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IDENTIFICATION OF NON-LINEAR
UNITY FEEDBACK SYSTEMS

by

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ABSTRACT

An algorithm for the identification of non-linear systems which can be described by a model consisting of a linear system in cascade with a non-linear element followed by another linear system in a unity negative feedback loop is presented. Cross-correlation techniques are employed to identify the system in terms of the individual linear and non-linear subsystems. Parameterisation of both the linear and non-linear elements is discussed and the results of a simulation study are included to demonstrate the validity of the algorithm.

1. INTRODUCTION

Although a large class of non-linear systems can be characterized by the functional Volterra series[1,2,3], identification techniques based on this representation provide a black-box description which gives very little insight regarding the structure of the process under investigation. For systems which are composed of linear subsystems and memoryless non-linearities the Volterra kernels can often be factored and related to the components of the original system. Ideally, identification should be in terms of these individual elements of the system such that the structure of the process is preserved and the error which results when a system is characterized by a truncated Volterra series is avoided.

Several authors[4,5,6,7,8,9,10,11] have studied the identification of non-linear open-loop systems consisting of cascade connections of linear dynamic and static non-linear elements but the identification of such systems under unity feedback appears to have been largely neglected. Characterization of non-linear feedback systems using Wiener G-functionals can of course be readily achieved using the technique of Lee and Schetzen[12] but a truncation error will be incurred and the structure of the original system is lost. In the present study an identification algorithm which provides estimates of the component subsystems for a class of non-linear feedback systems and which preserves the structure of the original process is presented. The class of systems considered includes systems which can be described by a general model, consisting of a linear system in cascade with a static non-linearity and a second linear element, under unity feedback. The algorithm represents an extension of previous research[10,13] which showed that when the input belongs to the class of separable processes identification of the linear and non-linear elements in the general model can be decoupled using correlation analysis.

A brief review of the theory of separable processes is presented in the next section and the identification of open loop systems which can be described by a general model is summarised in section 3. In section 4 the Volterra series representation of non-linear unity feedback systems is derived and the Volterra kernels are related to the component subsystems of the original process. An algorithm for the identification of the component subsystems is presented in section 5 using the results derived in previous sections. Special cases are discussed and simulated examples are included to illustrate the validity of the algorithm.

2. SEPARABLE RANDOM PROCESSES

The statistical properties of a stationary random process after transmission through an instantaneous non-linear device has been studied by several authors. Bussgang[14] has shown that for two Gaussian signals the cross-correlation function taken after one of them has undergone non-linear amplitude distortion is proportional to the cross-correlation function taken before the distortion. This result was extended by Luce[15] and later generalised by Nuttall[16] to include all inputs which belong to the separable class of random processes.

Let $p(\alpha,\beta;\tau)$ be the joint probability density function for the two stationary random processes $\alpha(t)$ and $\beta(t)$, and define

$$g(\beta,\tau) = \int_{-\infty}^{\infty} \alpha p(\alpha,\beta;\tau)\,d\alpha \qquad (1)$$

If the g-function separates as

$$g(\beta,\tau) = g_1(\beta)\,g_2(\tau) \;\forall\; \beta,\tau \qquad (2)$$

then $\alpha(t)$ is separable with respect to $\beta(t)$[16]. Contrary to Nuttall's original definition, equation (2) includes both the a.c. and d.c. components of the signal[13]; this definition simplifies the analysis in later sections.

Fortunately, the separable class of random processes is fairly wide and includes the Gaussian process, sine wave process, phase or frequency modulated process, squared Gaussian process etc.

Previous research[13] has shown that Nuttall's results can be extended to include separability under linear transformation. Thus if $\alpha(t)$ is separable with respect to $\beta(t)$, then $\alpha(t)$ is also separable with respect to the output of any linear filter with $\beta(t)$ as input.

By analysing the system illustrated in Fig.1, where $F[\cdot]$ is the transfer characteristic of an instantaneous non-linear element, Nuttall[16] proved that the separability of $x_1(t)$ with respect to $x_2(t)$ is a necessary and sufficient condition for the invariance property

$$\phi^F_{y_1y_2}(\tau) = C_F\phi_{x_1x_2}(\tau) \;\forall\; F \text{ and } \tau \qquad (3)$$

to hold where C_F is a constant. When $x_1(t) = x_2(t) = x(t)$ which is a separable process equation (3) relates the input-output cross-correlation function to the input autocorrelation function

$$\phi^F_{x_1y_2}(\tau) = C_{F_x}\,\phi_{xx}(\tau) \;\forall\; F \text{ and } \tau \qquad (4)$$

where C_{Fx} is given by

$$C_{Fx} = \frac{1}{\phi_{xx}(0)} \int xF[x]p(x)dx \qquad (5)$$

Provided $x^2(t)$ is separable with respect to $x_1(t) = x_2(t) = x(t)$ this result can be extended[13] to include the case when the top lead in Fig.1 contains a square law device,

$$\phi_{x_1^2 y_2}(\tau) = C_{FFx}\phi_{x^2x^2}(\tau) \;\forall\; F \text{ and } \tau \qquad (6)$$

where

$$C_{FFx} = \frac{1}{\phi_{x^2x^2}(0)} \int x^2F[x]p(x)dx \qquad (7)$$

3. ANALYSIS OF THE GENERAL MODEL

Before non-linear feedback systems can be considered it is necessary to establish an identification procedure[10,13] for the open-loop system illustrated in Fig.2. This system will be referred to as the general model and consists of a linear system $h_1(t)$ in cascade with a zero-memory non-linear element

$$y(t) = \gamma_1 x(t) + \gamma_2 x^2(t) + \ldots \gamma_k x^k(t) \qquad (8)$$

and another linear system $h_2(t)$.

From the Convolution theorem

$$z_2(t) = \int h_2(\theta)y(t-\theta)d\theta \qquad (9)$$

and

$$y(t) = F\{ \int h_1(\tau_1)u_2(t-\tau_1)d\tau_1\}$$

$$= \int Q(t,\tau_1)u_2(t-\tau_1)d\tau_1 \qquad (10)$$

where $Q(t,\tau_1)$ is a function of t and τ_1 only and is defined[13] as

$$Q(t,\tau_1) = \gamma_1 h_1(\tau_1)+\gamma_2 h_1(\tau_1)\int h_1(\tau_2)u_2(t-\tau_2)d\tau_2$$

$$+ \dots \gamma_k h_1(\tau_1)\int \dots \int h_1(\tau_2) \dots h_1(\tau_k)u_2(t-\tau_2)$$

$$\dots u_2(t-\tau_k)d\tau_2 \dots d\tau_k \qquad (11)$$

Combining equations (9) and (10) the output of the general model can be expressed as

$$z_2(t) = \int\int h_2(\theta)Q(t-\theta,\tau)u_2(t-\theta-\tau_1)d\theta d\tau_1 \qquad (12)$$

and the output cross-correlation function can be defined

$$\phi_{z_1 z_2}(\varepsilon) = \int\int h_2(\theta)\ \overline{Q(t-\theta,\tau_1)u(t-\theta-\tau_1)u_1(t-\varepsilon)d\tau_1 d\theta} \qquad (13)$$

The validity of the invariance property for the non-linear element in Fig.2 can be established using the results of the previous section. Thus providing $u_1(t)$ is separable with respect to $u_2(t)$, then $u_1(t)$ is separable with respect to $x(t)$[13] and from eqn (3)

$$\phi_{z_1 y}(\sigma) = C_{FG}\phi_{u_1 x}(\sigma)\ \forall\ F \text{ and } \sigma \qquad (14)$$

An expression for $\phi_{z_1 y}(\sigma)$ can be obtained from equation (10)

$$\phi_{z_1 y}(\sigma) = \overline{\int Q(t,\tau_1)u_2(t-\tau_1)u_1(t-\sigma)d\tau_1} \qquad (15)$$

and by definition

$$\phi_{u_1x}(\sigma) = \int h_1(\tau_1)\overline{u_2(t-\tau_1)u_1(t-\sigma)d\tau_1} \qquad (16)$$

