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An Approach to Optimal Control For Engineering Undergraduates. 1

By

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Abstract

The paper illustrates an approach to the teaching of optimal control in the presence of hard constraints at an undergraduate level. The treatment is rigorous but simple and includes concepts essential for postgraduate studies and the introduction of numerical optimization methods.

1. Introduction

The teaching of optimal control theory[1,2] to undergraduate engineering students is a fairly recent phenomenum which presents several difficulties to the lecturer concerned:-

(1) The choice of a mathematical level of presentation consistent with undergraduate engineering mathematics courses.

(2) The choice of a conceptual level consistent with the need to produce students capable of adapting the material to suite a larger class of problem solving situations.

(3) The case of optimal control in the presence of hard control constraints requires a combination of mathematical rigour and interpretation convincing to the good student but at a level attainable by weaker students.

(4) The ordering of the material in a manner such that the more advanced concepts are introduced one by one. For example, it has been found that the students react more sympathetically if the solution of more difficult two-point-boundary- value-problems is postponed until they have investigated simpler problems (the linear cost problem) and because used to the minimization of the Hamiltonian in the presence of hard constraints.

These two papers outline the approach adopted for and the observations made during a lecture course on optimal control for final year control engineering undergraduates in the Department of Control Engineering of the University of Sheffield. The course was designed to reach a compromise between the above requirements and to emphasize the following points:-

(a) The mathematical source and interpretation of the costate and Hamiltonian.

(b) The implicit iterative nature of the two-point-boundary- value-problem (TPBVP)

(c) The difficulties in the solution of the TPBVP due to control constraints and the dependence of the solution on the constraints.

(d) Existence and uniqueness of the optimal control as a function of the performance criterion and the control constraints.

(e) The need to apply common sense when applying all but the trivial results of optimal control theory.

The course was split into two parts,

(i) The application of variational calculus to optimal control.

(ii) Control constraints and Pontriagins Minimum Principle.

The students responded very well to the variational approach and the concept of the TPBVP with its implicit iterative nature. Difficulties were found however when the concept of control constraint was introduced. In order to overcome these difficulties the second part of the course was subdivided into two main parts.

(a) The solution of linear cost optimal control problems (see section 2)

(b) The solution of minimum energy/minimum fuel problems.

The linear cost problem was found to be a very good vehicle to give the students practice in the solution of simple TPBVP's where the Hamiltonian must be minimized with respect to a control restraint set and to provide a convincing foundation for the derivation of the mathematical source and meaning of the costate and Hamiltonian. The minimum energy problems illustrated to the students the various difficulties arising when it is necessary to minimize the Hamiltonian and satisfy state and costate boundary conditions simultaneously.

The papers present novel derivations of the Minimum Principle for these two classes of optimization problems. The approach is rigorous but simple and has the advantage of giving the students enough insight into the source of the concepts involved to enable a sensible discussion of iterative optimization methods in the solution of the linear quadratic optimization problems in the presence of control constraints. One student successfully completed a final year project implementing a gradient algorithm for the solution of a nuclear reactor optimization problem.

The mathematical machinery required for the approach is simple integral calculus, elementary matrix algebra and the concept of the inner product of two nx1 column vectors $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ defined by

$$\langle x,y \rangle = \sum_{j=1}^{n} x_i y_j \qquad (1)$$

and the simple properties (if λ is a scalar)

$$\langle x,y+z \rangle = \langle x,y \rangle + \langle x,z \rangle \qquad (2)$$

$$\langle x,\lambda y \rangle = \lambda \langle x,y \rangle \qquad (3)$$

$$\langle x,y \rangle = \langle y,x \rangle \qquad (4)$$

and, if A is a matrix with transpose A^T, then

$$\langle x, Ay \rangle = \langle A^T x, y \rangle \tag{5}$$

2. The Linear Cost Problem

The approach was restricted to the analysis of linear systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad x(o)=x_o \tag{6}$$

where u(t) is restricted to lie in a given control restraint set Ω.
The performance criterion (T fixed) is

$$J(u) = \langle \alpha, x(T) \rangle + \int_o^T \{ \langle \beta(t), x(t) \rangle + g(u(t),t) \} \, dt \tag{7}$$

Here A(t),B(t) are nxn and nxm matrices, $\alpha, \beta(t)$ are nx1 vectors and g is a scalar function of both u(t) and t. All functions are assumed to be suitably continuous in their arguments.

