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An Approach to Optimal Control for Engineering Undergraduates II

by

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Abstract

The paper continues an approach to the teaching of optimal control to engineering undergraduates and extends the linear cost results to the case of minimum energy/minimum fuel type problems. The treatment is rigorous but simple and gives direct insight into essential optimization concepts and the difficulties arising in numerical optimization methods.

1. Introduction

In a companion paper[1] an outline of an approach to the teaching of optimal control theory in the presence of hard constraints to undergraduate engineering students at the University of Sheffield was presented, and introductory material presented on the optimization of linear systems with performance criterion of the form (T fixed)

$$J(u) = \langle \alpha, x(T) \rangle + \int_0^T \{ \langle \beta(t), x(t) \rangle + g(u(t),t) \} dt$$

The advantages of initially considering this restricted class of optimization problem were demonstrated to be as follows:-

(1) The approach combines rigorous but simple undergraduate engineering mathematics with fundamental techniques used in the derivation of numerical optimization algorithms to attain a conceptual level convincing to the good students but attainable in problem solving by the weaker students. This technique avoids the 'statement without proof' approach of most introductory texts[2].

(2) The linear cost problem has a particularly simple structure[1] which allows the student to practice the idea of Hamiltonian minimization in the presence of a variety of control constraints before he moves in to the more difficult problems requiring simultaneous Hamiltonian minimization and satisfaction of the terminal state and costate boundary conditions (e.g. minimum energy problems).

(3) The mathematical source and interpretation of the costate and Hamiltonian are simple offshoots of the analysis and simple problems soon give the student a feel for the dependence of the optimal controller on the control constraints.

All these points prepare the student for a general statement of the Minimum Principle at a postgraduate level and hopefully convince him that the more indigestible parts do in fact have a firm mathematical foundation.

This paper completes the discussion by extending the approach to the solution of minimum energy optimization problems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad , \qquad x(o) = x_o \ , \ x(T) = x_f \qquad (1)$$

where $u(t)$ is restricted to lie in a given control restraint set Ω. The performance criterion (T fixed) is

$$J(u) = \int_o^T g(u(t),t)dt \qquad (2)$$

The approach is similar to that of the linear cost problem[1] and emphasizes the way that the costate and Hamiltonian conditions arise naturally from the analysis. Subsequent problem solving leads naturally to an elementary discussion of numerical optimization ideas and illustrate the following points:-

(1) Difficulties arising in the TPBVP due to the need to simultaneously satisfy the terminal state boundary conditions and also minimize the Hamiltonian.

(2) The possible non-existence of the optimal control due to the control constraints.

(3) The need to apply common sense when applying all but the more straightforward results of optimal control theory.

2. Minimum Energy/Minimum Fuel Optimization Problems

If $u_1(t)$, $u_o(t)$ are admissible controllers generating state trajectories $x_1(t), x_o(t)$ satisfying the state boundary conditions, then we can compare[1] the performances of the two controllers by considering

$$J(u_1) - J(u_o) = \int_o^T \{g(u_1(t),t) - g(u_o(t),t)\}dt \qquad (3)$$

If $p(t)$ is an arbitrary differentiable vector function then, from the state equations,

$$\int_0^T \langle p(t), A(t)\{x_1(t) - x_o(t)\} + B(t)\{u_1(t) - u_o(t)\} - \{\dot{x}_1(t) - \dot{x}_o(t)\}\rangle dt = 0 \tag{4}$$

Using the simple properties of the inner product[1] and integration by parts in the last term of equation (4) we obtain,

$$\int_0^T \langle A^T(t)p(t) + \dot{p}(t),\ x_1(t) - x_o(t)\rangle dt$$

$$+ \int_0^T \langle B^T(t)p(t),\ u_1(t) - u_o(t)\rangle dt - \Big[\langle p(t), x_1(t) - x_o(t)\rangle\Big]_o^T = 0 \tag{5}$$

