On the Stable Degree of Graphs

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Abstract. We define the stable degree $s(G)$ of a graph G by $s(G) = \min_U \max_{v \in U} \mathrm{d}_G(v)$, where the minimum is taken over all maximal independent sets U of G. For this new parameter we prove the following. Deciding whether a graph has stable degree at most k is NP-complete for every fixed $k \geq 3$; and the stable degree is hard to approximate. For asteroidal triple-free graphs and graphs of bounded asteroidal number the stable degree can be computed in polynomial time. For graphs in these classes the treewidth is bounded from below and above in terms of the stable degree.

1 Introduction

An *asteroidal triple*, or AT for short, is a set of three pairwise non-adjacent vertices in a graph such that any two of them are connected by a path that avoids the neighbourhood of the third. Graphs without asteroidal triples are *AT-free* [7]. This applies for instance to interval graphs, the intersection graphs of intervals on the real line. More precisely, interval graphs are exactly the chordal AT-free graphs [13]. Unlike the subclass of interval graphs, the whole class of AT-free graphs is not contained in the class of perfect graphs. For instance, C_5, the chordless cycle on five vertices, is AT-free, but not perfect. AT-free graphs form an interesting class of graphs due to their structural properties and also when studying the complexity on AT-free graphs for problems being NP-complete in general [5, 12].

An independent set of vertices in a graph is *asteroidal* if each three-element subset forms an AT [11]. The maximal size of an asteroidal set in a graph is its *asteroidal number*. Lots of the polynomial time algorithms for AT-free graphs, *i.e.* graphs of asteroidal number at most two, generalise to graphs of bounded asteroidal number [6, 5].

The *treewidth* is a parameter that measures the tree-likeness of a graph. Definitions are given in 2.2. Lots of the polynomial time algorithms for trees generalise to graphs of bounded treewidth. This applies to all problems that can be defined in monadic second order logic [8]. The *pathwidth* is a parameter similar to treewidth. For AT-free graphs, both parameters coincide [14], but are still hard to compute [1].

We introduce a new parameter, the stable degree of a graph. In Sections 3 and 4 we bound the treewidth in terms of the asteroidal number and stable degree.

In Section 5 we show that the stable degree is hard to compute in general, but if we restrict the input to AT-free graphs, or even graphs of bounded asteroidal number, then the stable degree can be computed by a polynomial time algorithm. As an immediate consequence, this enables new constant-factor approximations for the treewidth of AT-free graphs and graphs of bounded asteroidal number. In both cases these approximation algorithms are not better than the best known algorithms [3, 4], see Section 5.5.

2 Preliminaries

For a vertex v of a graph $G = (V, E)$ let $\mathrm{N}_G(v) = \{u \mid \{u, v\} \in E\}$ denote its *open neighbourhood*. The *closed neighbourhood* of v is $\mathrm{N}_G[v] = \{v\} \cup \mathrm{N}_G(v)$. Both concepts generalise to sets $U \subseteq V$ as follows: $\mathrm{N}_G[U] = \bigcup_{v \in U} \mathrm{N}_G[v]$ and $\mathrm{N}_G(U) = \mathrm{N}_G[U] \setminus U$. The *degree* of a vertex is the cardinality of its open neighbourhood, $\mathrm{d}_G(v) = |\,\mathrm{N}_G(v)|$. We omit the subscript G for neighbourhoods and degrees if there is no ambiguity about the graph G.

The set U is *independent* in G if $U \cap \mathrm{N}(u) = \varnothing$ for all $u \in U$, and U is *dominating* in G if $\mathrm{N}[U] = V$. An *independent dominating set* is both independent and dominating. An independent set is maximal (with respect to set inclusion) if and only if it is dominating.

2.1 Degrees

We introduce a new graph parameter based on the notion of degree. The *stable degree* of a graph G is defined by

$$s(G) = \min_{U} \max_{v \in U} \mathrm{d}_G(v)$$

where the minimum is taken over all maximal independent sets U of G.