Combining the results of equations (14), (15) and (16), $\phi_{z_1z_2}(\varepsilon)$ can be expressed as

$$\phi_{z_1z_2}(\varepsilon) = \phi_{u_1z_2}(\varepsilon) = C_{FG}\int\int h_2(\theta)h_1(\tau_1)\phi_{u_1u_2}(\varepsilon-\theta-\tau_1)d\theta d\tau_1 \qquad (17)$$

For the special case when $u_1(t) = u(t)$, $u_2(t) = u(t)+b$, $z_2'(t) = z_2(t)-\overline{z_2(t)}$, where u(t) is the practical realization of a zero mean Gaussian white process with a spectral density of 1 watt/cycle $\{\int\phi_{uu}(t)dt = 1\}$ and b is a non-zero mean level, equation (17) reduces to

$$\phi_{uz_2'}(\varepsilon) = C_{FG}\int\int h_2(\theta)h_1(\tau_1)\phi_{uu}(\varepsilon-\theta-\tau_1)d\theta d\tau_1$$

$$= C_{FG}\int h_1(\tau_1)h_2(\varepsilon-\tau_1)d\tau_1 \qquad (18)$$

The constant C_{FG} can be evaluated by expanding $\phi_{u_1y}(\sigma)$ in equation (14) using the non-linear characteristic equation (8) and equating terms to give

$$C_{FG} = \gamma_1+2\gamma_2 b\int h_1(\theta)d\theta+3\gamma_3\int h_1^2(\theta)d\theta+3\gamma_3 b^2\int\int h_1(\tau_1)h_1(\tau_2)d\tau_1 d\tau_2+\ldots \qquad (19)$$

Providing the linear subsystem $h_1(t)$ is stable, bounded inputs bounded outputs, C_{FG} is a finite constant and equation (18) is valid.

Following the derivation of the first order cross-correlation function, an expression for the second order correlation function

$$\phi_{u_1^2z_2}(\varepsilon) = \overline{u_1^2(t-\varepsilon)z_2(t)} \qquad (20)$$

can be obtained using the invariance property of equation (6), providing $u_1^2(t)$ is separable with respect to $u_2(t)$. For the special case when $u_1(t) = u(t)$, $u_2(t) = u(t)+b$ where u(t) is the practical realization of a zero mean white Gaussian process, it can readily be shown[13] that $\phi_{u^2 z_2'}(\varepsilon)$ reduces to

$$\phi_{u^2 z_2'}(\varepsilon) = 2C'_{FFG} \int h_2(\theta)h_1^2(\varepsilon-\theta)d\theta \tag{21}$$

The constant C'_{FFG} can be evaluated by expanding equation (6) to give

$$C'_{FFG} = \gamma_2+3\gamma_2 b \int h_1(\tau)d\tau+\ldots \tag{22}$$

Inspection of equations (18) and (21) shows that correlation analysis effectively decouples the identification[10,13] of the open-loop general model into two distinct steps; identification of the linear subsystems and characterization of the non-linear element. A least squares algorithm which decomposes equations (18) and (21) to provide unbiased estimates of the pulse transfer functions associated with the individual linear subsystem impulse response has been derived in a previous publication[10]. Once the linear systems have been identified estimates of the coefficients in the polynomial representation of the non-linearity can be readily computed.

4. ANALYSIS OF NON-LINEAR FEEDBACK SYSTEMS

The identification procedure developed in the previous section can be applied to unity feedback non-linear systems if the form of the Volterra kernels can be related to the component subsystems of the original process[17,18]. This can be achieved by applying the operator calculus developed by Brilliant[19] and George[20]. Consider the unity feedback non-linear system illustrated in Fig.3 and the equivalent open loop system $\underline{G}$ where $\underline{A}$ is a non-linear system with a known functional expansion. Using the notation of George[19]

$$\underline{G} = \underline{A}* (\underline{I}-\underline{G}) \tag{23}$$

Defining $\underline{K} = \underline{I}-\underline{G}$ (24)

such that $\underline{K}_1 = \underline{I}-\underline{G}_1$

$\vdots$

$\underline{K}_\ell = -\underline{G}_\ell$ for $\ell \geq 2$ (25)

equation (23) can be expressed as

$$\underline{G} = \underline{A} * \underline{K} \tag{26}$$

Since $\underline{G}$, $\underline{A}$ and $\underline{K}$ are the sums of homogeneous functional operators, equation (26) can be rewritten as

$$\sum_{\ell=1}^{\infty} \underline{G}_\ell = \left(\sum_{\ell=1}^{\infty} \underline{A}_\ell\right) * \left(\sum_{j=1}^{\infty} \underline{K}_j\right)$$

$$= \sum_{\ell=1}^{\infty} \sum_{n=1}^{\ell} \sum\nolimits_Q \underline{A}_n \circ (\underline{K}_{i_1} \cdot \underline{K}_{i_2} \cdots \underline{K}_{i_n}) \tag{27}$$

where the summation $\sum_Q$ is taken over all the integers $i_1, i_2 \ldots i_n$ such that $i_1+i_2+\ldots i_n = \ell$ and $1 \leq i_q \leq \ell$[21].

Equating operators of equal order

$$\underline{G}_\ell = \sum_{n=1}^{\ell} \sum\nolimits_Q \underline{A}_n \circ (\underline{K}_{i_1} \cdot \underline{K}_{i_2} \cdots \underline{K}_{i_n}) \tag{28}$$

which is equivalent to

$$\underline{G}_\ell(t_1, t_2 \ldots t_\ell) = \sum_{n=1}^{\ell} \sum\nolimits_Q \int du_1 \ldots \int du_n$$

$$\underline{A}_n(u_1, \ldots u_n) \underline{K}_{i_1}(t_1-u_1, t_2-u_1, \ldots t_{i_1}-u_1)$$

$$\ldots \underline{K}_{i_n}(\ldots, t_\ell-u_n) \tag{29}$$

Equation (28) can be simplified to give the Volterra kernels of the equivalent open-loop system as

$$\underline{G}_\ell = [\underline{I}+\underline{A}_1]^{-1} * \{\sum_{n=2}^{\ell} \sum_Q \underline{A}_n \circ (\underline{K}_{i_1} \cdots \underline{K}_{i_n})\} \tag{30}$$

for $\ell \geq 2$, and

$$\underline{G}_1 = [\underline{I}+\underline{A}_1]^{-1} * \underline{A}_1 \tag{31}$$

for $\ell = 1$.

From equations (24) and (30) the second and third order kernels are evaluated as

$$\underline{G}_2 = [\underline{I}+\underline{A}_1]^{-1} * \underline{A}_2 * [\underline{I}-\underline{G}_1] \tag{32}$$

$$\underline{G}_3 = [\underline{I}+\underline{A}_1]^{-1} * \{\underline{A}_3 * [\underline{I}-\underline{G}_1]-2\underline{A}_2 \circ ((\underline{I}-\underline{G}_1).\underline{G}_2)\} \tag{33}$$

When the non-linear system $\underline{A}$ has the structure of the general model as illustrated in Fig.4 where B(t) and C(t) are stable bounded-inputs bounded outputs, then

$$\underline{A}_n = \gamma_n \underline{C} \circ (\underline{B}^n) \qquad \text{for } n \leq k \tag{34}$$

$$\underline{A}_n = 0 \qquad \text{for } n > k \tag{35}$$

and

$$\underline{G}_\ell = [\underline{I}+\gamma_1 \underline{C}*\underline{B}]^{-1} * [\sum_{n=2}^{\ell} \sum_Q \gamma_n (\underline{C} \circ (\underline{B}^n)) \circ (\underline{K}_{i_1} \cdots \underline{K}_{i_n})] \qquad \text{for } \ell \geq 2 \tag{36}$$

$$\underline{G}_1 = [\underline{I}+\gamma_1 \underline{C}*\underline{B}]^{-1} * (\gamma_1 \underline{C}*\underline{B}) \tag{37}$$

George[20] has shown that the input-output relation is unique for this class of feedback system.