Adding the left-hand-side of this equation to the right-hand-side of equation (3), and noting that the state boundary conditions imply that

$$\Big[\langle p(t),\ x_1(t) - x_o(t)\rangle\Big]_o^T = 0 \tag{6}$$

we obtain

$$J(u_1) - J(u_o) = \int_0^T \langle A^T(t)p(t) + \dot{p}(t),\ x_1(t) - x_o(t)\rangle dt$$

$$+ \int_0^T \{H(x_o(t),p(t),u_1(t),t) - H(x_o(t),p(t),u_o(t),t)\}dt \tag{7}$$

where the Hamiltonian function is defined by

$$H(x,p,u,t) = g(u,t) + \langle p, A(t)x + B(t)u\rangle \tag{8}$$

Using the arguments of the previous paper[1] we choose the vector function $p(t)$ to eliminate the dependence of $J(u_1) - J(u_o)$ on $x_1(t)$ i.e.

$$\dot{p}(t) = -A^T(t)p(t) \tag{9}$$

which is the standard costate equation for this class of optimization problem. The students were rather surprised that the boundary conditions on the costate do not fall naturally out of the analysis. They were easily convinced by the argument that to specify the costate boundary conditions when both initial and final states are prespecified would over-specify the TPBVP.

Using equation (9), equation (7) reduces to

$$J(u_1) - J(u_o) = \int_o^T \{H(x_o(t),p(t),u_1(t),t) - H(x_o(t),p(t),u_o(t),t)\}dt \quad (10)$$

so that, if u_o is an optimal controller then, for any other admissible control u_1 we must have

$$J(u_1) \geqslant J(u_o) \quad (11)$$

It now follows naturally that a sufficient condition for $u_o(t)$ to be an optimal controller is that

$$H(x_o(t),p(t),u_o(t),t) = \min_u H(x_o(t),p(t),u,t) \quad (12)$$

where the minimization is performed with respect to the control restraint set Ω. The proof that this is necessary is non-trivial and the students had to accept this point.

The TPBVP to be solved for the optimal controller now becomes

$$\dot{x}_o(t) = A(t)x_o(t) + B(t)u_o(t) \qquad x_o(o) = x_o,\ x_o(T) = x_f \quad (13)$$

$$\dot{p}(t) = -A^T(t)p(t) \qquad p(T) \text{ unspecified} \quad (14)$$

$$H(x_o(t),p(t),u_o(t),t) = \min_u H(x_o(t),p(t),u,t) \quad (15)$$

It was helpful to the students at this point of the course to distribute a step by step solution method. The wording of the sheet is summarized below and illustrates how computer solution of the TPBVP arises naturally in the discussion:-

STEP 1: Solve the costate equations for a guess boundary condition $p(o) = \pi$.

STEP 2: For this guess costate solution, calculate the controller $u_\pi(t)$ which minimizes the Hamiltonian in $0 \leqslant t \leqslant T$.

STEP 3: Using π as a parameter vector, adjust π until $u_\pi(t)$ drives $x(o)$ to x_f in $0 \leqslant t \leqslant T$. In analytical studies this can sometimes be achieved algebraically. In more complex problems (e.g. high state dimension $n \geqslant 2$) computer solution may be necessary using a search technique on π.

It is emphasized that, at this point in the course, the students are well versed in the idea of minimizing the Hamiltonian and are hence in a good position to tackle the problem of simultaneously satisfying the terminal state boundary condition.

3. A Typical Worked Example

$$\dot{x}_1(t) = x_2(t) \qquad x_1(o) = 0\,, \quad x_1(1) = 1$$

$$\dot{x}_2(t) = u(t) \qquad x_2(o) = x_2(1) = 0 \tag{16}$$

$$|u(t)| \leqslant M \tag{17}$$

$$J(u) = \int_o^1 |u(t)|dt \tag{18}$$

The costate equations are

$$\dot{p}_1(t) = 0$$

$$\dot{p}_2(t) = -p_1(t) \tag{19}$$

Assume the boundary conditions $p_1(o) = \pi_1$, $p_2(o) = \pi_2$ from which

$$p_2(t) = \pi_2 - \pi_1 t \tag{20}$$

The Hamiltonian is

$$H = p_2(t)u(t) + p_1(t)x_2(t) + |u(t)| \tag{21}$$

Using graphical arguments the students readily appreciate that the controller minimizing the Hamiltonian is