We recall some parameters of a graph $G = (V, E)$ with complement $(V, \overline{E})$:

minimum degree $\delta(G) = \min\{\mathrm{d}_G(v) \mid v \in V\}$
2nd smallest degree $\delta_2(G) = 0$ if $|V| \leq 1$ and
$\delta_2(G) = \min\{\mathrm{d}_G(v) \mid v \in V \wedge \exists u \in V \setminus \{v\} \left(\mathrm{d}_G(u) \leq \mathrm{d}_G(v)\right)\}$ otherwise
degeneracy $d(G) = \max\{\delta(G[U]) \mid U \subseteq V\}$
Ramachandramurthi-bound $\gamma_{\mathrm{R}}(G) = |V| - 1$ if G is a complete graph and
$\gamma_{\mathrm{R}}(G) = \min\{\max\{\mathrm{d}_G(u), \mathrm{d}_G(w)\} \mid \{u, w\} \in \overline{E}\}$ otherwise
maximum degree $\Delta(G) = \max\{\mathrm{d}_G(v) \mid v \in V\}$

For all graphs G the following inequalities hold: $\delta(G) \leq \delta_2(G) \leq d(G) \leq \Delta(G)$, $\delta_2(G) \leq \gamma_{\mathrm{R}}(G) \leq \Delta(G)$ and $\delta_2(G) \leq s(G) \leq \Delta(G)$ [15]. For more information on these parameters and their use in lower bounding the treewidth of graphs we refer to [2].

2.2 Tree decomposition

A pair (X, T) is a *tree decomposition* of a graph $G = (V, E)$ if $T = (I, F)$ is a tree and $X : I \to 2^V$ maps the *nodes* of T to *bags*, *i.e.* subsets of V, such that

- for all $v \in V$ there is an $i \in I$ such that $v \in X(i)$,
- for all $e \in E$ there is an $i \in I$ such that $e \subseteq X(i)$
- for all $v \in V$, $T(v)$ is connected, where $T(v)$ is the subgraph of T induced by the $i \in I$ with $v \in X(i)$.

The *width* of (X, T) is $\max\{|X(i)| \mid i \in I\} - 1$, and the *treewidth* $\mathrm{tw}(G)$ of G is the minimal width of a tree decomposition of G.

The *pathwidth* $\mathrm{pw}(G)$ of G is the minimal width of a tree decomposition (X, T) of G where T is a path. For all AT-free graphs G we have $\mathrm{tw}(G) = \mathrm{pw}(G)$ by a result from [14].

In [15] Ramachandramurthi showed that γ_{R} is a lower bound on the treewidth. In Section 4 we use his idea to prove a lower bound in terms of the stable degree.

2.3 Asteroidal sets

A set $A \subseteq V$ is *asteroidal* in $G = (V, E)$ if for every vertex $u \in A$ there is a connected component $G[C]$ of $G - \mathrm{N}[u]$ containing $A \setminus \{u\}$. Consequently every asteroidal set is independent, and every independent set of size at most two is asteroidal. By $\mathrm{an}(G)$ we denote the *asteroidal number* of G that is the maximum cardinality of an asteroidal set in the graph G.

For different and non-adjacent vertices u and v of $G = (V, E)$ let $C(u, v)$ induce the connected component of $G - \mathrm{N}[u]$ containing v. We can use this notation to characterise asteroidal sets: an independent set A is asteroidal if and only if $C(u, v) = C(u, w)$ holds for every triple of different vertices $u, v, w \in A$.

The *interior* of an asteroidal set A in (V, E) is the subset of $V \setminus \mathrm{N}[A]$ of vertices that belong to the same connected component of $G - \mathrm{N}[u]$ as $A \setminus \{u\}$ for all $u \in A$. For $|A| > 1$ let $C(u, A)$ denote the set of vertices in this connected component, *i.e.* $C(u, A) = C(u, v)$ for all $v \in A \setminus \{u\}$. This enables us to define interior $I(A)$ formally by $I(A) = \bigcap_{u \in A} C(u, A)$. Furthermore we set $I(\varnothing) = V$ and $I(\{u\}) = V \setminus \mathrm{N}[u]$ for each vertex $u \in V$. A subset A of an asteroidal set B is asteroidal too, and we have $I(B) \cup (B \setminus A) \subseteq I(A)$ since $C(u, A) = C(u, B)$ for all $u \in A$.

A subset $A \subseteq D$ is a *cell* of the independent set D if A is asteroidal and $I(A) \cap D = \varnothing$. For two cells A and B of D, $A \subseteq B$ implies $A = B$ because $B \setminus A \subseteq I(A) \cap D$.

3 Upper bound on treewidth

Theorem 1. *For all non-empty graphs G we have* $\mathrm{tw}(G) < \mathrm{an}(G) \cdot s(G)$.