5. IDENTIFICATION OF THE LINEAR ELEMENTS

The Volterra series expansion of the general model under unity feedback, equations (36) and (37), can be represented as shown in Fig.5. Although the series is an infinite operator series the known structural form of the first two kernels can be exploited to develop an identification algorithm which identifies the system in terms of the individual subsystems, C(t), B(t) and F[•], and thus avoids the truncation error associated with a finite Volterra series representation.

If the Volterra expansion in Fig.5 is truncated to 'n' terms the output $z_2(t)$ can be expressed as

$$z_2(t) = \sum_{j=1}^{n} w_j(t) \tag{38}$$

where $w_j(t)$ is the contribution of the j'th kernel to the output

$$w_j(t) = \int d\tau_1 \int d\tau_2 \ldots \int d\tau_j G_j(\tau_1,\ldots\tau_k) u_2(t-\tau_1)\ldots u_2(t-\tau_k) \tag{39}$$

Define the output cross-correlation function

$$\phi_{z_1 z_2}(\sigma) = \overline{u_1(t-\sigma) z_2(t)} = \sum_{j=1}^{n} \phi_{u_1 w_j}(\sigma) \tag{40}$$

Inspection of equations (38), (39) and (40) shows that for a given functional form of $u_2(t)$ the form of the term $\phi_{u_1 w_j}(\sigma)$ is fixed but its amplitude is proportional to the j'th power of $u_2(t)$. Thus for a series of experiments with inputs $\alpha_i u_2(t)$ where $\alpha_i \neq \alpha_\ell$ for all $i \neq \ell$ the output correlation function $\phi_{z_1 z_{\alpha_i}}$ is given by

$$\phi_{z_1 z_{\alpha_i}}(\sigma) = \sum_{j=1}^{n} \alpha_i^{\,j} \phi_{u_1 w_j}(\sigma) \,, \quad \text{for } i = 1,2\ldots n \tag{41}$$

where z_{α_i} is the response of the feedback system to the input $\alpha_i u_2(t)$.

Equation (41) can be expressed as[22]

$$\begin{pmatrix} \phi_{uz_{\alpha_1}}(\sigma) \\ \cdot \\ \cdot \\ \cdot \\ \phi_{uz_{\alpha_n}} \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & & \\ \vdots & \vdots & \ddots & \\ 0 & \cdot & & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_n & & \alpha_n^{n-1} \end{pmatrix} \begin{pmatrix} \phi_{uw_1}(\sigma) \\ \cdot \\ \cdot \\ \cdot \\ \phi_{uw_n}(\sigma) \end{pmatrix} \tag{42}$$

where the diagonal matrix on the rhs is non-singular providing $\alpha_i \neq 0 \; \forall \; i$, and the second matrix is the transpose of the Vandermonde matrix[23] which is non-singular for $\alpha_i \neq \alpha_\ell$. Thus for any value of σ, equation (42) has a unique solution for $\phi_{uw_j}(\sigma)$ for $j = 1,2...n$.

Consider the situation when $u_1(t) = u(t)$, $u_2(t) = \alpha_i(u(t)+b)$, where $u(t)$ is the practical realization of a zero mean white Gaussian process and b is a non-zero mean level. From equations (12), (40) and (42)

$$\phi_{u_1w_1}(\sigma) = \overline{u(t-\sigma)w_1(t)} = \int G_1(\tau)\overline{\{u(t-\tau)+b\}u(t-\sigma)}\,d\tau$$

$$= G_1(\sigma) \tag{43}$$

Thus $\phi_{u_1w_1}(\sigma)$ is directly proportional to $G_1(\sigma)$ the first order Volterra kernel given by equation (37). Since the identification will normally be performed with the aid of a digital computer equation (37) can be expressed as

$$Z\{G_1(\sigma)\} = \frac{Ng_1(z^{-1})}{Dg_1(z^{-1})} = \frac{\gamma_1 C(z^{-1})B(z^{-1})}{1+\gamma_1 C(z^{-1})B(z^{-1})} \tag{44}$$

Using a least squares routine to estimate the parameters in $Ng_1(z^{-1})$ and $Dg_1(z^{-1})$ equation (44) can be rearranged to provide estimates of the numerator and denominator

$$\gamma_1 C(z^{-1})B(z^{-1}) = \frac{Ng_1(z^{-1})}{Dg_1(z^{-1})-Ng_1(z^{-1})} \tag{45}$$

$$1+\gamma_1 C(z^{-1})B(z^{-1}) = \frac{Dg_1(z^{-1})}{Dg_1(z^{-1})-Ng_1(z^{-1})} \tag{46}$$

By filtering the output data w_{α_j}, $j = 1,2...n$, with the estimate $\{1+\gamma_1 C(z^{-1})B(z^{-1})\}$ the kernels in equation (36) can be simplified

$$\underline{G}_{\ell r} = \sum_{n=2}^{\ell} \sum_Q \gamma_n (\underline{C} \circ (\underline{B}^n)) \circ (\underline{K}_{i_1} \cdot \underline{K}_{i_2} \cdots \underline{K}_{i_n}) \quad \text{for } \ell \geq 2 \tag{47}$$

and

$$\underline{G}_{1r} = \gamma_1 \underline{C} * \underline{B} \tag{48}$$

where the subscript r is used to signify the reduced kernels. The second order cross-correlation functions $\phi_{u^2 w'_{rj}}(\sigma)$, $j = 1,2...n$, can then be evaluated using the procedure of equation (42), where $w'_{rj}(t) = w_{rj}(t)-\overline{w}_{rj}(t)$.

Inspection of Fig.5 shows that the reduced second order Volterra kernel has the structure of the general model[10,13] where $F\{\cdot\} = \gamma_2(\cdot)^2$. Defining $\underline{H}_1 = \underline{B}*\underline{K}_1 = \underline{B}*[\underline{I}-\underline{G}_1]$, and using the results for the general model equation (21), the second order cross-correlation function of the output of the second kernel is

$$\phi_{u^2 w'_{r2}}(\sigma) = 2C'_{FFG} \int C(\theta)H_1^2(\sigma-\theta)\,d\theta \tag{49}$$

where $C'_{FFG} = \gamma_2$.

If equation (49) is evaluated in discrete time estimates of the parameters in the pulse transfer function

$$Z\{\phi_{u^2 w'_{r2}}(\sigma)\} = 2\gamma_2 C(z^{-1})L(z^{-1}) \tag{50}$$

can be obtained using least squares. An estimate of the pulse transfer functions of the component linear subsystems $\mu_1 B(z^{-1})$ and $\mu_2 C(z^{-1})$ can be obtained by decomposing the results of equations (45) and (50) using a multistage least squares algorithm derived previously[10], where μ_1 and μ_2 are constants.

Thus by computing the first order correlation function for the first Volterra kernel and the second order correlation function for the second Volterra kernel and exploiting the structural features of these estimates the individual linear subsystems in Fig.4 can be estimated to within constant scale factors. Notice that if the output of the feedback system Fig.4 is corrupted by an additive noise process u(t), providing this is statistically independent of u(t), then $E[u(t-\sigma)n(t)] = E[u^2(t-\sigma)n(t)] = 0 \;\forall\; \sigma$ and the results of equations (41), (43), (49) are unaffected and the estimates remain unbiased.

6. IDENTIFICATION OF THE NON-LINEAR ELEMENT

Once the linear subsystems have been identified estimates of the coefficients $\gamma_j' = (\mu_1{}^j \mu_2)^{-1} \gamma_j$ $j = 1,2\ldots k$ associated with the polynomial representation of the non-linear element, equation (8), can be computed sequentially.

