8. Construction first-order deadbeat model

X = plant step response

O = deadbeat model response



1.0
(a)
t* = log_e 2
0
time
1.0
(b)
E
3/8
0
time
1.0
E
21/32
(c)
3/8
0
time
1.0
E
(d)
3/8
0
time

Fig. 9

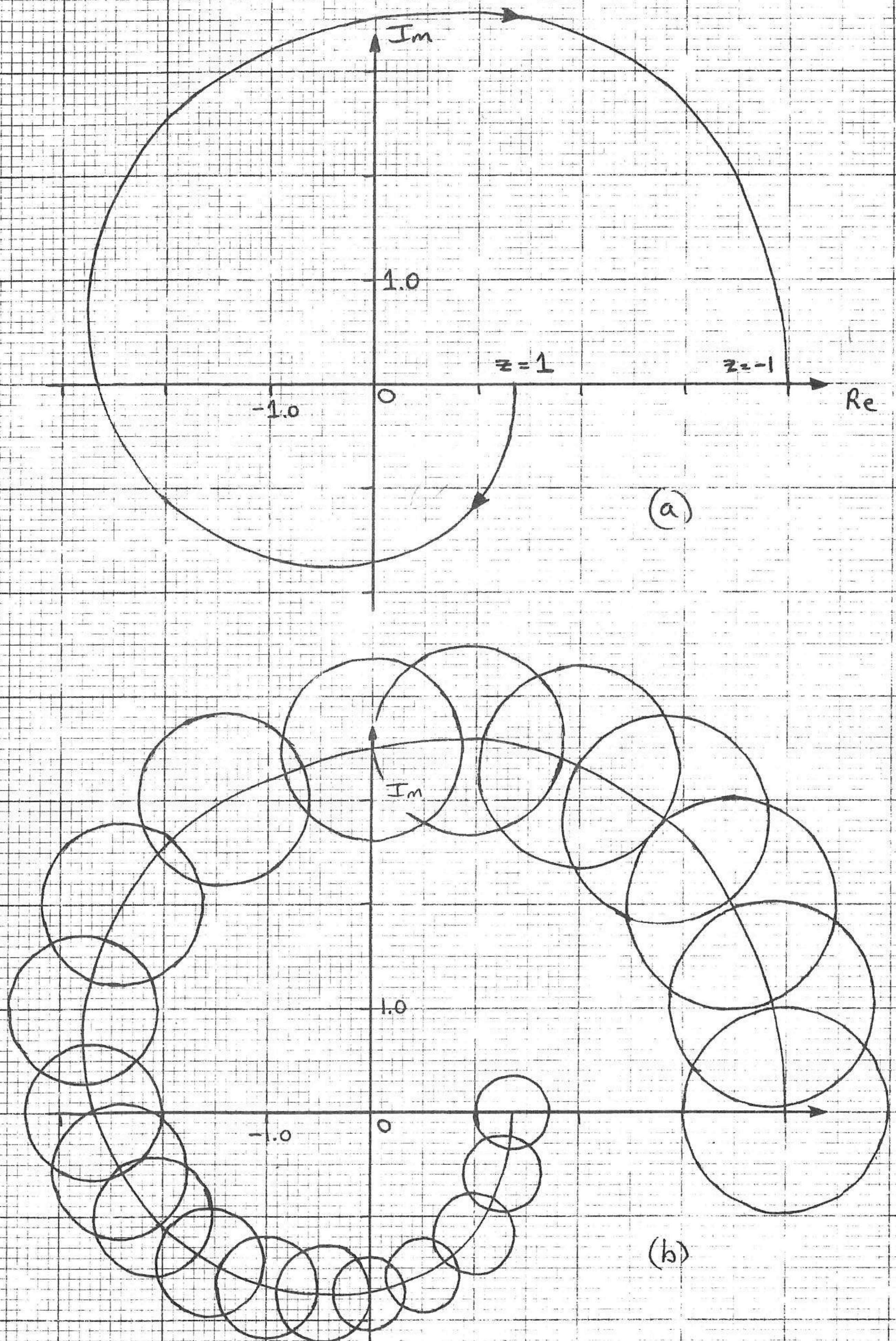


Fig 10.