For $G = (V, \varnothing)$ we have $\mathrm{tw}(G) \leq 0$, $\mathrm{an}(G) = \min\{2, |V|\}$ and $s(G) = 0$.

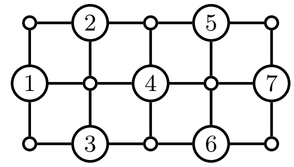

Fig. 1. $A = \{1, 2, 6, 7\}$ is an asteroidal set of this graph with interior $I(A) = \{3, 4, 5\}$. $A \cup \{4\}$ is independent but not asteroidal. Its cells are the sets $\{1, 2, 4\}$ and $\{4, 6, 7\}$.

Proof. Let D be a maximal independent set of $G = (V, E)$, and let $\mathcal{C}$ be the collection of cells of D. We construct a tree-decomposition (X, T) of G with $T = (\mathcal{C} \cup D, F)$ and X defined by

$$X(A) = \mathrm{N}(A) \quad \text{for all } A \in \mathcal{C} \qquad\qquad X(v) = \mathrm{N}[v] \quad \text{for all } v \in D\,.$$

If D was chosen such that $s(G) = \max\{\mathrm{d}_G(v) \mid v \in D\}$, we have $|X(v)| \leq \mathrm{d}_G(v) + 1$ for all $v \in D$, which implies $|X(v)| \leq s(G) + 1$. For all $A \in \mathcal{C}$ we have $|X(A)| \leq |A| \cdot s(G)$, and hence $|X(A)| \leq \mathrm{an}(G) \cdot s(G)$ since A is asteroidal. So the width of (X, T) will be less than $\mathrm{an}(G) \cdot s(G)$, since $E \neq \varnothing$ implies $s(G) \geq 1$.

It remains to show that for each D there is an F such that (X, T) is indeed a tree-decomposition of G. Since D is a maximal independent set of G we have $V = \bigcup_{v \in D} \mathrm{N}[v]$ and therefore $V = \bigcup_{i \in \mathcal{C} \cup D} X(i)$. We prove that (X, T) has the remaining properties of a tree-decomposition by induction on $|\mathcal{C}|$.

In the base case D is an asteroidal set of G. So we have $\mathcal{C} = \{D\}$. We make T a star with centre D and a leaf u for each vertex $u \in D$. Let $\{u, v\}$ be an edge of G. If there is a vertex $w \in \{u, v\} \cap D$ then we have $\{u, v\} \subseteq X(w)$. Otherwise there are vertices c and d in D that are adjacent to u and v because D is a dominating set of G. In this case we have $\{u, v\} \subseteq X(D)$. Next we prove that, for every vertex $v \in V$, the bags containing v induce a subtree $T(v)$ of T. This is obvious for $v \in D$ because $X(v)$ is the only bag containing v. Each vertex $v \in V \setminus D$ belongs to the central bag $X(D)$ and since T is a star, the subgraph $T(v)$ is connected.

In the inductive step there is a vertex $v \in D$ such that different connected components of $G - \mathrm{N}[v]$ contain vertices in D. That is, D is not asteroidal. Let $B_1, B_2, \ldots, B_k$ induce the connected components of $G - \mathrm{N}[v]$. For $j = 1, 2, \ldots, k$ we define $G_j = G[\mathrm{N}[v] \cup B_j]$, $D_j = \{v\} \cup (B_j \cap D)$, and $\mathcal{C}_j$ to be the set of cells of D_j in G_j. We have $D = \bigcup_{j=1}^{k} D_j$ and $\mathcal{C} \supseteq \bigcup_{j=1}^{k} \mathcal{C}_j$. Consider an asteroidal set $A \subseteq D$ that is not asteroidal in any G_j. Then A contains vertices in different connected components of $G - \mathrm{N}[v]$, which implies $v \in I(A)$. That is, $A \notin \mathcal{C}$ and therefore $\mathcal{C} = \bigcup_{j=1}^{k} \mathcal{C}_j$.

By induction hypothesis there is, for each $j = 1, 2, \ldots, k$, a set F_j of edges such that $T_j = (\mathcal{C}_j \cup D_j, F_j)$ is a tree, and the pair (X_j, T_j) is a tree-decomposition of G_j. Let $T = (\mathcal{C} \cup D, F)$ be the tree defined by $F = \bigcup_{j=1}^{k} F_j$.