$$u_o(t) = -M \text{ dez } p_2(t) \tag{22}$$

where the function dez is given by

$$\text{dez } x = \begin{cases} 1 & ; \quad x > 1 \\ 0 & ; \quad |x| < 1 \\ -1 & ; \quad x < -1 \\ \text{indeterminate if} & x = \pm 1 \end{cases} \tag{23}$$

and hence as $p_2(t)$ is a linear function of time (equation 20), and if $\pi_1 \neq 0$ then $u_o(t)$ must take a form similar to one shown in figure 1. It is at this stage in the analysis that the lecturer involved can demonstrate that common sense plays an important role in the solutions of practical optimization problems. Note that the boundary conditions on $x_2(t)$ demand that

$$\int_o^1 u_o(t)dt = 0 \tag{24}$$

so that, if u(t) is not identically zero (which is automatically satisfied if we are to satisfy the state boundary conditions), then $u_o(t)$ must have both positive and negative pulses on the interval $0 \leqslant t \leqslant 1$ i.e. $u_o(t)$ must be of the form indicated in (a) or (b) of Fig. 1.

Write a trial controller in the form

$$u(t) = \begin{cases} kM & ; \quad 0 \leqslant t < t_1 \\ 0 & ; \quad t_1 \leqslant t < t_2 \\ -kM & ; \quad t_2 \leqslant t \leqslant 1 \end{cases} \tag{25}$$

where $k = \pm 1$. Then the boundary conditions in $x_2(t)$ imply that

$$\int_o^1 u(t)dt = kMt_1 - kM(1 - t_2) = 0 \tag{26}$$

i.e.

$$t_1 + t_2 = 1 \tag{27}$$

so that the control pulses are symmetrically placed about $t = \frac{1}{2}$. The corresponding $x_2(t)$ trajectory is

$$x_2(t) = \begin{cases} kMt & ; \quad 0 \leqslant t \leqslant t_1 \\ kMt_1 & ; \quad t_1 \leqslant t \leqslant 1-t_1 \\ kMt_1 - kM(t-t_1+1) & ; \quad 1-t_1 \leqslant t \leqslant 1 \end{cases} \tag{28}$$

From the state boundary condition on $x_1(t)$

$$x_1(1) = \int_o^1 x_2(t)dt = 1 > 0 \tag{29}$$

i.e. $k = 1$ and, from equations (28), (29),

$$Mt_1^2 + Mt_1(1 - 2t_1) = 1 \tag{30}$$

i.e. $$M(t_1^2 - t_1 + \frac{1}{M}) = 0 \tag{31}$$

or $$t_1 = \tfrac{1}{2}\{1 - \sqrt{1 - \frac{4}{M}}\} \tag{32}$$

Combining equations (25) and (32) with $k = 1$ we see that the optimal controller has been found without an explicit solution of the costate equations. This example illustrates to the students how the minimum principle can indicate the form of the optimal controller, leaving the actual calculation of the control to a parameter search on 'switching times'.

As a final point, equation 32 indicates that the 'switch time' t_1 is well defined in $0 \leq t \leq \frac{1}{2}$ for all $M \geq 4$, but that t_1 becomes complex if $M < 4$. The students readily appreciate that this situation corresponds to the case when no optimal controller exists due to the fact that no admissible controller $u(t)$ can transfer the initial state to the final state in the prescribed time. The observation is a useful introduction to the idea that control constraints do affect the attainable performance and impose a minimum time for the accomplishment of a given task.

4. A Typical Problem Sheet Example

The example of section 3 is rather complex for examination purposes and is restricted to the lectures themselves or tutorials. For the purpose of problem sheets, the following type of example has been found to illustrate fundamental principles without requiring too long a time for completion.

$$\dot{x}(t) = -x(t) + u(t) \qquad x(o) = 0,\ x(1) = 1 \tag{33}$$

$$|u(t)| \leq M \tag{34}$$

$$J(u) = \int_0^1 |u(t)|\,dt \tag{35}$$

The costate equation is

$$\dot{p}(t) = p(t) \tag{36}$$

Assuming an initial condition $p(o) = -\pi$ it follows that $p(t) = -\pi e^t$. The Hamiltonian is

$$H = |u(t)| + p(t)u(t) - p(t)x(t) \tag{37}$$

with minimizing controller, if $k = \pm 1$,

$$u_o(t) = -M \text{ dez } p(t)$$

$$= \begin{cases} 0 & 0 \leqslant t < t_1 \\ kM & t_1 \leqslant t \leqslant 1 \end{cases} \tag{38}$$

From physical considerations we require $k = +1$. The problem can now proceed using t_1 is a parameter to satisfy the terminal state boundary condition.

