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A NOTE ON NONLINEAR OBSERVERS

by

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ABSTRACT

A general nonlinear exponential observer is presented for nonlinearly perturbed nonlinear systems. This is done by using the nonlinear variation of constants formula.

1. Introduction

In linear feedback control theory, it is important to be able to estimate the states of a system. Luenberger (1964, 1966) gave a general method for designing observers for linear systems which converge exponentially to the true states. Kou et al (1975) gave a generalization of these results to certain types of nonlinear systems. However, their results are based essentially on the stabilization of semilinear equations by the observation, using Lyapunov type results.

In this paper we shall use the nonlinear variation of constants formula to derive exponential observers for nonlinear perturbations of nonlinear systems, which generalize certain of the results of Kou (1975). The basic theory is presented in section 2, where we use the matrix function

$$\lambda(A) = \sup_{\lambda \varepsilon \mathrm{spec}\{(A+A^T)/2\}} \lambda \quad ,$$

which we assume can be made strictly negative by an appropriate combination of the observations. Finally in section 3 we shall present two simple examples.

2. General Theory

We shall consider a system governed by the equation

$$\dot{x}(t) = f(x,t) \quad , \quad x(t_o) = x_o \tag{2.1}$$

where f satisfies sufficient conditions so that (2.1) has a unique global solution for each x_o in some open set $\Omega \subseteq R^n$. Thus, for example, we could assume f to be measurable in t and Lipschitz Continuous in x for all t and $x \in \Omega$.

Suppose that equation (2.1) describes a physical system, the state x of which is not observable directly, but only through the nonlinear observation

$$y(t) = B(x(t)) \qquad (2.2)$$

Our object is to find a function $\beta : R^n \times R^n \times R^+ \to R^n$ such that the system

$$\dot{\hat{x}}(t) = f(\hat{x},t) + \beta(y, B\hat{x}(t),t) \quad , \quad \hat{x}(t_o) = \hat{x}_o \qquad (2.3)$$

(which again we assume to have unique global solutions, for $\hat{x}_o \in \Omega$) has solutions which converge as $t \to \infty$ to appropriate solutions of the unobservable system (2.1).

<u>Definition 2.1</u> The system (2.3) is an <u>exponential observer for system (2.1)</u> of type K,ε if, for x_o , $\hat{x}_o \in \Omega$, we have

$||x(t) - \hat{x}(t)|| \leq K(||x_o - \hat{x}_o||)e^{-\varepsilon t}$ where $\hat{x}(t)$, $x(t)$ are the solutions of (2.1) (2.3), respectively, with initial conditions x_o, $\hat{x}_o$.

In order to study the solutions of equation (2.3) we shall use the nonlinear variation of constants formula, due to Alekseev (1961); see also Brauer (1966). Let π denote the set $I \times I \times \Omega$, where $I = [t_o,\infty)$, and denote the solution of (2.1) with initial condition x_o at $t = t_o$ by $x(t;t_o,x_o)$. Then $x(t;t_o,x_o)$ is differentiable on π and the matrix

$$\Phi(t;t_o,x_o) = \frac{\partial}{\partial x_o}\left[x(t;t_o,x_o)\right]$$

is the fundamental solution of the equation

$$\dot{z}(t) = f_x\left[x(t;t_o,x_o),t\right]z. \qquad (2.4)$$

Equation (2.4) is known as the variational system of equation (2.1). Then, we have (Brauer, 1966):

<u>Lemma 2.2</u> Let $\hat{x}_o \in \Omega$. Then, for all $t \geq t_o$ such that $x(t;t_o,\hat{x}_o) \in \Omega$, $\hat{x}(t;t_o,\hat{x}_o) \in \Omega$, the solutions of (2.1) and (2.3) (with the same initial condition $\hat{x}_o$) are related by

$$\hat{x}(t;t_o,\hat{x}_o) - x(t;t_o,\hat{x}_o)$$
$$= \int_{t_o}^{t} \Phi(t;s,\hat{x}(s;t_o,\hat{x}_o))\beta(y,B\hat{x}(s;t_o,\hat{x}_o),s)ds. \quad \square$$

Now let $x(t)$ be a fixed solution of equation (2.1) with initial condition $x(o) = x_o$. In order to study the asymptotic behaviour of the difference $|x(t) - \hat{x}(t)|$ we subtract equation (2.1) form (2.3). Then,

$$\dot{\hat{x}}(t) - \dot{x}(t) = f(\hat{x},t) - f(x,t) + \beta(Bx,B\hat{x},t). \tag{2.5}$$

We shall first prove the following simple result in the case where B is a linear map from R^n to R^m.