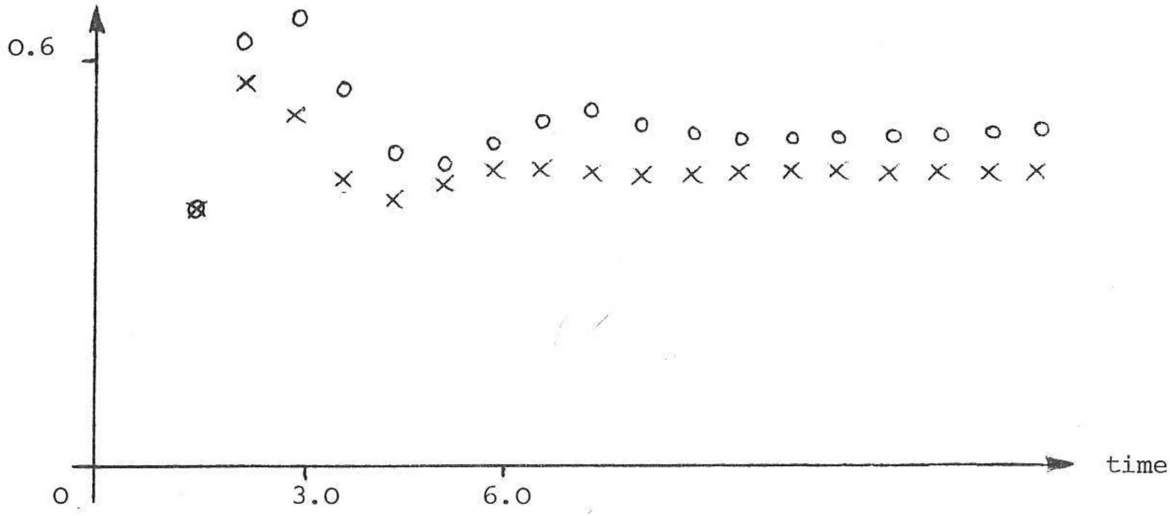
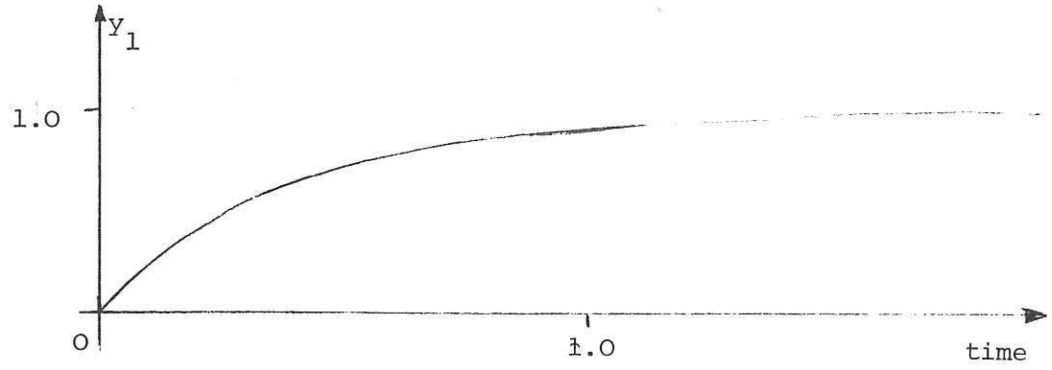


Fig. 11 Unit step responses of the real (O) and approximating (X) closed-loop systems