From equation (45) $\mu_1\mu_2 = \gamma_1$ and hence $\gamma_1' = 1.0$. The second order Volterra kernel can be synthesised using the estimates of the linear subsystems

$$\underline{G}_2{}^s = [\underline{I}+\gamma_1 \underline{\hat{C}}*\underline{\hat{B}}]^{-1}*(\mu_2\underline{\hat{C}})\circ(\mu_1{}^2\underline{\hat{B}}^2).[(\underline{I}-\underline{\hat{G}}_1)^2] \tag{51}$$

where $\underline{G}_2 = \gamma_2' \; \underline{G}_2{}^s$

$$\gamma_2 = \gamma_2{}'\mu_1{}^2\mu_2$$

and $$\hat{w}_2(t) = \gamma_2{}' \, \underline{G}_2{}^s[u_2(t)] \quad (52)$$

By definition

$$\phi_{uw_2}(\sigma) = \gamma_2{}' \, \overline{\underline{G}_2{}^s[u_2(t) \; u(t-\sigma)]}$$

$$= \gamma_2{}' \, \phi_{uz\underline{G}_2{}^s}(\sigma) \quad (53)$$

where $\phi_{uw_2}(\sigma)$ has been estimated using equation (42). Introducing an estimation error $e(\sigma)$ and considering N points of the sampled cross-correlation functions in equation (53)

$$\underline{\phi}_{uw_2}(\sigma) = \gamma_2{}' \, \underline{\phi}_{uz\underline{G}_2{}^s}(\sigma) + \underline{E} \quad (54)$$

and a least squares estimate of $\gamma_2{}'$ can be computed as

$$\hat{\gamma}_2{}' = \{\underline{\phi}^T_{uz\underline{G}_2{}^s} \, \underline{\phi}_{uz\underline{G}_2{}^s}\}^{-1} \, \underline{\phi}^T_{uz\underline{G}_2{}^s} \, \underline{\phi}_{uw_2} \quad (55)$$

Following the same procedure for the third Volterra kernel, this can be synthesised as

$$\underline{G}_3{}^s = \underline{G}_{31}^s + \underline{G}_{32}^s \quad (56)$$

where

$$\underline{G}_{31}^s = [\underline{I}+\gamma_1\hat{\underline{C}}*\hat{\underline{B}}]^{-1}(\mu_2\hat{\underline{C}})\circ(\mu_1{}^3\hat{\underline{B}}^3)\circ[(I-\hat{G}_1)^3] \quad (57)$$

$$\underline{G}_{32}^s = [\underline{I}+\gamma_1\hat{\underline{C}}*\hat{\underline{B}}]^{-1} * [-2(\mu_2\hat{\underline{C}})\circ(\mu_1{}^2\hat{\underline{B}}^2)\circ(\gamma_2{}'\hat{\underline{G}}_2{}^s(\underline{I}-\hat{\underline{G}}_1))] \quad (58)$$

$$\underline{G}_{31} = \gamma_3{}' \, \underline{G}_{31}^s \quad (59)$$

and $\gamma_3 = \gamma_3{}'\mu_1{}^3\mu_2$

Combining equations (56), (57) and (58) the output of the third Volterra kernel can be expressed as

$$\hat{w}_3(t) = \gamma_3' \, \underline{G}^s_{31}[u_2(t)] + G^s_{32}[u_2(t)] \tag{60}$$

By definition

$$\phi_{uw_3}(\sigma) = \gamma_3' \phi_{uzG^s_{31}}(\sigma) + \phi_{uzG^s_{32}}(\sigma) \tag{61}$$

where $\phi_{uw_3}(\sigma)$ is known from equation (42) and hence the least squares estimate of γ_3' is given by

$$\gamma_3' = \{\underline{\phi}^T_{uzG^s_{31}} \underline{\phi}_{uzG^s_{31}}\}^{-1} \underline{\phi}^T_{uzG^s_{31}} \{\underline{\phi}_{uw_3} - \underline{\phi}_{uzG^s_{32}}\} \tag{62}$$

The coefficients $\gamma_4' \ldots \gamma_k'$ can be readily evaluated in a similar manner and the identification is complete.

7. SPECIAL CASES

7.1 The Unity Feedback Wiener Model

The Wiener model[9] consists of a linear system in cascade with a continuous non-linear element. When the Wiener model is placed in a unity feedback loop, $\underline{C} = \underline{I}$ in equations (36) and (37) which reduce to

$$\underline{G}^w_\ell = [\underline{I}+\gamma_1\underline{B}]^{-1} * [\sum_{n=2}^{\ell} \sum_Q \gamma_n (\underline{B}^n) o (\underline{K}_{i_1} \ldots \underline{K}_{i_n})] \qquad \text{for } \ell \geq 2 \tag{63}$$

and $$\underline{G}_1^w = [\underline{I}+\gamma_1\underline{B}]^{-1} * \gamma_1\underline{B} \tag{64}$$

Utilising the results of equations (43) and (64)

$$\phi^w_{uw_1}(\sigma) = G_1^w(\sigma) = [\underline{I}+\gamma_1\underline{B}]^{-1} * \gamma_1\underline{B} \tag{65}$$

from which the linear dynamics $\gamma_1\underline{B}$ can be estimated. The coefficients in the polynomial representation of the non-linear element can be readily evaluated by substituting $\underline{C}$ = I in the results of Section 6.

7.2 The Unity Feedback Hammerstein Model

The Hammerstein model[8,11] consists of a zero-memory non-linear element followed by linear dynamics. When the Hammerstein model is placed under unity feedback, setting $\underline{B} = \underline{I}$ in equations (36) and (37), the kernels in the Volterra expansion reduce to

$$\underline{G}_\ell^H = [\underline{I}+\gamma_1\underline{C}]^{-1} * [\sum_{n=2}^{\ell} \sum_{Qn} \gamma_n \underline{C} \circ (\underline{K}_{i_1} \ldots \underline{K}_{i_n})] \quad \text{for } \ell \geq 2 \tag{66}$$

and $$\underline{G}_1^H = [\underline{I}+\gamma_1\underline{C}]^{-1} * \gamma_1\underline{C} \tag{67}$$

From equations (43) and (67)

$$\phi_{uw_1}^H(\sigma) = G_1^H(\sigma) = [\underline{I}+\gamma_1\underline{C}]^{-1} * \gamma_1\underline{C} \tag{68}$$

hence the linear dynamics $\gamma_1 C_1$ can be identified and the results of Section 6 can be employed to estimate the non-linear characteristic.

8. SIMULATION RESULTS

The identification procedure outlined above was used to identify a general, Wiener and a Hammerstein model under unity feedback. All the models were simulated on an ICL 1906S digital computer and in each case 10,000 data points were generated by recording the response to a Gaussian input sequence.

A general model consisting of a linear system

$$B(z^{-1}) = \frac{0.2z^{-1}}{1-0.88z^{-1}} \tag{69}$$

in cascade with a non-linear element

$$y(t) = x(t) + 0.4x^2(t) + 0.2x^3(t) \tag{70}$$

and another linear system

$$C(z^{-1}) = \frac{0.3z^{-1}}{1-0.7z^{-1}} \tag{71}$$

in a unity negative feedback loop was simulated. The system response was recorded for eight amplitude levels of input, $\alpha_i u_2(t)$, $i = 1,2...8$ where $u_2(t)$ is a white Gaussian process N{0.4,0.8} and $\alpha_j = \alpha_{j-1} - 0.04$, $\alpha_1 = 1.0$. A comparison of the estimated impulse responses and the theoretical weighting sequences of the linear subsystems are illustrated in Fig.6 and the estimated parameters are summarised in Table 1.