Using the normal technique in the derivation of numerical algorithms, the costs $J(u_1)$, $J(u_o)$ of using two controllers $u_1(t)$ and $u_o(t)$ respectively are compared by calculating the difference $J(u_1) - J(u_o)$ ie.

$$J(u_1)-J(u_o) = \langle \alpha, x_1(T)-x_o(T) \rangle + \int_o^T \{ \langle \beta(t), x_1(t)-x_o(t) \rangle + g(u_1(t),t) - g(u_o(t),t) \} \, dt \tag{8}$$

Now, for any differentiable vector function $p(t)$, equation (6) implies that

$$\int_o^T \langle p(t), A(t)\{x_1(t)-x_o(t)\} + B(t)\{u_1(t)-u_o(t)\} - \{\dot{x}_1(t)-\dot{x}_o(t)\} \rangle \, dt = 0 \tag{9}$$

or, using the simple properties of the inner product and integration by parts in the last term in equation 9,

$$\int_o^T \langle A^T(t)p(t) + \dot{p}(t), x_1(t)-x_o(t) \rangle \, dt + \int_o^T \langle B^T(t)p(t), u_1(t)-u_o(t) \rangle \, dt - [\langle p(t), x_1(t)-x_o(t) \rangle]_o^T = 0 \tag{10}$$

Adding the left-hand side of this expression to equation (8), and noting that $x_1(o)=x_o(o)=x_o$,

$$J(u_1)-J(u_o) = \langle \alpha - p(T),\ x_1(T)-x_o(T) \rangle$$
$$+ \int_o^T \langle \beta(t)+A^T(t)p(t)+\dot{p}(t),\ x_1(t)-x_o(t) \rangle\ dt$$
$$+ \int_o^T \{ \langle B^T p(t), u_1(t)-u_o(t) \rangle + g(u_1(t),t)-g(u_o(t),t) \}\ dt \qquad (11)$$

At this stage in the analysis the students react sympathetically to the argument that the arbitrary function p(t) can be chosen to simplify this expression as much as possible by eliminating the dependence upon the state trajectory $x_1(t)$. This argument is consistent with the normal technique used in the derivation of numerical optimization algorithms in that, given a trial controller $u_o(t)$ and its state trajectory $x_o(t)$, it is helpful if we can estimate the effect of using an updated controller $u_1(t)$ without calculating the corresponding state trajectory $x_1(t)$.

Using the above argument, p(t) must satisfy the differential equation

$$\dot{p}(t) = -A^T(t)p(t) - \beta(t) \qquad (12)$$

with terminal boundary condition

$$p(T) = \alpha \qquad (13)$$

These equations are easily identified with the costate equations of the Minimum Principle. The above analysis illustrates how the idea of costate variables arises naturally in the problem using elementary mathematical manipulations and by appealing to the practical computational aspects of optimization theory.

Substituting equations (12) and (13) into equation (11) and defining the Hamiltonian function

$$H(x,p,u,t) = \langle \beta(t),x \rangle + g(u,t) + \langle p,\ A(t)x + B(t)u \rangle \qquad (13)$$

it follows that

$$J(u_1)-J(u_o) = \int_o^T \{ H(x_o(t),p(t),u_1(t),t)-H(x_o(t),p(t),u_o(t),t) \}\ dt \qquad (15)$$

At this stage in the analysis it is assumed that $u_o(t)$ is an optimal controller for the process. Hence, for any other controller $u_1(t)$,

$$J(u_1) \geqslant J(u_o) \qquad (16)$$

or $$\int_o^T \left\{ H(x_o(t),p(t),u_1(t),t) - H(x_o(t),p(t),u_o(t),t) \right\} dt \geqslant 0 \qquad (17)$$

Using graphical arguments the students readily appreciate that for these conditions to hold, it is necessary that, for all $o \leqslant t \leqslant T$,

$$H(x_o(t),p(t),u_1(t),t) = \min_u \; H(x_o(t),p(t),u,t) \qquad (18)$$

where the minimization is performed with respect to the restraint set Ω. Moreover, in this case the condition is also sufficient for u_o to be the optimal controller. This analysis demonstrates how the Hamiltonian minimization condition arises natürally in optimization problems.

The two point boundary value problem to be solved for the optimal controller $u_o(t)$ can now be summarized as

$$\dot{x}_o(t) = A(t)x_o(t) + B(t)u_o(t) \;, \quad x_o(o)=x_o$$

$$\dot{p}(t) = -A^T(t)p(t) - \beta(t) \;, \quad p(T) = \alpha$$

$$H(x_o(t),p(t),u_o(t),t) = \min_u H(x_o(t),p(t),u,t) \;, \; o \leqslant t \leqslant T \qquad (19)$$

and can be solved in the following step by step manner:-

STEP ONE: Solve the costate equations backwards in time for $p(t)$. Because of the particular structure of this problem the costate is independent of the state and the controllers.

STEP TWO: Given $p(t)$ perform the minimization of the Hamiltonian for $o \leqslant t \leqslant T$. Note that the resulting controller $u_o(t)$ is independent of the state trajectory $x_o(t)$.