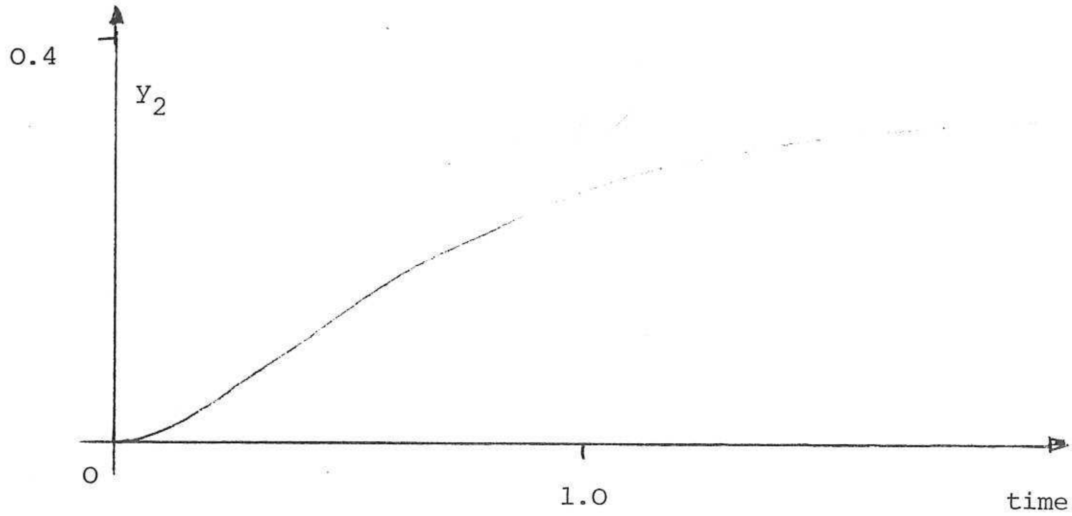


Fig. 12

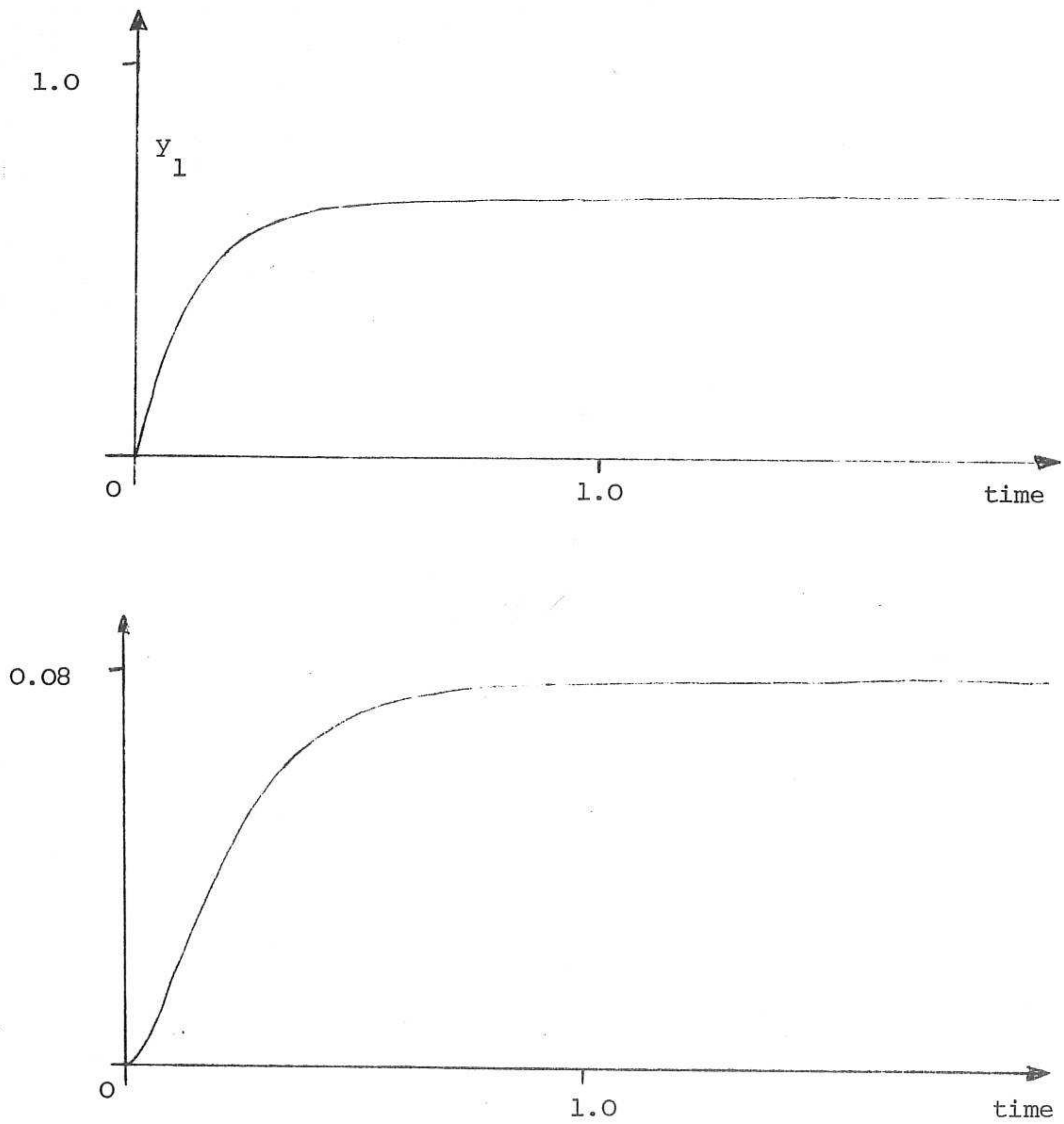


