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Modelling discrete soil reinforcement in numerical limit analysis

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Soil reinforcement is widely used in geotechnical engineering. While there are various means of accounting for the presence of soil reinforcement in limit analysis and limit equilibrium type calculations, these are often highly problem specific. In this paper a general means of incorporating soil reinforcement within numerical limit analysis calculations is presented. A key feature of this implementation is that the reinforcement is modelled 'in parallel' with the soil model such that allows the soil to flow past the nails as might occur for soil nailing. To illustrate this the Discontinuity Layout Optimization (DLO) numerical analysis procedure is used, and the efficacy of the approach is evaluated via application to reinforced slope problems involving rigid soil nails under plane-strain conditions. The analyses are calibrated against a two-part wedge analysis method such as presented in British Standard BS8006:1995 or AASHTO LRFD Bridge design specifications. It is shown that the DLO-based procedure produces identical results only when the two-part wedge collapse mechanism is prescribed in advance (achieved by artificially strengthening the soil except along predefined failure planes). A more critical mechanism is otherwise predicted, with the soil strength at collapse required to be approximately 10% higher than predicted by the two-part wedge method (or alternatively soil nail lengths required to be approximately 20% greater).

KEYWORDS: Discontinuity layout optimization, limit analysis, soil nails, soil reinforcement, reinforced slope, verification.

1 Introduction

Soil reinforcement is a widely used technique in geotechnical engineering. To model the effect of soil reinforcement, various means by which traditional limit analysis or limit equilibrium analysis formulations can be modified have been presented in the literature. For example, a reinforcement element is often idealised using a pre-computed pull-out force acting to stabilise the construction (Duncan 1996), and 'method of slices' type calculations are often used to assess stability (Bishop 1955; Spencer 1967; Janbu 1973). In this case forces which approximate the available tensile and shear resistance provided by the reinforcement are applied to each slice, with the tensile resistance being computed from the length of reinforcement which lies beyond the failure mechanism. These forms of analysis can be carried out by hand, or can be automated in a spreadsheet or simple computer program.

For soil nail type reinforcement, Juran et al. (1990) developed this methodology further to model combined tensile and shear forces in the nail by incorporating nail deflections on both sides of the slip surface. In the analysis the maximum shear and tensile forces are calculated by means of a kinematic limit analysis approach, but the modified Bishop method (Bishop 1955) is still used to identify the failure mechanism. Zhu et al. (2005) also developed a method for distributing the normal stresses acting on nails crossing a slip surface, following the method proposed by Morgenstern and Price (1965). This allowed a continuous stress distribution to be modelled, leading to a more accurate approximation of the required strength and pull-out resistance of the soil nail. Sheahan and Ho (2003) presented a simplified approach designed to approximate the failure mechanisms observed experimentally in full-scale soil nailed wall tests, such as in Clouterre test wall no. 1 (Plumelle et al. 1990) and in the Amherst test wall (Sheahan 2000). This approach was similar to the two-wedge mechanism used to assess the stability of slopes, with various trial wedge angles considered to establish the critical mechanism.

In BS8006:1995 a two-part wedge method (Stocker et al. 1979) is recommended for reinforced soil slopes. The method is essentially the same as the simplified AASHTO (2002) tie back wedge analysis for geosynthetic reinforced soil walls (Allen et al. 2003; Bathurst et al. 2005). The factor of safety is defined by comparing the restoring and disturbing forces for a prescribed failure mechanism. Soil nails act to increase the restoring force and thus are able to increase the factor of safety of the slope. Two main categories of failure are analysed for a given problem, external and internal stability. External stability is concerned with the overall stability of the reinforced area of the slope against bearing and tilt failure, forward sliding, and slip failure around the reinforced area. Internal stability relates to the reinforcement itself, and its ability to resist tensile failure and slip failure relative to the surrounding soil. In reality problems will seldom fall purely into one of these categories, and a combined or 'compound' failure will often occur. Thus the two-part wedge analysis is concerned entirely with analysing the compound stability of a slope. (Although both two-part wedge and log spiral methods are described in BS8006:1995, for sake of brevity only the former will be considered in this paper.)

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Rather than modifying existing hand-type analysis methods to account for the presence of soil reinforcement, in this paper a means of modelling soil reinforcement within numerical limit analysis procedures is described, with the Discontinuity Layout Optimization (DLO) numerical analysis procedure proposed by Smith and Gilbert (2007) here used to provide illustrative solutions for a soil nail reinforced slope, and with results compared with those obtained using the two-part wedge method.

The specific aims of the paper are to:

1. Propose a general approach for modelling soil reinforcement using numerical limit analysis, ensuring this is consistent with current theoretical representations.
2. Describe a plane strain DLO formulation based on this general approach.
3. Verify the DLO formulation against a soil nail reinforced slope problem also analysed using British Standard BS8006:1995.
4. Undertake an outline parametric study using the DLO formulation, highlighting areas where current simple models may fail to capture certain aspects of behaviour.

The paper is concerned only with analysis at the ultimate limit state (ULS) and thus issues relating to reinforcement flexibility and strain compatibility will not be covered.

2 MODELLING REINFORCEMENT IN NUMERICAL LIMIT ANALYSIS PROCEDURES

2.1 Background

Considering traditional limit analysis or limit equilibrium methods, a number of observations can be made:

1. Typically analyses only consider a few specific collapse mechanisms.
2. Reinforcement is assumed to possess a finite pull-out strength (*i.e.* resistance to movement along its length relative to the soil).
3. Reinforcement is assumed to have the effect of applying a stabilising force in the direction of its line (or plane) of placement relative to a predefined sliding wedge or slip circle computed from the pull-out strength and the length of reinforcement beyond the slip mechanism.
4. The resistance to movement of the reinforcement normal to its plane of placement can potentially be modelled, but this is normally neglected in the simpler models (*e.g.* the two-part wedge method).
5. External and internal stability are considered separately.

In contrast, in a general approach it should be possible to:

1. Determine the critical failure mechanism, without relying on predefined mechanism geometries.
2. Model longitudinal (pull-out) and lateral resistance.
3. Model the effect of reinforcement on soil failure, regardless of where the reinforcement is located within the soil body.
4. Eliminate the distinction between external and internal stability (*i.e.* the critical stability state should be identified, whether this involves internal failure, external failure, or some combination of both).

The above can potentially be achieved when using a numerical limit analysis procedure.

2.2 Numerical limit analysis procedures

At the present time, there appear to be two main general purpose means of performing numerical limit analysis calculations, Finite Element Limit Analysis (FELA) (*e.g.* Lysmer 1970, Sloan 1988, Makrodimopoulos and Martin 2006) and Discontinuity Layout Optimization (DLO) (*e.g.* Smith and Gilbert 2007, Gilbert et al. 2010). The ULS can also be determined using conventional elasto-plastic FE analysis, by iterating to a collapse state. However this paper is concerned only with direct limit analysis approaches, such as FELA and DLO, in which the collapse state can be analysed directly using optimization techniques, often with relatively minimal computational effort.

Numerical limit analysis methods can be formulated from either a kinematic or static equilibrium viewpoint. In this paper a kinematic approach will be adopted. This involves prescribing a velocity field which covers the entire problem domain. Mathematical optimization (e.g. linear programming) can then be used to select the set of field variables which correspond to the minimum energy velocity field, and also the critical collapse mechanism.