5. Conclusions

Together with a companion paper, this paper has outlined an approach to the teaching of optimal control in the presence of hard constraints to undergraduate engineering students. The course is designed to avoid the 'statement without proof' techniques of typical undergraduate texts[(2)] by providing a rigorous treatment of a restricted but useful class of problems using only typical undergraduate engineering mathematics. The advantage of this formulation is a convincing introduction to general principles for the better student at a level attainable in problem solving by the weaker students. In addition, the approach gives a simple insight into the mathematical source and meaning of such concepts as costate, Hamiltonian, existence etc. The author believes that the techniques and concepts used and discussed in the course lay a firm foundation for postgraduate studies of the 'general' Minimum Principle and the discussion of gradient-type numerical optimization methods.

After an initial period coming to grips with the new concepts and matrix manipulations, the students received the course enthusiastically (one student undertaking a project to implement a gradient algorithm for the solution of a nuclear reactor optimization problem). The linear cost problem[1] gave them practice in the idea of Hamiltonian minimization for a variety of control constraints and although worked examples similar to that of section 3 initially created difficulties, practice with problem sheet examples similar to that of section 4 soon eased the situation. Together with several problem-sheets, handouts were prepared summarizing the essential elements of each technique for problem-solving. In this way the main conclusions of the analysis could be separated and comparisons readily made between the various types of problem.

References

(1) D. H. Owens: 'An approach to optimal control for engineering undergraduates I', Department of Control Engineering, Sheffield University, Research Report No. 21.

(2) H. A. Prime: 'Modern Concepts in Control Theory', McGraw-Hill, London, 1969.

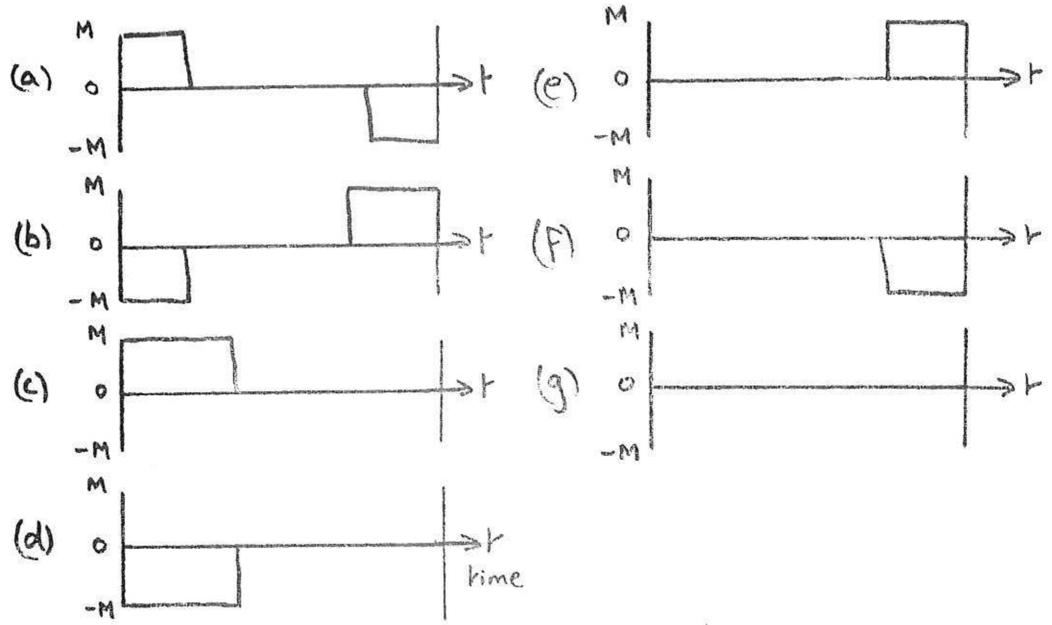

Fig. 1. Candidates for the Optimal Controller.