We show that (X, T) is a tree-decomposition of G. For each edge $\{u, w\}$ of G there is an index j such that $\{u, w\}$ is an edge of G_j. By induction hypothesis there is an $i \in \mathcal{C}_j \cup D_j$ such that $\{u, w\} \subseteq X(i)$. Finally we show that $T(w)$ is a tree for every vertex $w \in V$. This is obvious for $w = v$ because $X(v)$ is the only bag containing v. For each vertex $w \neq v$ that is not adjacent to v there is a unique index $j \in \{1, 2, \ldots, k\}$ such that $w \in B_j$. All the bags containing w are contained in B_j, and by induction hypothesis the indices $\{i \mid w \in X(i)\}$ induce a subtree $T_j(w)$ of T_j. Clearly $T_j(w)$ is the subtree $T(w)$ of T. If w is adjacent to v then the elements of $I_j(w) = \{i \in \mathcal{C}_j \cup D_j \mid w \in X(i)\}$ induce a subtree $T_j(w)$ of T_j for every $j \in \{1, 2, \ldots, k\}$. Since $v \in I_j(w)$ for each j, the union of all the $T_j(w)$ is the tree $T(w)$, which is a subtree of T. □

Corollary 1. *For all non-empty AT-free graphs G we have* $\mathrm{pw}(G) < 2 \cdot s(G)$.

Proof. For all AT-free graphs G we have $\mathrm{pw}(G) = \mathrm{tw}(G)$ [14]. □

4 Lower bound on treewidth

In Lemma 1 we give the treewidth of chain graphs, which form a subclass of AT-free graphs. This result is used in the proof of Theorem 2, which provides a lower bound on the treewidth of a graph in terms of its stable degree and its asteroidal number.

4.1 Chain graphs

A connected bipartite graph $G = (A, B, E)$ is a *chain graph* if the vertices in A can be numbered $a_1, a_2, \ldots, a_p$ such that $\mathrm{N}(a_{i-1}) \supseteq \mathrm{N}(a_i)$ holds for all indices i with $1 < i \leq p$.

Let $G = (A, B, E)$ be a chain graph with $A = \{a_i \mid 1 \leq i \leq p\}$ and $B = \{b_j \mid 1 \leq j \leq q\}$ as above. We define $\Pi(G)$ to be the set of all pairs (s, t) with $1 < s \leq p$ and $1 \leq t < q$ such that $(a_s, b_{t+1}, a_{s-1}, b_t)$ is a P_4 of G, but not a C_4.

Lemma 1. *For every chain graph G with $\Pi(G) \neq \varnothing$ we have*
$\mathrm{tw}(G) = \min\{\mathrm{d}(a_s) + \mathrm{d}(b_t) - 1 \mid (s, t) \in \Pi(G)\}$.

We omit the proof due to space restrictions. A chain graph $G = (A, B, E)$ with $\Pi(G) = \varnothing$ is complete bipartite. In this case we have $\mathrm{tw}(G) = \min\{|A|, |B|\}$.

4.2 Construction

A tree decomposition is *small* if no bag is contained in another bag. If (X, T) is not small then T has an edge $\{i, j\}$ such that $X(i) \subseteq X(j)$ or $X(i) \supseteq X(j)$. We can *contract* the edge $\{i, j\}$ to obtain tree decomposition of the same graph and the same width, but with smaller index set I. To do so we choose a new index $l \notin I$, define $X(l) = X(i) \cup X(j)$, replace I by $\{l\} \cup I \setminus \{i, j\}$, and modify T such that $\mathrm{N}(l) = \mathrm{N}(\{i, j\})$. Iteration leads to a small tree decomposition.

Lemma 2. *Let (X, T) be a small tree decomposition of a graph G. Then G has a vertex that is contained in exactly one bag.*

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  input  : A tree decomposition (X,T) of width t of a graph G = (V,E) with
           an(G) ≤ a
  output : A maximal independent set D of G with d_G(v) ≤ at^2 for all v ∈ D
1 begin
2     D ← ∅;
3     while V ≠ ∅ do
4         while there is a contractible edge of (X,T) do contract it;
5         choose a vertex v ∈ V that appears in exactly one bag of (X,T);
6         D ← D ∪ {v}; V ← V \ N_G[v];
7         for i ∈ I do X(i) ← X(i) \ N_G[v]
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Theorem 2. *For all non-empty graphs G we have $s(G) \leq \mathrm{an}(G) \cdot \mathrm{tw}(G)^2$.*

Proof. Let $G = (V, E)$ be a graph and let (X, T) be its tree decomposition of width w. We consider the set $D \subseteq V$ constructed by the algorithm above. Throughout the algorithm (X, T) is a tree decomposition of the shrinking graph G, and the width of (X, T) does not increase.