In this context reinforced soil can be considered as a homogeneous medium with anisotropic properties, as for example proposed by de Buhan et al. (1989), and later implemented into a finite element limit analysis model by Yu and Sloan (1997). Alternatively, and as will be assumed here, individual reinforcing elements can be modelled explicitly.

2.3 Soil-reinforcement: pull-out resistance

Almost all methods of limit analysis or limit equilibrium described in the current literature treat reinforcement as possessing a fixed pull-out strength T (force/unit length/unit width) defined by the physical properties of the reinforcement and the surrounding soil. For soil reinforcement the pull-out strength, T can be estimated from the skin friction on the reinforcement, per unit length and width, as follows:

$$T = \alpha(c' + \sigma'_v \tan \phi'_{mob})a \tag{1}$$

where:

α = interaction coefficient
c' = drained cohesion intercept
σ'_v = vertical effective stress
ϕ'_{mob} = mobilised angle of friction between the soil reinforcement and soil mass

and where a is equal to 2 for sheet reinforcement to account for the upper and lower surfaces of the sheet. For essentially one-dimensional (1D) reinforcement such as soil nails, $a = \pi D n$ where D is the effective diameter of the soil nail, and n is the number of soil nails per unit width. Equation (1) may also be written in the following generic form:

$$T = \bar{T}_c + \bar{T}_q \sigma'_v \tag{2}$$

In most methods, the vertical effective stress σ'_v is for simplicity taken as a 'simple static' stress *i.e.* the effective weight per unit area of material immediately above a given point. In reality this stress will alter as the system approaches failure. Additionally the use of the vertical stress component in the calculations is questionable for inclined reinforcement, or when the reinforcement is beneath an inclined soil surface. However it is a pragmatic assumption and for sake of simplicity will be adopted in this paper.

2.4 Soil-reinforcement: lateral resistance

In this context lateral resistance is defined as the resistance to lateral displacement of the reinforcement through the soil. In many analyses this is ignored, and for sheet reinforcement it is normally assumed to be infinite (though see discussion in Section 2.5). For essentially one-dimensional reinforcement elements such as soil nails, lateral resistance can be considered in a similar way as for piles, giving the following general equation for lateral resistance N (force/per unit length/unit width):

$$N = nD(c'N_c + N_q \sigma'_v) \tag{3}$$

where N_c and N_q are analogous to the well known Terzaghi bearing capacity parameters. Equation (3) may also be written in the following general form:

$$N = \bar{N}_c + \bar{N}_q \sigma'_v \tag{4}$$

2.5 Modelling of reinforcement using a numerical limit analysis procedure

When using a general numerical analysis procedure it is not usually possible to model the effects of reinforcement based on *a priori* knowledge of the collapse mechanism. This is because reinforcement can modify the form of the critical mechanism. This implies that the reinforcement must be an inherent part of the model. In a plane strain analysis, two possible approaches present themselves:

1. Model reinforcement as a sheet material directly within the soil model. This may be represented as a thin two-dimensional sheet of a specific material type, or may be represented using a special one-dimensional sheet element.

2. Model reinforcement as a one-dimensional element in parallel with the soil model, where the reinforcement velocities are represented independently of the soil velocities but are linked to them through an appropriate energy dissipation equation.

In this paper it is suggested that the second approach is adopted, for the following reasons:

1. Considering for example a soil nail, it is not possible to realistically model this as as a 'sheet' material. This is because in reality a collapsing soil mass could flow around the soil nail. Thus a soil nail and a solid metallic sheet placed in the ground cannot be treated in the same way, since the latter will not allow soil to flow through it, requiring that soil deformation is either parallel to it, or that any perpendicular component of movement can only be accommodated by bending.

2. Paradoxically, flexible sheet reinforcement is often considered as a hypothetically 'rigid' reinforcement material, with no lateral resistance ($N = 0$), *e.g.* in the two-part wedge method. Thus specific modelling of bending is not required and only tensile pull-out is taken into account. This is justified because a flexible sheet has no resistance to bending, so will deform with the soil, and will dissipate negligible energy when perpendicular components of movement are involved. Parallel movements will however mobilise shear resistance and lead to dissipation of energy.

For sake of brevity, only rigid reinforcement elements (*e.g.* soil nails that are not expected to bend) will be considered in this paper; however incorporation of bending strength is relatively straightforward. Each reinforcement element can be considered free to displace independently of the deformation of the main soil mass, by assigning 3 independent velocity variables to the element: u, v, and ω, which are the x, and y direction velocities (of its centre) and rotational velocity respectively. These can be linked to the main analysis via an energy dissipation equation. Knowing the velocity of the reinforcement and of the surrounding soil in terms of the problem variables, it is possible to determine the relative velocities between soil and reinforcement at any point. Let these velocities be s_r parallel to the reinforcement and n_r perpendicular to the reinforcement. The rate of work (dW) done at any point over a short length dl is thus

$$dW = (T|s_r| + N|n_r|).dl \tag{5}$$

This can be integrated along the reinforcement length and included in the main limit analysis energy equation. Since the overall energy must be minimised, the optimization procedure will find the optimal combination of reinforcement and soil mass movements that minimise energy dissipation, taking numerous potential soil/reinforcement interaction modes into account. To facilitate the integration, it is most straightforward to model the reinforcement along the boundary of elements in FELA or a special discontinuity in DLO as will be discussed below.

3 Discontinuity Layout Optimization

3.1 Introduction

The Discontinuity Layout Optimization (DLO) numerical analysis procedure (Smith and Gilbert 2007) can be used to identify the critical translational failure mechanism and associated upper bound collapse load factor for any geotechnical stability problem, to a user specified numerical resolution. In the procedure the problem is discretised using nodes distributed across the problem domain, with each node then connected to every other node by a potential slip-line. Each potential slip-line i is assigned problem variables defining the relative normal (n_i) and slip (s_i) velocities across that line. Compatibility is maintained by constraining the vector sum of the relative velocities of potential slip-lines meeting at each node to zero in the x and y directions, using equations (6) and (7) respectively:

$$\sum_{i=1}^{m} (s_i\alpha_i - n_i\beta_i) = 0 \tag{6}$$

$$\sum_{i=1}^{m} (s_i\beta_i + n_i\alpha_i) = 0 \tag{7}$$

where α_i and β_i are direction cosines of slip-line i and where m is the number of potential slip-lines meeting at the given node.

Rate of energy dissipation can also be determined from these relative velocities. (However note that when using optimisation to find a solution it is necessary to introduce 'plastic multiplier' variables which are constrained to be non-negative. This is to avoid the rate of energy dissipation from becoming negative, as might be the case if the relative slip value s_i was used directly; refer to Smith and Gilbert (2007) for more details). By minimising the energy dissipation, while ensuring compatibility is enforced, the set of variables corresponding to the critical collapse mechanism can be determined. In general s_i and n_i will be found to be zero for most potential slip-lines in the problem, with the remaining potential slip-lines defining the geometry of the critical collapse mechanism.

3.2 Implementation of soil reinforcement in DLO

In order to incorporate soil reinforcement in a DLO limit analysis formulation, it is necessary to be able to calculate the relative movements between reinforcement and soil, along the entire length of the reinforcement. An efficient means of achieving this is to terminate all potential slip-lines within the soil mass at nodes positioned along the length of the reinforcement, *i.e.* preventing slip-lines from crossing the reinforcement without passing through a node. With this approach a relatively high nodal density should be employed to maintain good accuracy.