STEP THREE: Calculate the state trajectory $x_o(t)$ using $u_o(t)$.

The simple structure of this TPBVP enables the student to concentrate upon the meaning of equation (18) and to obtain expertize on its application to different systems and a class of control restraint sets Ω without the usual difficulties arising from the need to satisfy the costate boarding conditions and minimize the Hamiltonian simultaneously. The particular examples emphasized are

(1) $g(u,t)=\frac{1}{2}\langle u,Ru\rangle$, control unconstrained, R positive definite.

(2) $g(u,t)=\frac{1}{2}u^2$, $|u|\leqslant M$

(3) $g(u,t)=|u|$, $|u|\leqslant M$

(4) $g(u,t)=0$, $|u|\leqslant M$

which relate the results to previous results from variational calculus and provides a lead into techniques useful in the minimum fuel/minimum energy problem and time-optimal control.

3. Interpretation of the Hamiltonian Function and Costate

An additional feature of the linear cost problem and the simple but rigorous derivation of the Minimum Principle in this case, is that it provides an ideal means of illustrating the meaning of the Hamiltonian function and costate. Consider the case of n=m=1, and let $u_o(t)$ and $u_1(t)$ be admissable controllers such that

$$u_1(t) = \begin{cases} u_o(t) & ; \quad o\leqslant t< t_1 \\ u_o(t)+\Delta u & ; \quad t_1\leqslant t< t_1+\delta \\ u_o(t) & ; \quad t_1+\delta\leqslant t\leqslant T \end{cases} \tag{20}$$

and let $x_o(t)$ and $x_1(t)$ be the corresponding state trajectories at x_o. Schematic examples of these responses are shown in Fig. 1. It is noted that, whereas the control perterbation $u_1(t)-u_o(t)$ is non-zero only in the interval $t_1\leqslant t<t_1+\delta$, the state trajectory $x_1(t)-x_o(t)$ is, in general, non-zero for all $t > t_1$ i.e. the state carries information on the control function in a forward-time direction.

From equations (15) and (20)

$$J(u_1)-J(u_o) = \int_{t_1}^{t_1+\delta} \{ H(x_o(t),p(t),u_1(t),t)-H(x_o(t),p(t),u_o(t),t)\} \, dt \tag{21}$$

so that all information concerning the difference in cost is contained in the interval $t_1\leqslant t\leqslant t_1+\delta$. This observation rather surprises students until it is pointed out that the costate equation(12) has a terminal boundary condition and must be solved backwards in time. Interpreting this as a reverse-time information flow, it is readily understood that the costate collects all the information on the effect on the performance of the state perturbation for $t \geqslant t_1+\delta$ into the Hamiltonian function on the interval $t_1\leqslant t\leqslant t_1+\delta$

If δ is small, a further interpretation can be given to the Hamiltonian function as follows,

$$J(u_1)-J(u_o) \simeq \delta \{ H(x_o(t_1),p(t_1),u_o(t_1)+\Delta u,t_1)-H(x_o(t_1),p(t_1),u_o(t_1),t_1) \}$$

$$\simeq \delta \left. \frac{\partial H}{\partial u} \right|_{x_o(t_1),p(t_1),u_o(t_1),t_1} \Delta u \qquad (22)$$

That is, the gradient of H with respect to u represents the sensitivity of the cost to control perturbations - the larger the value of $\frac{\partial H}{\partial u}$ the greater the sensitivity. For the case of unconstrained optimal control ($\frac{\partial H}{\partial u} = 0$) these ideas lead naturally to the characterization of the optimal control as a controller producing a cost which is insensitive to admissible controller perturbations.

4. Typical Worked Examples

4.1 $$\dot{x}(t) = -x(t) + u(t) \quad , \quad x(o) = 1 \qquad (23)$$

$$|u(t)| \leq M \qquad (24)$$

$$J(u) = x(1) + \int_o^1 \tfrac{1}{2}u^2(t)\, dt \qquad (25)$$

Following the step by step procedure given in section 2, the costate equation is

$$\dot{p}(t) = p(t) \quad , \quad p(1) = 1 \qquad (26)$$

which is easily solved to obtain

$$p(t) = e^{t-1} \qquad (27)$$

The Hamiltonian function is (eqn 14)

$$H = p(t)\, u(t) - p(t)\, x(t) + \tfrac{1}{2}u^2(t) \qquad (28)$$

so that the optimal controller is obtained using graphical arguments as

$$u_o(t) = \begin{cases} -p(t) & ; \quad |p(t)| \leq M \\ -M\,\mathrm{sgn}\ p(t) & ; \quad |p(t)| > M \end{cases} \qquad (29)$$

By plotting this solution graphically as a function of M, the students can be given a preliminary insight into the dependence of optimal controls on the control constraints and the fact that optimal controllers do not necessarily touch the constraint boundary. Examples of this type are also an excellent preparation for minimum energy optimization problems.