Fig. 13 Closed-loop responses to a unit step demand in y_1

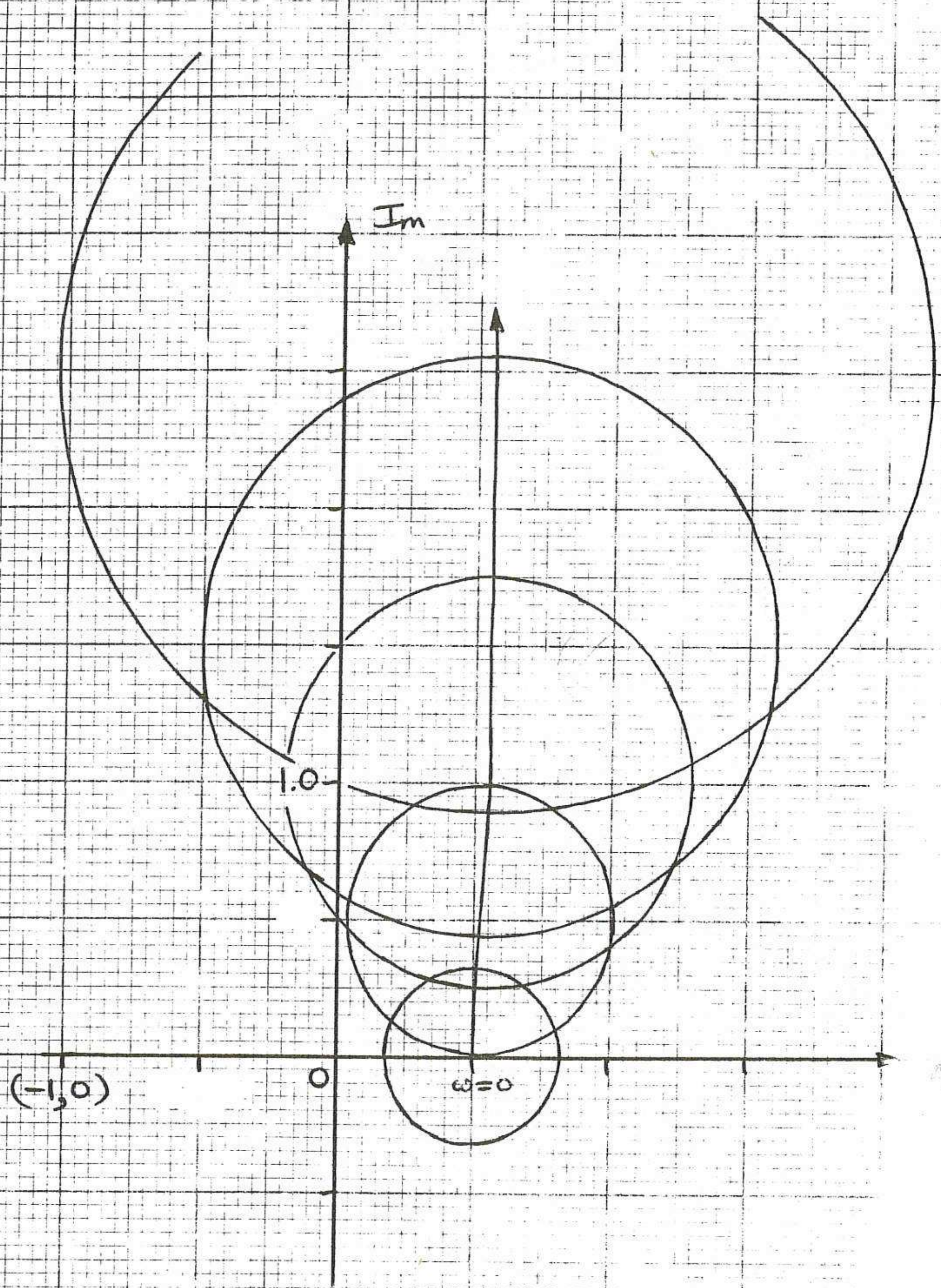


Fig 14