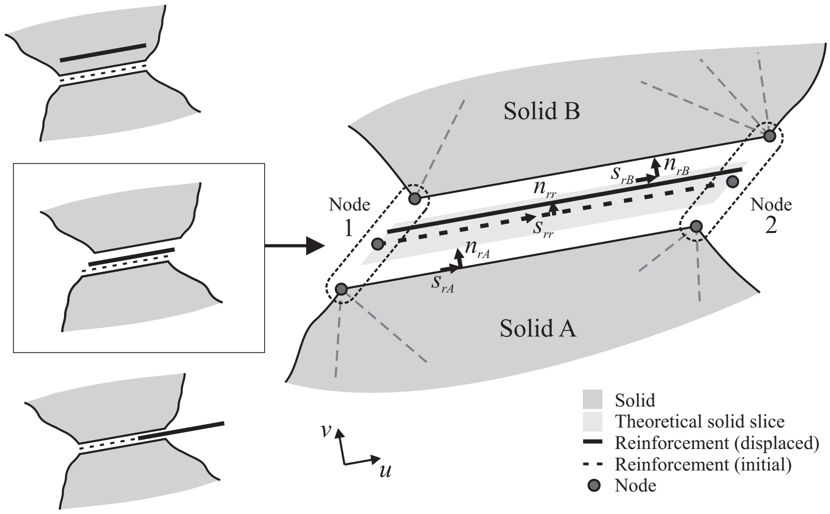


Figure 1: Modelling movement of section of reinforcement relative to surrounding soil using DLO.

Figure 1 shows potential displacement modes of a section of reinforcement moving relative to the surrounding soil and relative movements between soil and reinforcement in general terms. Movements at potential discontinuities lying above and below the reinforcement are shown. The relative movement of the reinforcement is calculated against a theoretical slice which represents the contact between the solids. Considering translational movement of the reinforcement (*e.g.* absolute reinforcement velocities u_r and v_r, aligned in local x and y directions respectively), appropriate new variables can be introduced to define the relative movement between the reinforcement and the theoretical soil slice:

$$s_{rr} = u_r - u_t \tag{8}$$

$$n_{rr} = v_r - v_t \tag{9}$$

where s_{rr} and n_{rr} represent relative slip and normal velocities of the reinforcement relative to the theoretical soil slice, which itself is moving with absolute velocity u_t and v_t in the local x and y directions respectively. (Note that although in DLO the slip-line velocities are usually relative, at the boundaries absolute values can readily be derived, making the calculations in (8) and (9) straightforward.)

Since the reinforcement is modelled in 'parallel' to the soil, compatibility of the soil is unaffected by the presence of the reinforcement. The additional work done within the system, due to the presence of a short segment of reinforcement, may be calculated using equation (5), now rewritten in terms of the new variables below:

$$dW = (T|s_{rr}| + N|n_{rr}|)\,.dl \tag{10}$$

Note that rather than directly using the absolute value terms of equation (10) when seeking a solution, these can be replaced with non-negative values to allow a standard linear programming solver to be used (*e.g.* providing that the total work is to be minimised then $|s_{rr}| \equiv s^+_{rr} + s^-_{rr}$, where $s_{rr} = s^+_{rr} - s^-_{rr}$, and where s^+_{rr}, $s^-_{rr} \geq 0$).

The relative movement between the reinforcement and the theoretical soil slice can also be calculated to incorporate other behaviour which can occur on the boundaries such as dilation:

$$s_{rA} = u_t - u_A \tag{11}$$

$$n_{rA} = v_t - v_A \tag{12}$$

where s_{rA} and n_{rA} represent relative slip and normal velocities of the theoretical soil slice to the edge of Solid A. Variables s_{rB} and n_{rB} can be defined in a similar way for Solid B.

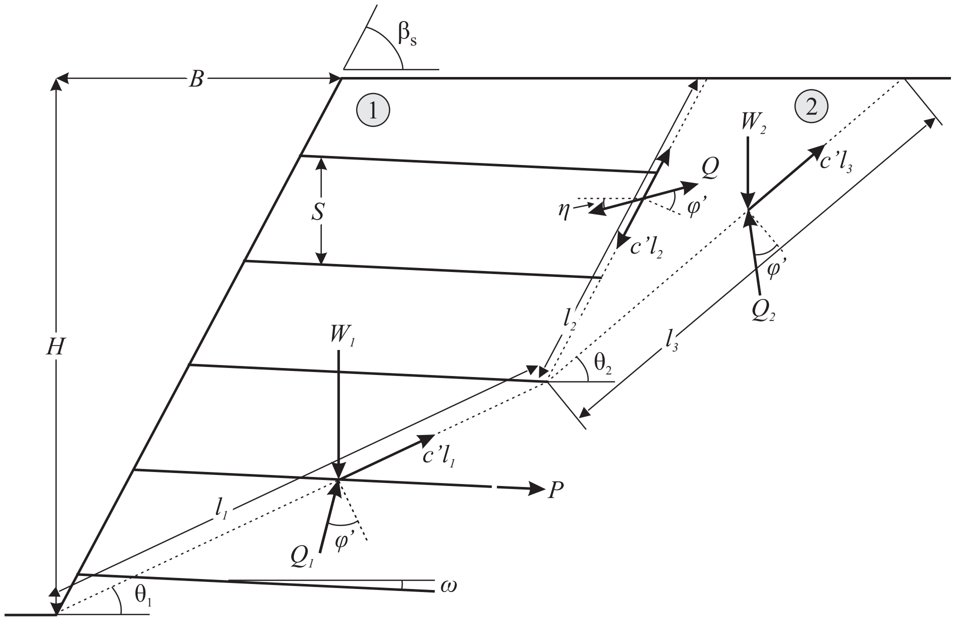


Figure 2: Two-part wedge analysis (including cohesion)

Also note that setting the pullout and lateral resistance of the reinforcement to zero will simply cause the reinforcement to be ignored, and will not insert a frictionless bound into the soil at the location of the reinforcement. Finally, although not considered here, it should be noted that modelling of rotation and/or yielding of the reinforcement can be be included simply by adding appropriate additional terms to equation (10).

3.3 Model validation

A simple reinforced vertical cut model was used to verify the model developed; full details are presented in Appendix A.

4 Analysis of a reinforced slope

4.1 Introduction

Consider the analysis of a simple reinforced slope, as shown in Figure 2. This will initially be analysed using the two-part wedge method described in BS8006:1995, with calculations undertaken using a spreadsheet. Two DLO models will then be developed, the first with the mechanism artificially restricted to comprise two soil wedges, the second with no such restriction. The analyses will be carried out in the context of soil nail reinforcement, though much of the theory is equally applicable to flexible reinforcement (*e.g.* geosynthetic sheets).

The analysis parameters are given in Table 1, and the geometry of the problem is shown in Figure 2. To facilitate objective comparison between the various models, partial factors were initially not applied. Also, soil nails with a length of 5 m were used in the initial analyses. The pullout force, P, was calculated based on the 'simple static' vertical effective stress experienced by each nail at its midpoint. The spacing of the nails has been increased beyond the guidelines given in the British Standard to allow a clearer picture of the failure mechanism to be obtained when using DLO. To account for this increased spacing, the diameter of the soil nails (taken to include the grout annulus) was also increased.

While a two-part wedge analysis permits modelling of a reinforced slope with $c' = 0$, this implicitly neglects possibility of localised failure between the reinforcement at the slope face. However a general unrestricted numerical analysis will identify such a collapse mode unless a suitable facing system is modelled. Consideration of this is however beyond the scope of this paper. Therefore to facilitate simple comparison between the two-part wedge model and an unrestricted DLO analysis, problems with small values of c' will additionally be considered, though use of a non-zero c' value in design is not generally a standard procedure.