A Wiener model under unity feedback consisting of a linear system

$$B(z^{-1}) = \frac{0.1z^{-1}}{1-1.62z^{-1}+0.7z^{-2}} \tag{72}$$

in cascade with the non-linear element

$$y(t) = x(t) + 0.7x^2(t) + 0.8x^3(t) \tag{73}$$

was simulated by recording the system response to a six level input, $\alpha_i u_2(t)$, $i = 1,2...6$ where α_i and $u_2(t)$ are defined as above. The theoretical weighting sequence and the estimated impulse response of the linear subsystem are compared in Fig.7.

A Hammerstein model consisting of the non-linear element

$$y(t) = x(t) + 0.4x^2(t) + 0.2x^3(t) \tag{74}$$

in cascade with the linear subsystem

$$C(z^{-1}) = \frac{0.1z^{-1}}{1-1.6z^{-1}+0.67z^{-2}} \tag{75}$$

under unity negative feedback was simulated for a six level input $\alpha_i u_2(t)$ where $u_2(t)$ is a white Gaussian process N{0.2,0.4}. A comparison of the estimated and theoretical linear subsystem impulse responses are illustrated in Fig.8.

Inspection of the estimated system parameters, summarised in Table 1 for all the models, clearly demonstrates the effectiveness of the algorithm.

9. CONCLUSIONS

An algorithm for the identification of non-linear unity negative feedback systems in terms of the individual component subsystems has been presented. When the input is separable and comprises the summation of a multilevel white Gaussian process and a non-zero mean the identification of the linear and non-linear elements can be decoupled using correlation analysis. Estimates of the linear subsystem pulse transfer functions can then be obtained using a multistage least squares algorithm and the coefficients in the polynomial representation of the non-linear element can be readily computed.

Although the algorithm utilizes the structural properties of the first two kernels in the Volterra series expansion for this class of system, characterization in terms of these kernels is avoided and truncation errors are not incurred. Identification in terms of the individual elements of the original system permits the components to be synthesised in a manner which preserves the system structure and provides valuable information for control. This approach overcomes many of the disadvantages associated with black-box identification techniques and provides a very concise description of the process.

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	Parameter	$n_{1,1}$	$n_{1,2}$	$d_{1,1}$	$d_{1,2}$	$n_{2,1}$	$n_{2,2}$	$d_{2,1}$	$d_{2,2}$	γ_1	γ_2	γ_3
General model	Estimate	0.1963	-	-0.8845	-	0.303	-	-0.6912	-	1.0934	0.4141	0.2579
	Theoretical	0.2	-	-0.88	-	0.30	-	-0.70	-	1.0	0.40	0.20
Wiener model	Estimate	0.1015	-0.00158	-1.62071	0.7002	-	-	-	-	1.0	0.679	0.895
	Theoretical	0.10	0.0	-1.62	0.70	-	-	-	-	1.0	0.70	0.80
Hammer- stein model	Estimate	-	-	-	-	0.1014	-0.00214	-1.6021	0.6711	1.0	0.399	0.211
	Theoretical	-	-	-	-	0.10	0.0	-1.60	0.67	1.0	0.40	0.20