The set D is independent in G because we remove in Line 6 the closed neighbourhood of v from G for every vertex v added to D. The algorithm terminates when $V = \varnothing$ holds. Therefore D is a maximal independent set of G.

To bound the degree of a vertex $v \in D$ we consider the sets $U = \mathrm{N}_G(v)$ and $W = \mathrm{N}_G(U)$, and define a partial order $\sqsubseteq$ on W such that $\mathrm{N}_G(w_1) \cap U \subset \mathrm{N}_G(w_2) \cap U$ implies $w_1 \sqsubseteq w_2$ for all vertices $w_1, w_2 \in W$. For different vertices $w_1, w_2 \in W$ with $\mathrm{N}_G(w_1) \cap U = \mathrm{N}_G(w_2) \cap U$ we ensure that $\sqsubseteq$ becomes antisymmetric by fixing $w_1 \sqsubset w_2$ or $w_2 \sqsubset w_1$ accordingly, for instance based on a given linear order on V.

The set U splits into new and old neighbours of v. The *new neighbours* are in the unique bag of (X, T) containing v when v is chosen. There are at most t new neighbours. The *old neighbours* are adjacent to v and a vertex w that was added to D before v. These old neighbours of v were new neighbours of w and removed from G together with w (Line 6).

Let $C \subseteq W$ be a maximal chain of $(W, \sqsubseteq)$, *i.e.* C is a set of $\sqsubseteq$-comparable vertices, and $\subseteq$-maximal with this property. Let B be the new neighbours in U of vertices in C. We define a bipartite graph $H = (B, C, F)$ with $F = E \cap \{\{b, c\} \mid b \in B, c \in C\}$. By maximality we have $v \in C$. Therefore H is a chain graph. We define subsets $B_1 \subseteq B$ and $C_1 \subseteq C$ as follows:

- If $\Pi(H) = \varnothing$ and $|B| \leq |C|$ then $B_1 = B$ and $C_1 = \varnothing$.
- If $\Pi(H) = \varnothing$ and $|B| > |C|$ then $B_1 = \varnothing$ and $C_1 = C$.
- If $(r, s) \in \Pi(H)$ and $\mathrm{tw}(H) = \mathrm{d}_H(b_r) + \mathrm{d}_H(c_s) - 1$ then $B_1 = \mathrm{N}_H(c_s)$ and $C_1 = \mathrm{N}_H(b_r)$.

In all three cases let $B_2 = B \setminus B_1$ and $C_2 = C \setminus C_1$. We have $|B_1| + |C_1| \leq t$ because H is a subgraph of G, which implies $\mathrm{tw}(H) \leq \mathrm{tw}(G)$. Moreover we have $|B_2| \leq t \cdot |C_1|$ since there is no edge of H with endpoints in B_2 and C_2, that is, all vertices in B_2 are new neighbours of vertices in C_1. This implies $|B| \leq t^2$ since $t \geq 1$ because of $E \neq \varnothing$.

Next let $A \subseteq W$ be an antichain of $(W, \sqsubseteq)$, *i.e.* A is a set of $\sqsubseteq$-incomparable vertices. For different vertices $w_1, w_2 \in A$ there is a vertex $u_2 \in \mathrm{N}(w_2) \cap U \setminus \mathrm{N}(w_1)$. It establishes a path (w_2, u_2, v) in $G - \mathrm{N}[w_1]$. Since such a path exists for all $w_2 \in A \setminus \{w_1\}$ we have $A \setminus \{w_1\} \subseteq C(w_1, v)$. Since this holds for all $w_1 \in A$ the set A is asteroidal in G. Consequently we have $|A| \leq a$ for every antichain A of $(W, \sqsubseteq)$.

By Dilworth's theorem W can be covered by k chains of of $(W, \sqsubseteq)$, where k is the maximum size of an antichain. U is the union of the B-sets for the chains in the cover. With $k \leq a$ this implies $d_G(v) \leq at^2$ for all $v \in D$. □

There might be a better lower bound on the stable degree:

Conjecture 1. For every graph G we conjecture $s(G) \leq \mathrm{an}(G) \cdot \mathrm{tw}(G)$.