Parameter	Value
Unit weight, γ	18 kN/m^3
Nail diameter, D	0.25 m
Interaction coefficient, α	1.0
Height, H	7.5 m
Slope batter, B	4 m
Nail spacing, S	1.5 m
Nail inclination, ω	0°
Horizontal nail spacing, n	1 m
T_c/c'	0.785 kN/m
$T_q/\tan\phi'$	0.785 m

Table 1: Reinforced slope analysis parameters

4.2 Two-part wedge model

In the two-part wedge analysis horizontal equilibrium of each wedge is considered, giving an inter-wedge force, Q, for each wedge which acts on the discontinuity between the wedges (equations (13) and (14)). Conventionally the two-part wedge model is used to calculate the required pull-out force in the soil nails, but, by equating equations (13) and (14), the balance of the inter-slice forces for any given set of soil conditions can be found. To calculate a factor of safety, the problem must be analysed at its point of failure. This can be achieved by varying either the strength or the unit weight of soil until the inter-wedge Q values are equal and at a minima, allowing the point of failure of the slope to be established. In this paper the angle of shearing resistance has been modified to bring about failure, allowing the ϕ' value required for stability to be established. This can be related to the actual value of ϕ' present in the soil to give a factor of safety on strength.

The British Standard presents equations for the case of $c' = 0$ only as follows:

$$Q_{\{\text{wedge 1}\}} = \frac{(W_1 + P\sin\omega)\tan(\phi' - \theta_1) + P\cos\omega}{\cos\eta - \sin\eta\tan(\phi' - \theta_1)} \tag{13}$$

$$Q_{\{\text{wedge 2}\}} = \frac{W_2\tan(\theta_2 - \phi')}{\cos\eta + \sin\eta\tan(\theta_2 - \phi')} \tag{14}$$

where the various parameters are as indicated in Figure 2.

Inclusion of a cohesion intercept term, c', furnishes the following more general equations:

$$Q_{\{\text{wedge 1}\}} = \frac{(W_1 + P\sin\omega - c'l_1\sin\theta_1 + c'l_2\sin\beta_s)\tan(\phi' - \theta_1) + P\cos\omega + c'l_1\cos\theta_1 - c'l_2\cos\beta_s}{\cos\eta - \sin\eta\tan(\phi' - \theta_1)} \tag{15}$$

$$Q_{\{\text{wedge 2}\}} = \frac{(W_2 - c'l_2\sin\beta_s - c'l_3\sin\theta_2)\tan(\theta_2 - \phi') - c'l_2\cos\beta_s - c'l_3\cos\theta_2}{\cos\eta + \sin\eta\tan(\theta_2 - \phi')} \tag{16}$$

Equations (15) and (16) were programmed into a spreadsheet to allow a variety of different wedge angles to be analysed. For each pair of wedge angles the required ϕ' value was calculated. The pair of wedge angles associated with the highest value of ϕ' in which the system is stable corresponds to the critical mechanism. Apart from the need to solve for both wedge angle and soil strength, the main difficulty with this analysis is that the pull-out resistance of each soil nail varies with the chosen wedge angle, due to the changed pull-out length. For a soil nail which intersects a slip line, the pull-out resistance of each end of the nail needs to be calculated based on the overburden lying above the mid-point of each individual section of nail either side of the intersection. This defines which end of the soil nail remains anchored (as the mobilised pull-out resistance is equal to the lower capacity of the two sections of the nail).

Figure 3 shows a contour plot of the required ϕ' for all combinations of θ_1 and θ_2 for a soil nail length of 5m. It can be seen that the solution space contains two local maxima. (Caution must be used with some solvers to ensure that the global maximum is found, in this case giving rise to the highest ϕ' required for stability.)

4.3 Simplified (restricted) DLO model

In order to permit direct comparison with the two-part wedge method a simplified DLO model was set up. This was achieved by defining the geometry of the wedges and then applying the properties of the soil to the interfaces between the wedges. The wedges themselves, as well as the surrounding soil, were modelled using a rigid material possessing the unit weight of the real soil. This ensured that slip-lines could only form along the wedge boundaries. Figure 4 shows an example of a forced two-part wedge mechanism implemented in the DLO procedure.

Figure 5 shows a comparison between the two-part wedge spreadsheet model prediction of required ϕ' for θ_1 angles of 25°, 30° and 35° and for all possible θ_2 angles and for $c' = 0$ kPa and $c' = 2.5$ kPa (note that in the case of the $c' = 2.5$ kPa simulations a corresponding increase in the value of T_c utilised for the soil nails was implemented). Selected DLO results are also plotted for comparison. It can be seen in the figures that the DLO results exactly match those generated by the two-part wedge method, except for the case where $\theta_1 > \theta_2$ where the difference between the two-part wedge method and the DLO analysis is 0.5% when $c' = 0$ kPa and 0.8% when $c' = 2.5$ kPa.

The increased difference is due to a reversal in the shear component of the inter-wedge force Q arising from the change in the kinematics of the mechanism in this case. This reversal is identified when using limit analysis but not in the standard two-part wedge limit equilibrium analysis. Modification of the two-part wedge limit equilibrium analysis to account for the reversal led, for example, to a required ϕ' of 25.24° when θ_1 was 35° and θ_2 of 30° with $c' = 0$ kPa. When c' was increased to 2.5 kPa the required ϕ' reduced to 21.65°. Both these angles are identical to the DLO results plotted in Figure 5.

As would be expected, the inclusion of a small degree of cohesion in the analysis leads to a decrease in the value of ϕ' required to ensure stability, in this example by 14% ($\theta_1 = 35°, \theta_2 = 40°$).

4.4 Standard (unrestricted) DLO analysis

A range of standard (*i.e.* unrestricted) DLO simulations were conducted with the goal of finding the ϕ' value required for stability, for a range of soil nail lengths, reinforcement properties and backfill weights. For each soil nail length the critical set of wedge angles was identified and the required ϕ' value recorded. The DLO models were set up and analysed using the LimitState:GEO 2.0 software (LimitState 2009) with a target number of 1000 nodes. The boundary nodal spacing was set to be half that within the solids, which is the default setting.

As stated earlier, to avoid a localised surface failure of the soil between reinforcement elements, a small c' value was used in all analyses. The smallest value of c' capable of inhibiting surface failure can be found by analysing a geometrically similar slope problem which extends between adjacent soil nails. In the current example the relevant problem comprises an un-reinforced slope of height 1.5m; in this case c' must be approx. 2.5 kPa to ensure stability. Figure 6(a) shows how this magnitude of cohesion affects the required ϕ' in a two-part wedge analysis. Hereafter the $c' = 2.5$ kPa results will be used to permit direct comparison against the DLO results. Please note that the authors are not proposing that cohesion be used in lieu of a more realistic facing, the inclusion of cohesion here is meant purely for validation purposes.

Figure 6(b) shows a plot of required ϕ' versus nail length for two-part wedge and unrestricted DLO analyses, now with $c' = 2.5$ kPa in both cases. In these analyses the lateral capacity of the nails was set to zero ($N = 0$). The two-part wedge method is shown to be unconservative when compared to the unrestricted DLO analysis, which indicates that a longer nail length (approx. 1.0-1.5 m) is required to ensure stability for the same problem parameters. Note that the unrestricted DLO analysis curve flattens out as ϕ' drops below 25° as local failure of the soil between the nails then occurs.