Table 1 Summary of the Identification Results

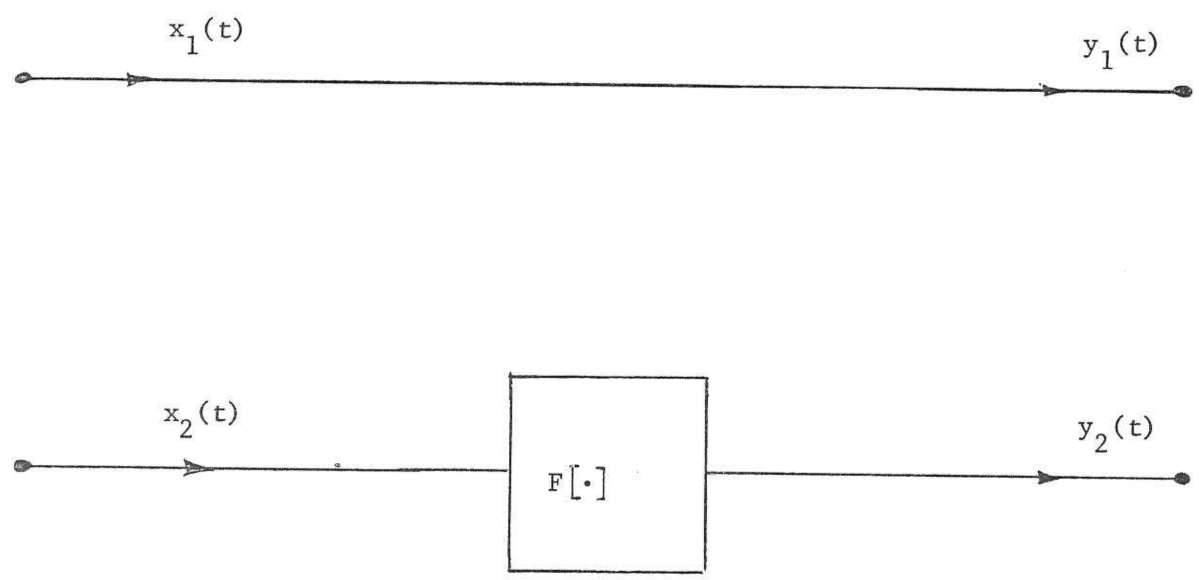


Fig. 1 Non-linear no-memory system

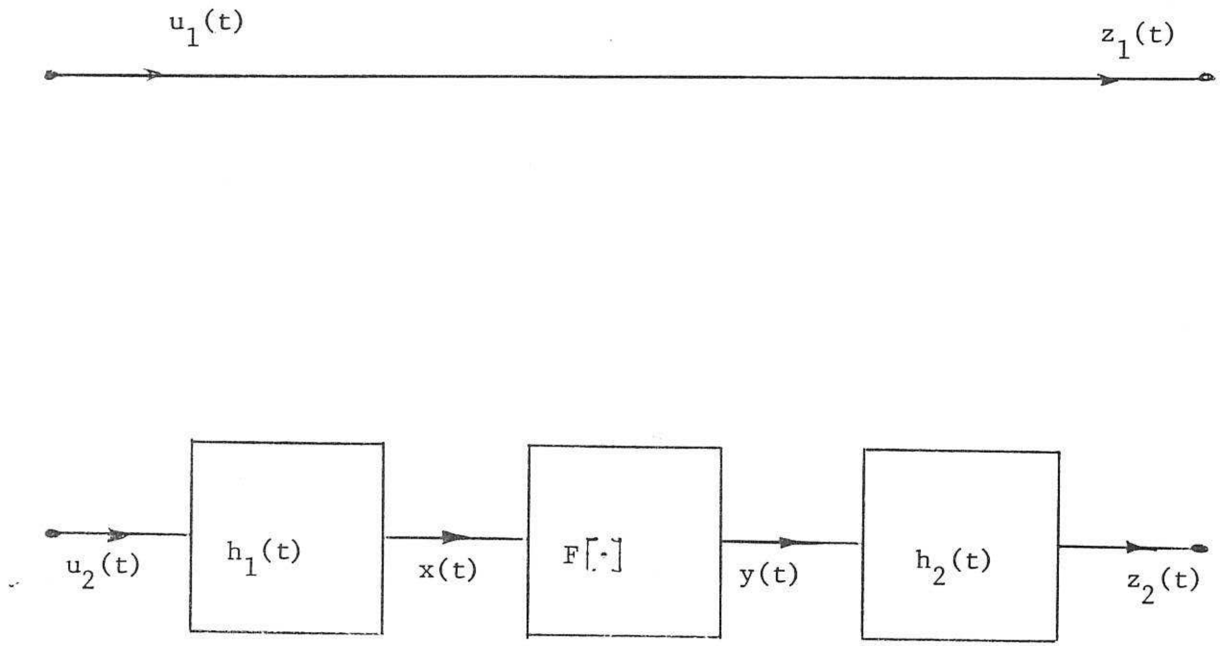


Fig. 2 The General Model

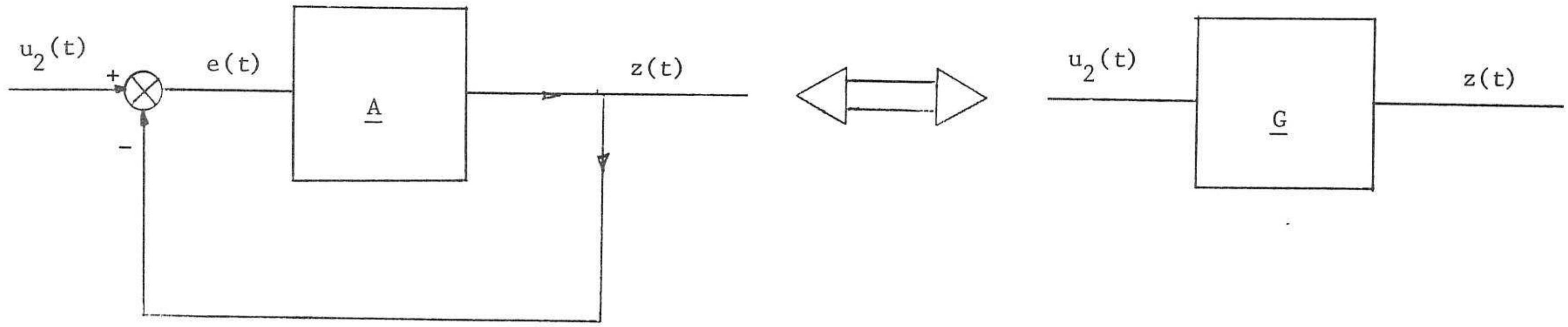


Fig. 3 Non-linear feedback system

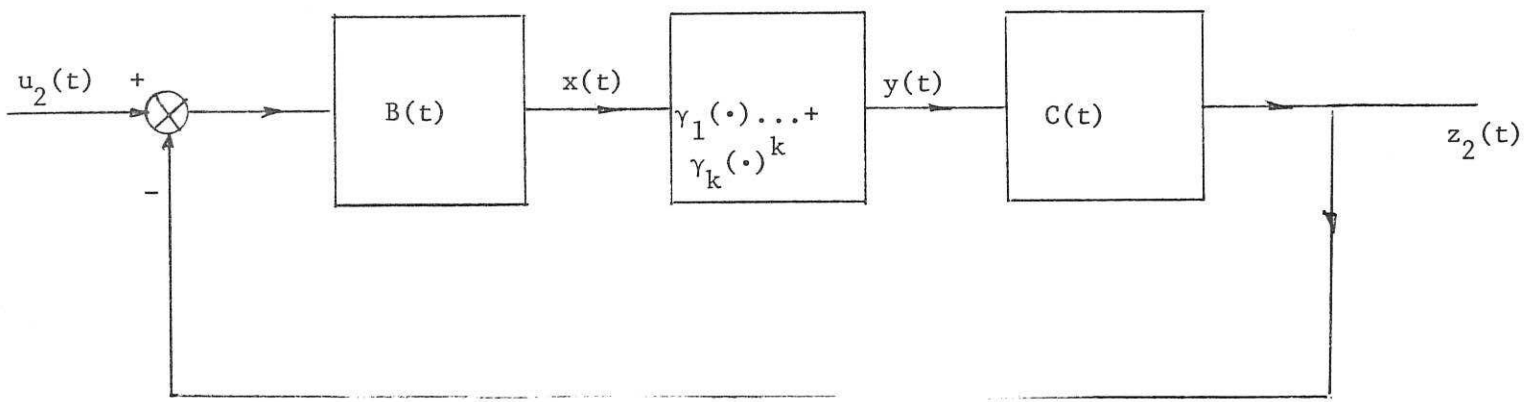


Fig. 4 The General Model under Unity Feedback

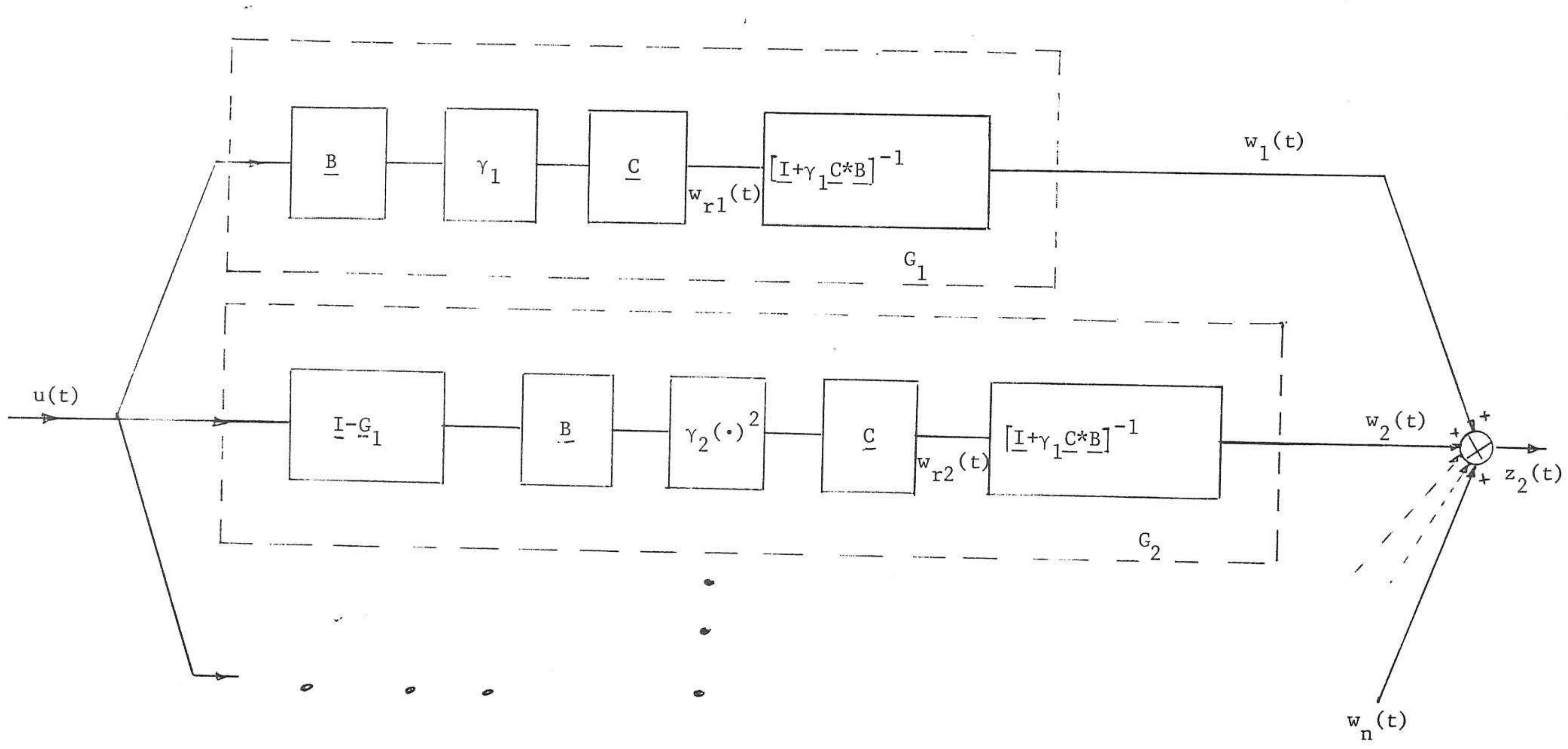
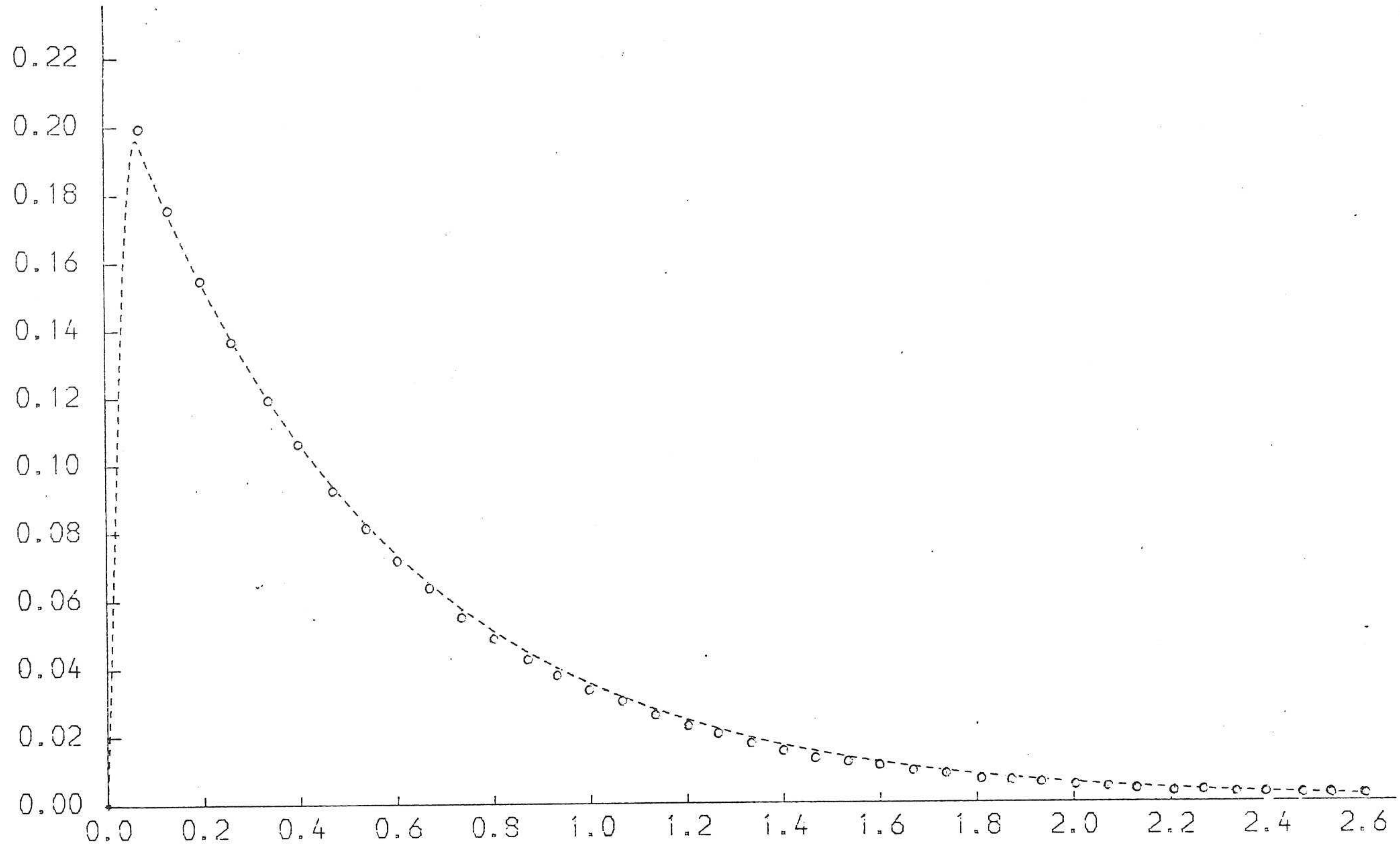