4.2 $$\dot{x}_1(t) = x_2(t) \qquad x_1(o) = 1$$

$$\dot{x}_2(t) = u(t) \qquad x_2(o) = 0 \tag{30}$$

$$|u(t)| \lesssim M \tag{31}$$

$$J(u) = -2x_1(1) + x_2(1) \tag{32}$$

The costate equations are

$$\dot{p}_1(t) = 0 \qquad p_1(1) = -2$$

$$\dot{p}_2(t) = -p_1(t) \qquad p_2(1) = 1 \tag{33}$$

so that $p_2(t) = 2t-1$ (34)

The Hamiltonian is

$$H = p_2(t)\,u(t) + p_1(t)\,x_2(t) \tag{35}$$

so that the optimal controller becomes

$$u_o(t) = -M\text{sgn}\ p_2(t) = \begin{cases} +M & o \leqslant t < \frac{1}{2} \\ -M & \frac{1}{2} < t \leqslant 1 \end{cases} \tag{36}$$

Practice with this type of problem can be a good preparation for later discussions of bang-bang types of optimal control and switching concepts.

5. Discussion

The first part of an approach to the teaching of optimal control in the presence of hard constraints to undergraduate engineering students has been presented in this paper. The course uses elementary material normally given in undergraduate mathematics courses and yet enables a rigorous deviation of the Minimum Principle for linear cost problems at a level convincing to the better students but at a level attainable by the weaker students.

The restriction of the material initially to a discussion of linear cost problems makes possible an interpretation of the concepts of costate and Hamiltonian and enables the student to master the Hamiltonian minimization ideas in the presence of a variety of control constraints without the normal difficulties associated with the simultaneous satisfying of the Hamiltonian minimization and costate or state boundary conditions. An important aspect of this rigorous but simple introduction tothe Minimum Principle is that the approach uses techniques fundamental to postgraduate studies of numerical optimization methods (eg. first-order gradient algorithms).

In the second part of this paper a similar treatment of the minimum energy/minimum fuel type problems is presented, and a detailed example described which illustrates these features of the TPBVP mentioned in section 1 which cannot be illustrated by the linear cost problem.

References

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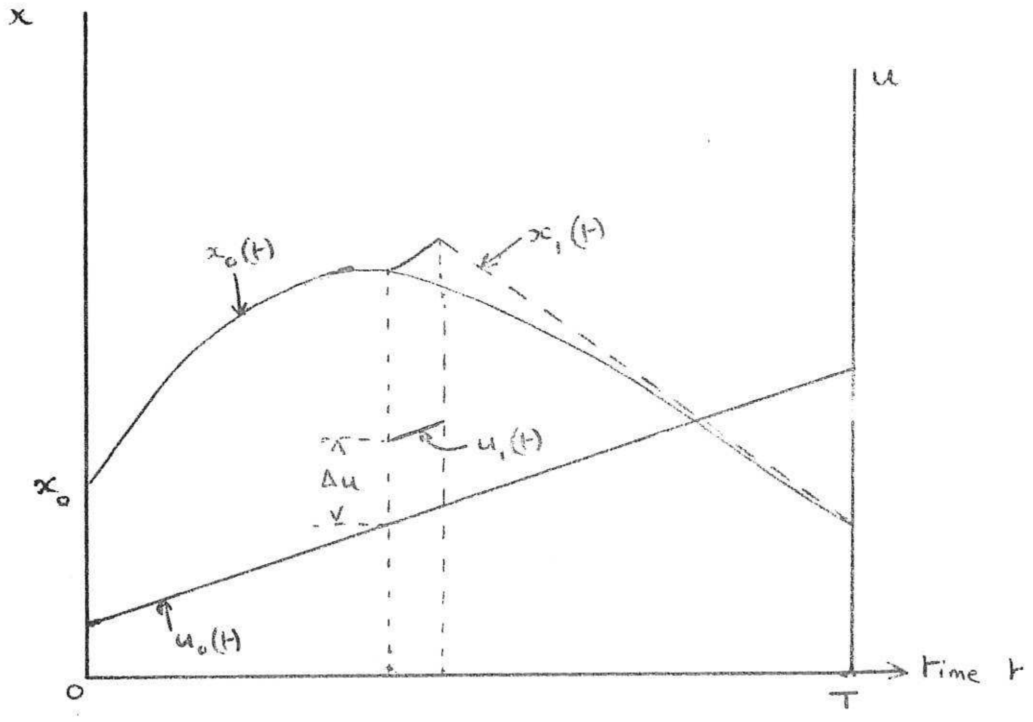

Fig. 1. The Effect of a Control Perturbation.