The failure mechanism for a standard (unrestricted) DLO analysis is shown in Figure 7(a), indicating that some soil is 'extruded' between adjacent reinforcement elements at lower levels. Nevertheless, the major slip surfaces extend through the nailed zone and then up to the surface behind the nailed zone, leading to a mechanism quite similar in form to a simple two-part wedge mechanism. However, one major advantage of using DLO is that it is easy to assess which assumptions and parameters have the greatest impact on the stability of the slope; this will be considered further in the next section.

4.5 Factors affecting the stability of the slope

4.5.1 Study 1: influence of lateral capacity

Results corresponding to the zero lateral capacity ($N = 0$) and infinite lateral capacity ($N = \infty$) analyses are plotted in Figure 6(b) and Figure 6(c) respectively. It is evident that a modest reduction in required length is predicted when an infinite lateral capacity is used, indicating that lateral capacity has relatively little influence on this type of analysis (assuming rigid nails).

4.5.2 Study 2: influence of assumed stress distribution

In the current DLO implementation the vertical stress used in the pull-out resistance calculation (Eq.(1)) is taken as an averaged value computed at the mid point of each nail. However the nail can be subdivided into

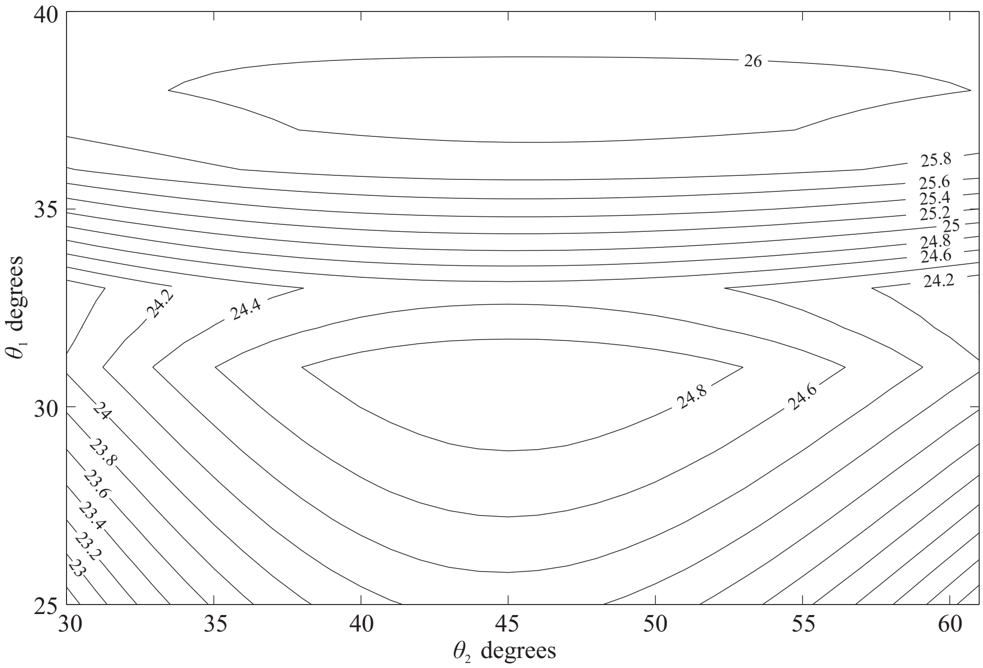


Figure 3: Contour plot of required ϕ' for soil nail length of 5m analysed with the two-part wedge method ($c' = 0$)

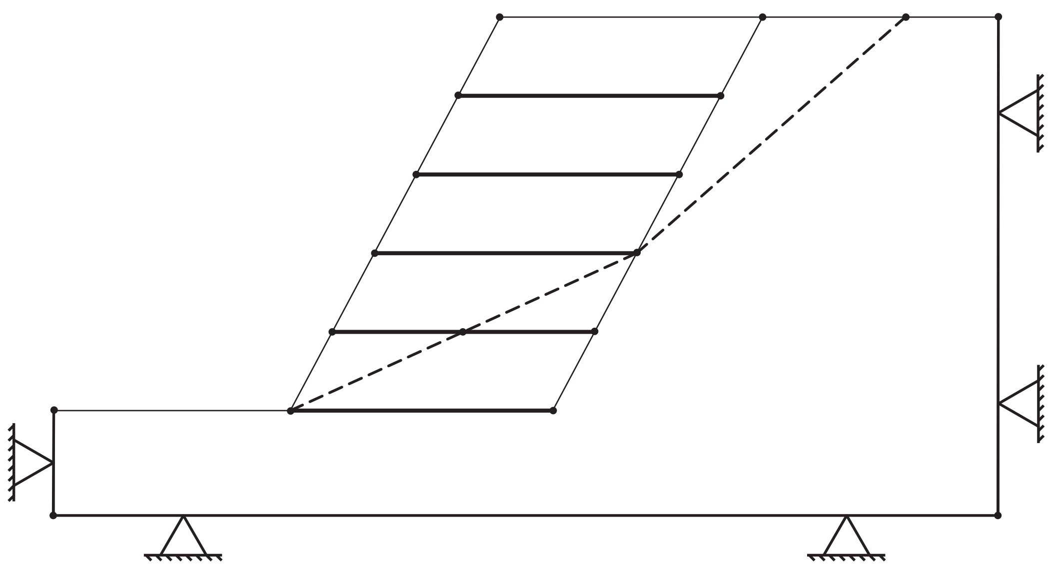


Figure 4: Forced twin wedge slip mechanism implemented in the DLO procedure

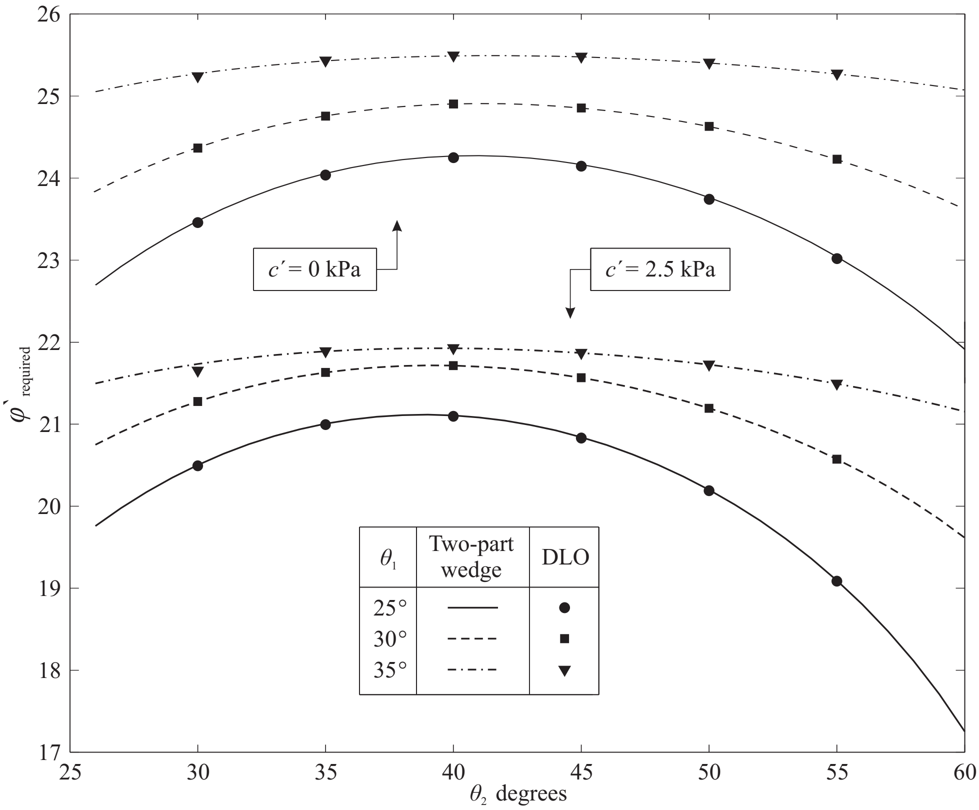


Figure 5: Comparison of required ϕ' for the two-part wedge method and the DLO method for varying wedge angles