Fig. 5 Volterra series expansion of a nonlinear feedback system



o o o Theoretical response $h_1(k)$ * * * Theoretical response $h_2(k)$

- - - Estimated response $\hat{h}_1(k)$ — - — Estimated response $\hat{h}_2(k)$

Fig. 6a A comparison of impulse responses for the general model

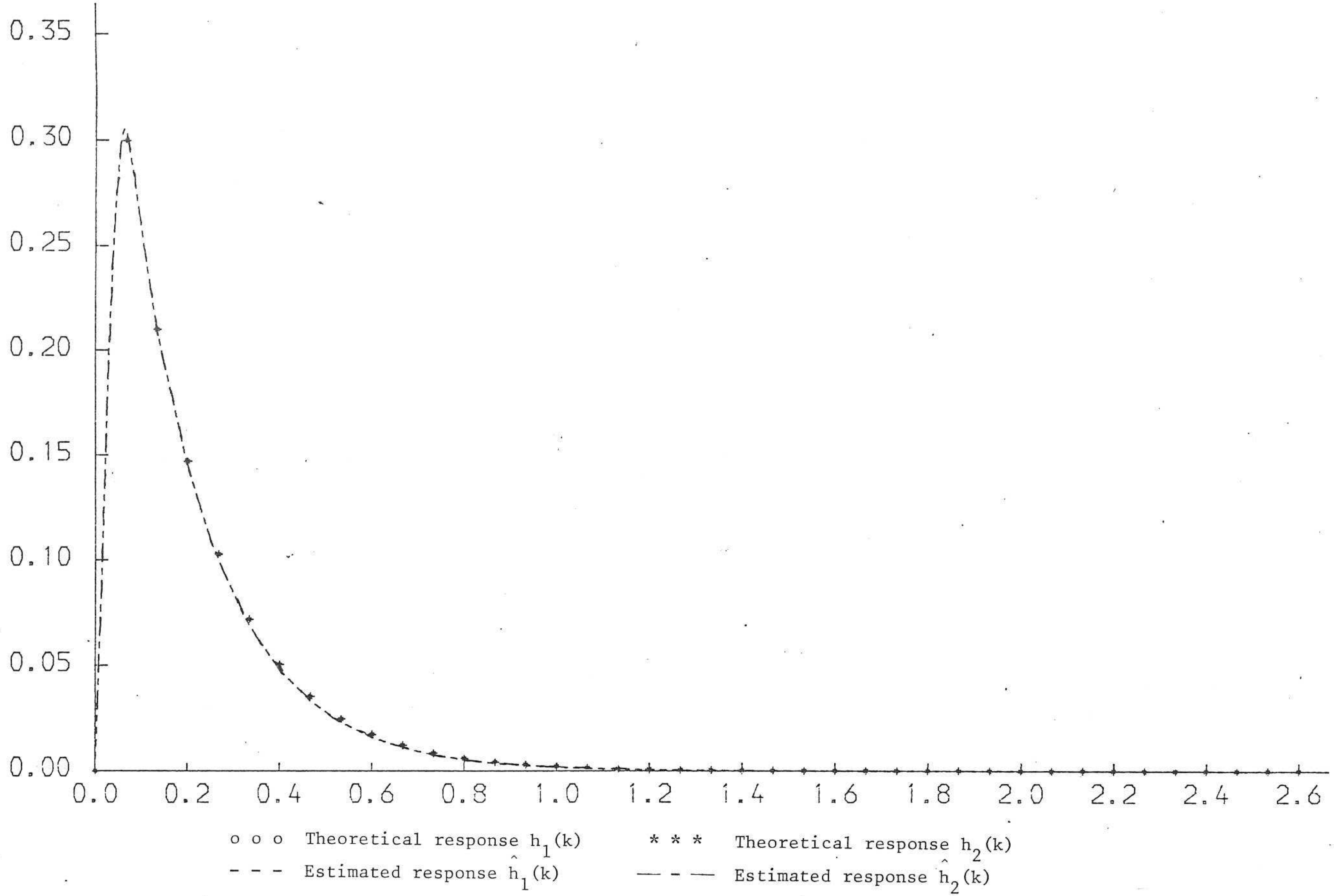
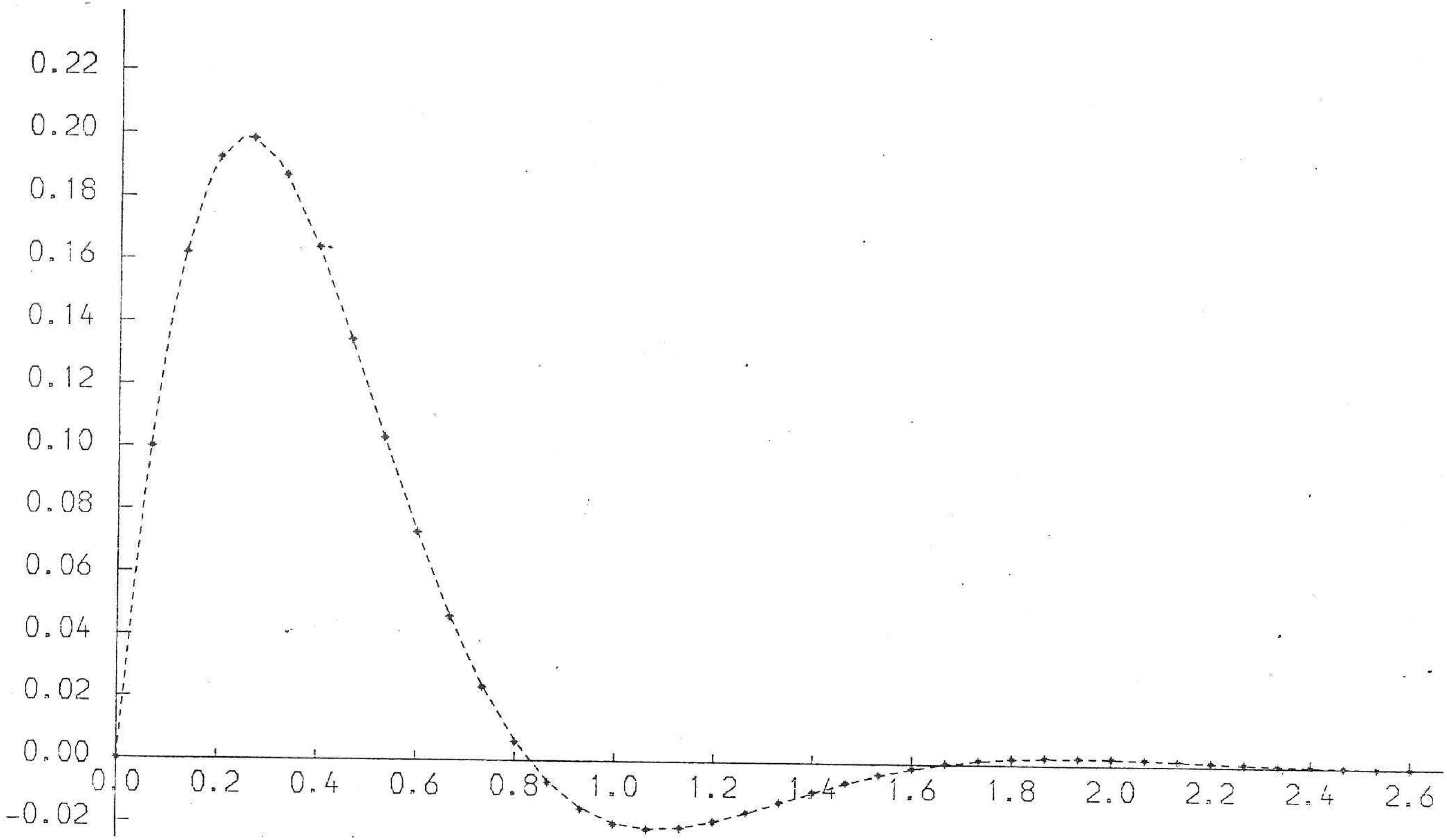