<u>Lemma 2.3</u> Let $f(x,t) = f_1(x,t) + A(t)x$ where $A(\cdot):R^+ \to \mathcal{L}(R^n,R^n)$ and f_1 is dissipative for all $t \geq o$, i.e.

$$< f_1(x,t) - f_1(y,t), x - y > \leq 0, \ t \geq 0, \ \forall x, y \in R^n$$

Then, if $\exists$ a matrix function $P(\cdot):R^+ \to \mathcal{L}(R^m,R^n)$ such that

$$\lambda(A(t) + P(t)B) \leq -\varepsilon < 0 \quad (\varepsilon > 0)$$

where $$\lambda(Q) = \sup_x \frac{<Qx,x>}{||x||^2},$$

the system (2.1) is exponentially observable.

<u>Proof</u>. From (2.5) with $\beta(Bx,B\hat{x},t) = P(t)\cdot(B(-x+\hat{x}))$, we have

$$\dot{\overline{\hat{x}-x}} = f_1(\hat{x},t) - f_1(x,t) + (A(t) + P(t)B)(\hat{x}-x)$$

i.e.

$$\tfrac{1}{2}\dot{\overline{||\hat{x}-x||^2}} = <f_1(\hat{x},t) - f_1(\hat{x},t), \hat{x}-x >$$

$$+ < (A(t) + P(t)B)(-x+\hat{x}), \hat{x}-x >$$

$$\leq \lambda(A(t) + P(t)B) \ ||\hat{x}-x||^2$$

$$\leq -\varepsilon||\hat{x}-x||^2$$

and so

$$||\hat{x}-x|| < e^{-\varepsilon t}||\hat{x}(o) - x(o)|| . \ \square$$

Note that this lemma gives a very simple proof of the exponential observability of the system

$$\dot{x}_1 = x_1, \ \dot{x}_2 = x_1 - 2x_2 + e^{-x_2}, \ y = x_1 + x_2,$$

considered by Kou, et al (1975). For, it is easy to check that $P = (-2,1)^T$ has the required property and that $f_1(x) = (o,e^{-x_2})^T$ is dissipative, since

$$(e^{-x_2} - e^{-\bar{x}_2})(x_2 - \bar{x}_2) \leq 0.$$

We now define, for any nonlinear map $N: R^n \to R^n$ the quantity

$$\mu_N = \sup_{\substack{x_1, x_2 \in R^n \\ x_1 \neq x_2}} < Nx_1 - Nx_2, x_1 - x_2 > ||x_1 - x_2||^{-2} .$$

In a sense, this represents the 'maximum slope' of N. A simple modification of lemma 2.3 proves the following result:

Lemma 2.4 Let $f, f_1, A, B, P, \varepsilon$ be as in lemma 2.3. Then if

$\mu_{f_1} \leq \delta < \varepsilon$ (for each fixed $t \geq t_o$) the system (2.1) is exponentially observable. □

In order to use the nonlinear variation of constants formula, we need to place a bound on the function Φ. Using the notation of equation (2.4), Brauer (1966) shows that if

$\lambda(f_x(t,x)) \leq \alpha(t)$ for all $t \geq t_o$ and all $x \in \Omega$, then

$$|\Phi(t, t_o, x_o)| < \exp\left(\int_{t_o}^{t} \alpha(u)\,du\right) \tag{2.6}$$

for all $t \geq t_o$, and all $x_o \in \Omega$.