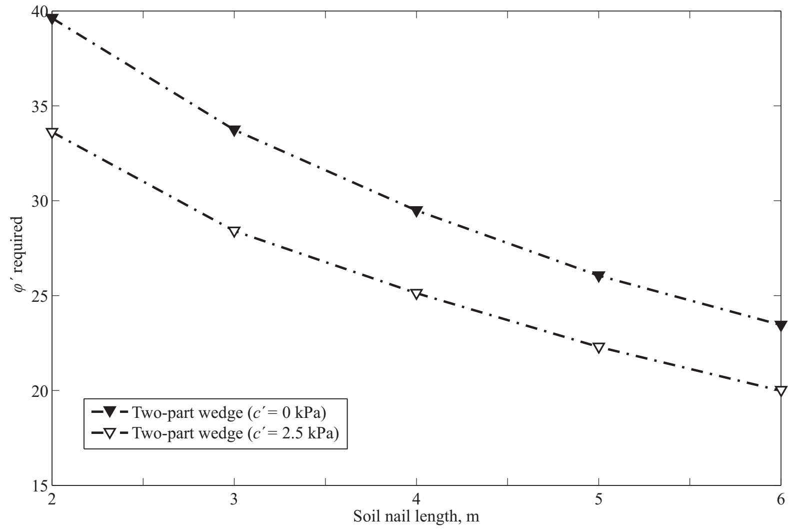
smaller lengths separated by vertices. Thus adding extra vertices along the nail length potentially gives a more representative pull-out resistance profile. To demonstrate this, analyses in which vertices were placed every 0.25 m along the nail lengths were undertaken, for both the lateral resistance cases considered previously (*i.e.* $N = 0$ and $N = \infty$). By adding extra vertices, the pull-out resistance close to the face of the slope is reduced, and when the lateral resistance is zero, this allows a localised full face failure as illustrated in Figure 7(b). This is reflected in Figure 6(b) where the plot for 'extra vertices' when $N = 0$ levels off at 32° as the stability of the face becomes the critical mechanism. For the corresponding $N = \infty$ case the full slope face is still locally stable and a larger global mechanism is mobilised (Figure 7(c)). This is reflected in Figure 6(c) by a much smaller increase in the required value of ϕ'. It should be noted that the majority of slip lines between the soil nails in Figure 7(c) are due to 'extrusion' of the material between the nails, which effectively act as rigid plates when the lateral capacity is infinite.

4.5.3 Study 3: influence of presence of soil nail at base of slope

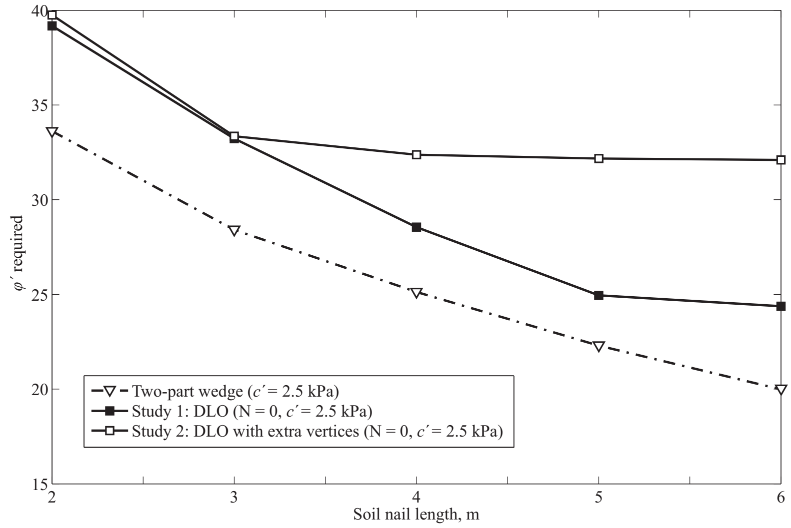
In the two-part wedge method the lowest soil nail is not incorporated directly into the analysis. Its contribution to stability was thus investigated using the standard (unconstrained) DLO analysis by simply removing the lowermost nail. By comparing Figure 7(a) and Figure 7(d) it is evident that the lowest nail acts to inhibit a partial 'bearing' failure, which otherwise extends the failure mechanism beyond the toe of the slope, and which leads to a slightly higher value of ϕ' being required to ensure stability. In this example $\phi' = 25.85°$ is required if the nail is not present, and $\phi' = 24.95°$ if it is included.

4.5.4 Study 4: influence of partial factors on adequacy

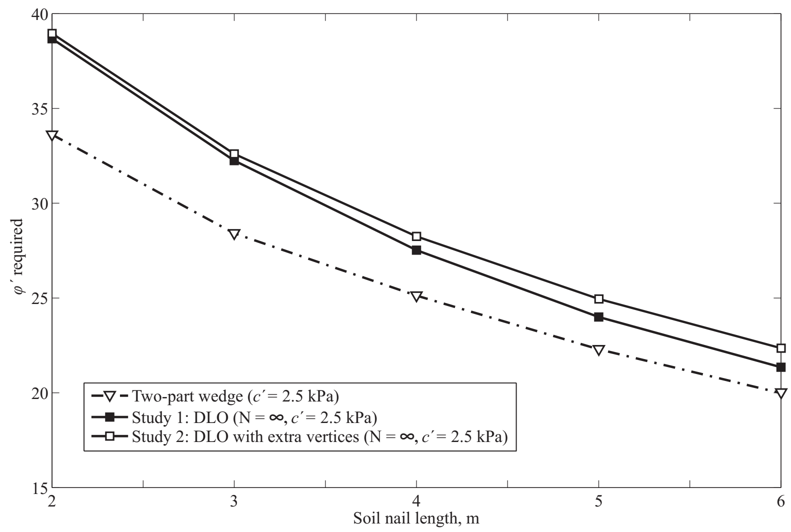
Partial factors of 1.5 on weight and 1.3 on soil nail pull-out force (P) were applied to the problem to assess the new required soil strength for a given soil nail length (in line with the British Standard). The factor on weight was applied to the soil weight acting as a disturbing action. It was not applied to the vertical stress used to compute the nail pullout resistance, in accordance with BS8006. Figure 8 shows the required soil strength for differing soil nail lengths, for a value of $c' = 3.5$ kPa (an increased value was utilised here to avoid face failure when using the factored quantities).