For AT-free graphs we can prove this conjecture:

Theorem 3. *For every AT-free graph G we have $s(G) \leq 2\,\mathrm{tw}(G)$.*

Proof. We assume a tree decomposition (X, T) of G where $T = (I, F)$ is a path with $I = \{1, 2, \ldots, \ell\}$ and $F = \{\{i-1, i\} \mid 1 < i \leq \ell\}$. We construct D by the algorithm as before and choose v always from the bag indexed by the maximum leaf of T. Let $l(v) = \min\{i \in I \mid v \in X(i)\}$ and $r(v) = \max\{i \in I \mid v \in X(i)\}$ for each vertex v. To prove $s(G) \leq 2\,\mathrm{tw}(G)$ it suffices to show $\mathrm{N}[v] \subseteq X(l(v)) \cup X(r(v))$ for all $v \in D$. Assume a neighbour $v' \in \mathrm{N}(v) \setminus (X(l(v)) \cup X(r(v)))$. Then v' and v belong to a bag $X(i)$ with $l(v) < i < r(v)$, contradicting the fact that v appears in exactly one bag when chosen. □

5 Computing the stable degree

5.1 Polynomial cases: $k \leq 2$

We define the decision problems SD and k-SD for every $k \in \mathbb{N}$ by

$$\mathrm{SD} = \{(G, k) \mid s(G) \leq k\} \qquad k\text{-SD} = \{G \mid s(G) \leq k\}\,.$$

Lemma 3. *The problem k-SD can be solved in polynomial time for $k \in \{0, 1, 2\}$.*

Proof. If C induces a component of G then $s(G) \leqslant \max\{\Delta(G[C]), s(G - C)\}$. On this observation we base the following reduction rule for k-SD:

Low-degree component: If C induces a component in G with $\Delta(G[C]) \leq k$ then we replace G by $G - C$ because $G \in k\text{-SD} \iff G - C \in k\text{-SD}$.

In a graph $G = (V, E)$ with $s(G) \leq k$ the set $D_k = \{v \in V \mid \mathrm{d}(v) \leq k\}$ is dominating. For $k = 0$ and $k = 1$ this necessary condition for $G \in k$-SD is also sufficient. For $k = 2$, more reduction rules are required:

Pendant vertex: Let $(x, y_1, y_2, y_3, \ldots, y_l, z)$ be a path in G with $\mathrm{d}(x) = 1$, $\mathrm{d}(y_i) = 2$ for $i = 1, 2, 3, \ldots, l$ and $\mathrm{d}(z) > 2$. Then we replace G by the graph $G' = G - \{x, y_1, y_2, \ldots, y_l, z\}$ because $G \in$ 2-SD $\iff$ $G' \in$ 2-SD.

Long path: Let $(x, y_1, y_2, \ldots, y_l, z)$ be a path in G with $\mathrm{d}(x) > 2$, $\mathrm{d}(y_i) = 2$ for $i = 1, \ldots, l$, $\mathrm{d}(z) > 2$ and $l \notin \{0, 2\}$. Then $G \in$ 2-SD if and only if $G - \{x, y_1, y_2, \ldots, y_k, z\} \in$ 2-SD.

Unique neighbour: Let (x, y_1, y_2, z) be a path in G, with $\mathrm{d}(x) > 2$, $\mathrm{d}(y_1) = 2$, $\mathrm{d}(y_2) = 2$ and $\mathrm{d}(z) > 2$. If $\mathrm{N}(z) \cap D_2 = \{y_2\}$ then $G \in$ 2-SD if and only if $G - \{y_1, y_2, z\} \in$ 2-SD.

If none of these reduction rules apply then the minimum degree $\delta(G)$ of $G = (V, E)$ is at least two. We will show that our necessary condition for $G \in k$-SD is also sufficient. Let $X = \{v \in V \mid \mathrm{d}(v) > 2\}$ and $Y = \{v \in V \mid \mathrm{d}(v) = 2\}$. Clearly, if there is a vertex in X without neighbour in Y then $G \notin$ 2-SD. Otherwise we will show that $G \in$ 2-SD.