Now let $f: R^n \to R^n$ be any analytic map, and let

$$\sum_{n=o}^{\infty} \frac{D^n f(x)}{n!} (\overbrace{x, \ldots, x}^{n \text{ times}})$$

be its Taylor series. Then we may write

$$f(\hat{x}) - f(x) = g(\hat{x}-x, x)(\hat{x}-x)$$

where $g: R^n \times R^n \to \mathcal{L}(R^n, R^n)$ and so $f(\hat{x}) - f(x)$ may be written as a function $\bar{f}(\bar{x}, x) \triangleq f(\hat{x}) - f(x)$, where $\bar{x} = \hat{x} - x$. To find an expression for g, write

$$f(\hat{x}) - f(x) = \sum_{n=1}^{\infty} \frac{D^n f(x) \bar{x}^{(n)}}{n!}$$

where

$$\bar{x}^{(n)} = (\underbrace{\bar{x}, \ldots, \bar{x}}_{n}) .$$

Hence,

$$f(\hat{x}) - f(x) = \{ \sum_{n=1}^{\infty} \frac{D^n f(x)}{n!} \bar{x}^{(n-1)} \} \bar{x}$$

where the term in brackets is an element of $\mathcal{L}(R^n, R^n)$ for each $x \in R^n$. Thus

$$g(\bar{x},x) = \sum_{m=0}^{\infty} \frac{1}{m+1} D^{n+1} f(x) \frac{\bar{x}^{(m)}}{m!}$$

Now $D_{\bar{x}} \bar{f}(\bar{x},x) = D_{\bar{x}} g(\bar{x},x)\bar{x} + g(\bar{x},x) = D_{\hat{x}} f(\hat{x})$,

for all $\hat{x} \in \Omega$.

In order to prove a nonlinear version of lemma 2.4 we shall need the following lemma of Brauer (1966).

Lemma 2.5 If (2.6) is satisfied and $x_0, y_0 \in \Omega$ where Ω is convex, then, for any solution $x(t;t_0,x_0)$ of (2.1) we have

$$\| x(t;t_0,y_0) - x(t;t_0,x_0) \| < \| y_0 - x_0 \| \exp \left(\int_{t_0}^{t} \alpha(u)du \right) . \quad \Box$$

We are now ready to prove our main result:

Theorem 2.6 Let Ω be convex and suppose that any solution $x(t;t_0,x_0)$ of (2.1) with $x_0 \in \Omega$ remains in Ω. Then, if $\exists$ a matrix P such that

$$\lambda(Df_1(x) + P\ D\ B(x)) \lesssim -\varepsilon < 0$$

for all $x \in \Omega$ and

$$||f_2(x,t) - f_2(y,t)|| \lesssim K||x - y||$$

for all $x,y \in \Omega$ and $t \in I$, where

$$f = f_1 + f_2,$$

and f_1, B are analytic, then equation (2.1) is exponentially observable if $K < \varepsilon$.

Proof From equation (2.5),

$$\dot{\hat{x}}(t) - \dot{x}(t) = f_1(\hat{x},t) - f_1(x,t) + PB\hat{x} - PBx + f_2(\hat{x},t) - f_2(x,t) \tag{2.7}$$

We now consider equation (2.7) as a perturbation of the equation

$$\dot{y} = F(y,t) \tag{2.8}$$

where $F(\hat{x}-x,t) = f_1(\hat{x},t) - f_1(x,t) + PB\hat{x} - PBx$

(Since f_1 and B are analytic, we have seen above that it is possible to write the right hand side in this form .) If Φ_1 is the variational solution of (2.8), then using lemma 2.2 we have,

$$\hat{x}(t) - x(t) = y(t) + \int_{t_o}^{t} \Phi_1(t;s,\hat{x}-x)(f_2(\hat{x},s)-f_2(x,s))ds$$

and so

$$||\hat{x}(t) - x(t)|| \leq ||y(t)|| + \int_{t_o}^{t} \exp(-\varepsilon(t-s))K||\hat{x} - x||ds .$$