(a) Effect of introducing small c' value on two-part wedge analysis

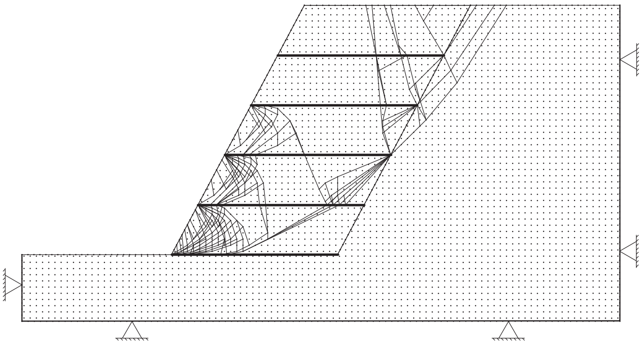


(b) Zero lateral resistance ($N = 0$) DLO analysis compared to two-part wedge analysis

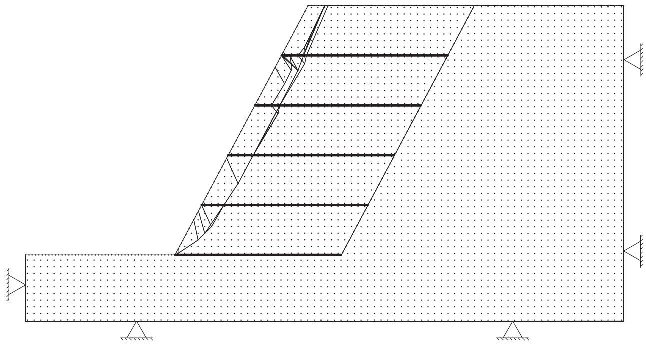


(c) Infinite lateral resistance ($N = \infty$) DLO analysis compared to two-part wedge analysis

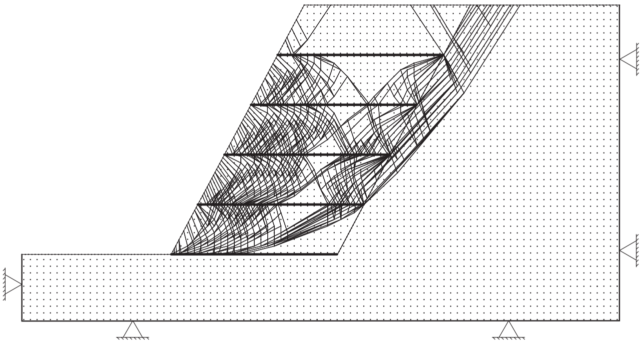
Figure 6: Required strength for given soil nail lengths for problem parameters given in Table 1



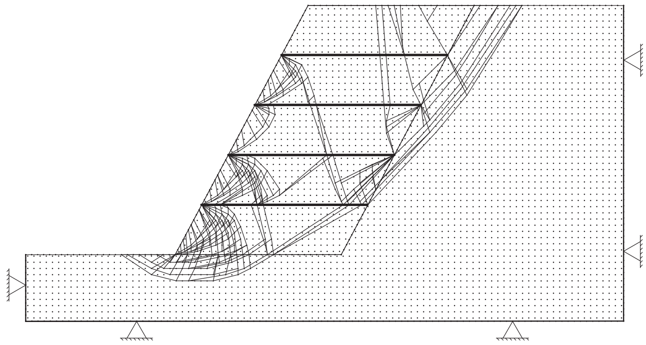
(a) Benchmark DLO analysis, $\phi'_{\text{required}} = 24.95°$



(b) Study 2: Extra vertices added along reinforcement, $N = 0$, $\phi'_{\text{required}} = 32.17°$



(c) Study 2: Extra vertices added along reinforcement, $N = \infty$, $\phi'_{\text{required}} = 24.95°$



(d) Study 3: Reinforcement omitted at base of slope, $\phi'_{\text{required}} = 25.85°$

Figure 7: Factors affecting slope stability (unrestricted DLO analyses, 5 m nail, $c' = 2.5$ kPa, other parameters as defined in Table 1)

Both the factored two wedge and unrestricted DLO ($N = 0$) analysis results suggest a margin of safety over that indicated by the unfactored two wedge analysis results. For example for c'=3.5kPa, $\phi' = 30°$, the unfactored two wedge and DLO nail lengths are ∼2.2m and ∼3.3m, while the factored two wedge and DLO lengths are ∼3.8m and ∼4.3m respectively.

4.6 Discussion

The studies of constrained mechanisms in this paper indicate that the proposed soil reinforcement limit analysis approach gives very similar results to the two-part wedge limit equilibrium approach, providing confidence in the new formulation. However, the more sophisticated unconstrained DLO analysis has highlighted additional factors that need to be considered that are either currently neglected or which are implicitly included in simpler analyses. While one of the advantages of DLO is that internal and external stability do not need to be checked separately, this does mean that a local face instability failure may be identified when c' is small (this is obviously not the case when using the two-wedge analysis method).

The need for explicit modelling of facing was avoided in this study by examining problems which possess a small c' throughout the soil body. For problems where c' is taken as zero, explicit facing would need to be modelled. Investigation of a suitable framework for modelling facing is is a significant study in itself and is beyond the scope of this paper, however one possible option is to model a thin surface layer of soil with non-zero c'.

For the example problem considered, the two-part wedge method was shown to provide non-conservative results. When BS8006 partial factors were introduced the two-wedge method was found to give a margin of safety comparable to that obtained using the unconstrained DLO model **without** partial factors. Nevertheless, an additional margin of safety can be expected to arise from ignoring both the contribution of facing and the lateral capacity of the soil nails.

Finally, in the current study tensile and bending failure of the nails has not been considered.

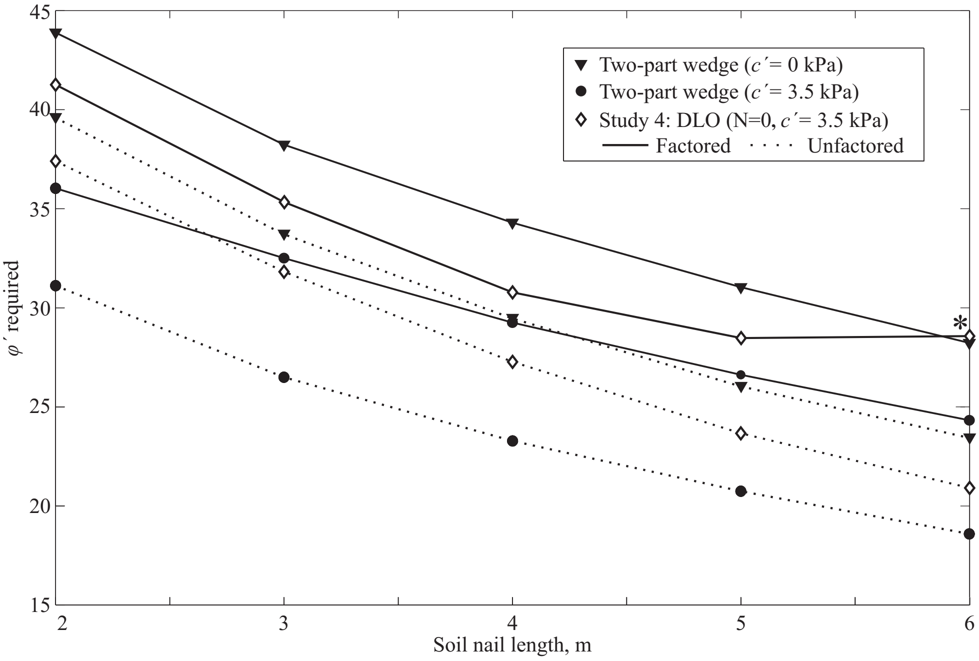


Figure 8: Study 4: Required strength for given soil nail lengths with factored parameters for two-part wedge analysis, for $c' = 3.5$ kPa. Note that the curve for the factored DLO analysis with $N = 0$ (*) flattens out beyond a length of 5m due to local face failure.