Fig. 6b A comparison of impulse responses for the general model



* * * Theoretical response

\- - - Estimated values

Fig. 7 A comparison of impulse responses for the Wiener model

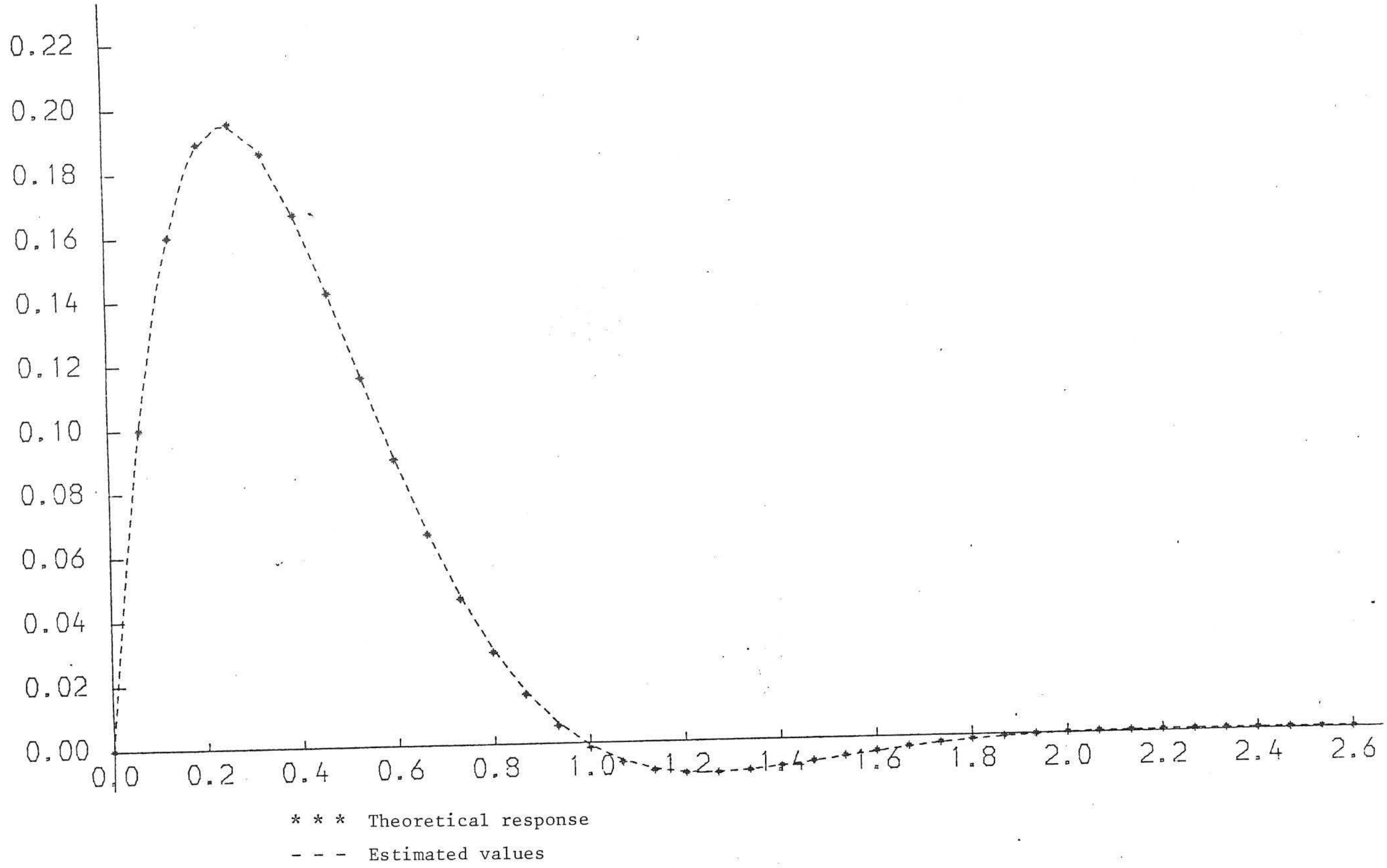


Fig. 8 A comparison of impulse responses for the Hammerstein model