However, by lemma 2.5, we have

$$||y(t)|| < ||\hat{x}(t_o) - x(t_o)||\exp(-\varepsilon(t-t_o)),$$

and so

$$||\hat{x}(t) - x(t)|| < ||\hat{x}(t_o) - x(t_o)||\exp\{(-\varepsilon(t-t_o))$$

$$+ \int_{t_o}^{t} \exp(-\varepsilon(t-s))K||\hat{x}-x||ds$$

and by Gronwall's lemma

$$||\hat{x}(t) - x(t)|| < ||\hat{x}(t_o) - x(t_o)||\exp\{(K-\varepsilon)(t-t_o)\}$$

and the result follows. □

<u>Remark</u> We have not proved a generalization of lemmas 2.3, 2.4 in theorem 2.6 since in the latter case we require a stronger condition on f_2 than was required in the former cases on f_1.

3. Examples

3.1 Consider the system

$$\left.\begin{aligned} \dot{x}_1 &= x_1 + f_1(x_1) \\ \dot{x}_2 &= x_1 - 2x_2 + e^{-x_2} + f_2(x_2) \end{aligned}\right\} = f(x_1,x_2) \tag{3.1}$$

$$y = x_1 + x_2,$$

where f_1, f_2 are differentiable with maximum slope $\leq \delta < 1$. Then, as we have seen above,

$$(e^{-x_2} - e^{-\bar{x}_2})(x_2 - \bar{x}_2) \leq 0$$

and if we put $P = (-2,1)^T$, then

$$\dot{\hat{x}}(t) = f(\hat{x}) + P(x_1 + x_2)$$

is an exponential observer for the system (3.1) by lemma 2.4, since

$$\lambda\left\{\begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}\right\} = \lambda\left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\} = -1 .$$

3.2 Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_1^3 + x_1x_2 + f_1(x_1) \\ \dot{x}_2 &= -x_2 - x_2^5 - 4\,x_1x_2 + f_2(x_2) \\ y &= x_1x_2 \end{aligned} \qquad (3.2)$$

where f_1, f_2 are Lipschitz continuous with constant $K < 1$.

Then

$$D\left\{\begin{pmatrix} -x_1-x_1^3 + x_1x_2 \\ -x_2-x_2^5 - 4x_1x_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix} x_1x_2\right\}$$

$$= \begin{pmatrix} -1 - 2x_1^2 + x_2 & x_1 \\ -4\,x_2 & -1 - 5x_2^2 - 4x_1 \end{pmatrix} + \begin{pmatrix} -x_2 & -x_1 \\ 4x_2 & 4x_1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 - 2x_1^2 & 0 \\ 0 & -1 - 5x_2^2 \end{pmatrix}$$

and

$$\lambda\left\{\begin{pmatrix} -1 - 2x_1^2 & 0 \\ 0 & -1 - 5x_2^2 \end{pmatrix}\right\} \leq -1$$

Hence, by theorem 2.6, the system

$$\begin{aligned} \dot{\hat{x}}_1 &= -\hat{x}_1 - \hat{x}_1^3 + \hat{x}_1\hat{x}_2 + f_1(\hat{x}_1) - (\hat{x}_1\hat{x}_2 - y) \\ \dot{\hat{x}}_2 &= -\hat{x}_2 - \hat{x}_2^5 - 4\hat{x}_1\hat{x}_2 + f_2(\hat{x}_1) + 4(\hat{x}_1\hat{x}_2 - y) \end{aligned}$$

is an exponential observer for the system (3.2).

4. Conclusions

We have presented a method for obtaining exponential observers for general nonlinear systems, and our results generalize some of those of Kou et al (1975). The latter paper considers basically nonlinear perturbations of linear systems, using Lyapunov theory. In order to obtain results in the case of nonlinear perturbations of nonlinear systems we have made use of the nonlinear variation of constants formula.

Of course, the basic system must be fairly restricted. It is necessary, we have seen, for the unperturbed system to be stabilizable by a suitable conbination of the observations, and that the nonlinear perturbation has uniformly bounded slope or is Lipschitz.

5. References

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