5 Conclusions

(a) A general means of incorporating soil reinforcement within numerical limit analysis calculations has been presented, and illustrated by implementing soil nail reinforcement within the Discontinuity Layout Optimisation (DLO) limit analysis procedure. This has been validated via application to simple benchmark element tests.

(b) A constrained DLO analysis has been shown to give identical results to those from a two-part wedge analysis of a soil nail reinforced slope, further validating the proposed approach. The two-part wedge analysis method was also extended to model soils possessing cohesion intercept c'.

(c) It has been found that the required nail lengths calculated using the two-part wedge method are significantly lower (by approx. 20%) than those calculated using a standard (unconstrained) DLO analysis. This indicates that the two-part wedge method produces nonconservative results. In practice this nonconservativism may be counteracted by a number of factors, such as:

- strengthening of the reinforced slope due to face reinforcement
- cohesion of the soil
- lateral capacity of the soil nails
- location of the lowest soil nail
- partial factors employed (which may be specific to this analysis approach)

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A APPENDIX A - Verification of model

A.1 General approach

The reinforcement element was verified by modelling a simple constrained collapse mechanism using DLO. In this case the failure mechanism was forced to follow a predetermined form to enable comparison with a simple hand analysis. In the following sections hand analysis of a simple collapse mechanism is followed by comparison with the DLO model.

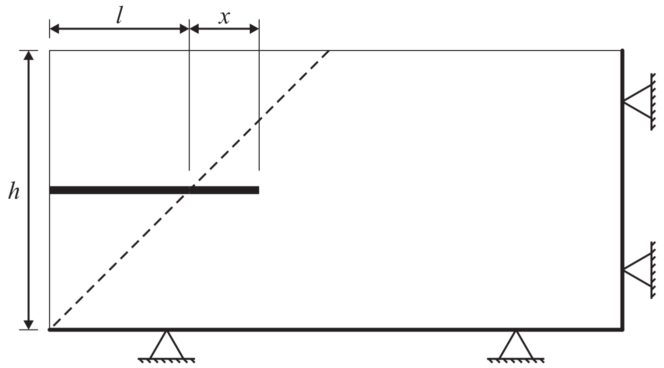

Figure 9: Soil nail reinforced vertical cut (undrained case)

A.2 Undrained case

Referring to Figure 9, suppose the wedge displaces to the left by a distance δ. Take the unit weight of the soil as γ and the undrained shear strength of the soil as c_u. Let T_s be the pull-out resistance per unit length in the soil, and N_s the lateral resistance per unit length in the soil. Assume that the failure mechanism consists of a simple 45^o sliding wedge as depicted *i.e.* that the values of T_s and N_s are small enough not to influence the failure mode.

The work equation must be determined in two cases:

1. The nail remains in the wedge:

$$\frac{1}{2}\gamma h^2\delta = c_u\sqrt{2}h\sqrt{2}\delta + xT_s\delta + xN_s\delta \tag{17}$$

2. The nail remains in the main soil body and the wedge moves past it:

$$\frac{1}{2}\gamma h^2\delta = c_u\sqrt{2}h\sqrt{2}\delta + lT_s\delta + lN_s\delta \tag{18}$$

Clearly the mode changes at the stage when $x = l$. Hence the unit weight γ_f of the soil required for failure is given by:

$$\gamma_f = 4\frac{c_u}{h} + \frac{2\min(l,x)}{h^2}(T_s + N_s) \tag{19}$$

A selection of sample solutions are presented in Table 2, for $c_u = 1.0$ and $h = 1.0$, comparing the theoretical value of γ_f to the DLO value γ_{fDLO}, indicating that the model is working correctly for the undrained case.

T_s	N_s	l	x	γ_f	γ_{fDLO}/γ_f
0.0	0.0	0.50	0.25	4.000	1.0
0.1	0.0	0.50	0.25	4.050	1.0
0.1	1.0	0.50	0.25	4.550	1.0
1.0	1.0	0.50	0.25	5.000	1.0
0.1	0.0	0.50	0.75	4.100	1.0
0.1	1.0	0.50	0.75	5.100	1.0
1.0	1.0	0.50	0.75	6.000	1.0

Table 2: Solutions to equation (19)

A.3 Drained case

For cohesionless soil in drained conditions, it is necessary to add a facing to the soil block to prevent local collapse. This is straightforwardly achieved using a rigid block as shown in Figure 10.

As before let T_s be the pull-out resistance per unit length in the soil, and N_s the lateral resistance per unit length in the soil. Let the rigid block be weightless and have smooth interfaces with the soil and base. Assume that the failure mechanism consists of a simple sliding wedge oriented at $45^o - \phi/2$ to the vertical as depicted *i.e.* that the values of T_s and N_s are small enough not to influence the failure mode. Let the wedge displace to

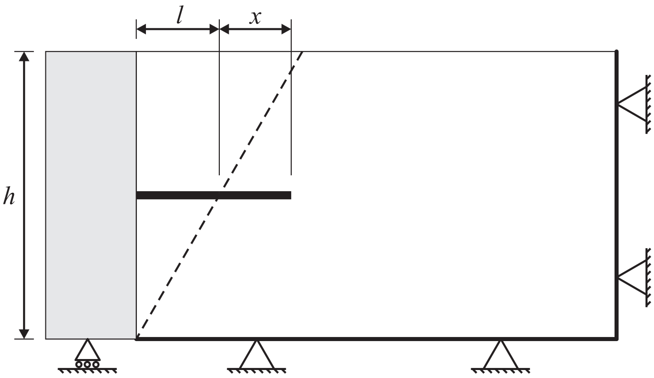

Figure 10: Soil nail reinforced vertical cut (drained case)

the left by a distance δ. Take the unit weight of the soil as γ and the drained shear angle of shearing resistance of the soil as ϕ. The work equation (assuming associative flow) must be determined in two cases:

1. The nail remains in the wedge:

$$\frac{1}{2}\gamma h^2 \delta \tan^2(45 - \phi/2) = xT_s\delta + xN_s\delta \tan(45 - \phi/2) \tag{20}$$

2. The nail remains in the main soil body and the wedge moves past it:

$$\frac{1}{2}\gamma h^2 \delta \tan^2(45 - \phi/2) = lT_s\delta + lN_s\delta \tan(45 - \phi/2) \tag{21}$$

Clearly the mode changes at the stage when $x = l$. Hence the unit weight γ_f of the soil required for failure is given by:

$$\gamma_f = \frac{2\min(l, x)}{h^2 \tan^2(45 - \phi/2)} \left(T_s + N_s \tan(45 - \phi/2)\right) \tag{22}$$

A selection of sample solutions are presented in Table 3, for $\phi = 30^o$ and $h = 1.0$, indicating that the model is working correctly for the drained case.

T_s	N_s	l	x	γ_f	$\gamma_f DLO/\gamma_f$
0.0	0.0	$\sqrt{3}/6$	0.25	0.000	1.0
0.1	0.0	$\sqrt{3}/6$	0.25	0.150	1.0
0.1	0.1	$\sqrt{3}/6$	0.25	0.237	1.0
1.0	1.0	$\sqrt{3}/6$	0.25	2.366	1.0
0.1	0.0	$\sqrt{3}/6$	0.50	0.173	1.0
0.1	0.1	$\sqrt{3}/6$	0.50	0.273	1.0
1.0	1.0	$\sqrt{3}/6$	0.50	2.732	1.0

Table 3: Solutions to equation (22)