We construct an auxiliary bipartite graph $H = (X \cup Z, F)$ where Z is the set of edges of $G[Y]$ and $F = \{\{x, z\} \mid x \in X, z \in Z, \mathrm{N}_G(x) \cap z \neq \varnothing\}$. We have $\mathrm{d}_H(x) \geq 2$ for all $x \in X$ and $\mathrm{d}_H(z) = 2$ for all $z \in Z$. This implies $|\,\mathrm{N}_H(A)| \geq |A|$ for all $A \subseteq X$ and therefore H has an X-saturating matching. The X-saturating matching of H corresponds to an X-saturating matching M of G. The M-saturated vertices in Y form an independent set that dominates X. Therefore this subset extends to a maximal independent subset of Y, and we have $G \in$ 2-SD. □

5.2 Hardness for $k \geq 3$

Lemma 4. *For every $k \geq 3$, the problem k-SD is NP-complete.*

Proof. Clearly the problem k-SD is in NP. To show the NP-hardness we reduce from a restricted version of SAT, where every boolean variable x appears in at most two clauses positively, that is as x, and in at most two clauses negatively as $\overline{x}$ [16].

Let φ be a formula in CNF with this property. For each variable x_i appearing in φ, $1 \leq i \leq n$, we create a truth assignment component which is a $K_{2,2}$ with partite sets $\{x_i^1, x_i^2\}$ and $\{\overline{x}_i^1, \overline{x}_i^2\}$. For the clause c_j of φ, $1 \leq j \leq m$, we create a satisfaction test component which consists of a single vertex c_j. We add the edge $\{x_i^l, c_j\}$ if the clause c_j contains the lth appearance of the positive literal x_i, and we create the edge $\{\overline{x}_i^l, c_j\}$ if the clause c_j contains the lth appearance of the negative literal $\overline{x}_i$. We complete the construction of the reduction graph G by adding all edges $\{c_j, c_l\}$ for $j \neq l$.

Every vertex v in a truth assignment component of G has degree at most three, and every vertex in a satisfaction test component has degree at least m. We may assume $k < n < m$.

Let $a : \{x_1, x_2, \ldots, x_n\} \to \{\mathsf{true}, \mathsf{false}\}$ be a satisfying truth assignment of φ. Then $D = \{x_i^l \mid a(x_i) = \mathsf{true}, l \in \{1, 2\}\} \cup \{\overline{x}_i^l \mid a(x_i) = \mathsf{false}, l \in \{1, 2\}\}$ is a maximal independent set of G, and therefore $s(G) \leq k$.

On the other hand, let D be a maximal independent set of G with $\mathrm{d}_G(v) \leq k$ for all $v \in D$. By this degree condition D contains only vertices from truth assignment components of G. Since D is independent, these are either x_i^1 and x_i^2 or $\overline{x}_i^1$ and $\overline{x}_i^2$. We define $a(x_i) = \mathsf{true}$ if $x_i^1 \in D$ and $a(x_i) = \mathsf{false}$ if $\overline{x}_i^1 \in D$. Assume that a clause c_j is not satisfied. Then the vertex c_j has no neighbour in D, contradicting the fact that D is maximal. □

In fact the reduction shows that the stable degree is hard to approximate: $\varphi \in \mathrm{SAT}$ implies $s(G) \leq 3$ and $\varphi \notin \mathrm{SAT}$ implies $s(G) \geq m$. Since SAT remains NP-complete when restricted to formulae with more than m clauses (for fixed value of m), we have the following lemma.

Lemma 5. *There is no polynomial time algorithm approximating the stable degree by a constant factor, unless* P = NP*.*

5.3 Bounded cliquewidth

For fixed values of k the problem k-SD can be formulated in MSOL. Therefore its restriction to graphs of bounded tree- or cliquewidth can be solved in linear time [8, 9].

5.4 Bounded asteroidal number

In this subsection we develop an algorithm computing the stable degree of graphs of bounded asteroidal number, such as (unit) interval graphs or AT-free graphs, which have unbounded tree- and cliquewidth. We start with technical lemmas on connected components and cells. Remember that $C(u, v)$ induces the connected component of $G - \mathrm{N}[u]$ containing v.

Lemma 6. *For all independent triples* $\{u, v, w\}$ *of a graph,* $C(v, u) \neq C(v, w)$ *implies* $C(v, u) \subseteq C(w, u)$ *and* $C(v, w) \subseteq C(u, w)$*.*

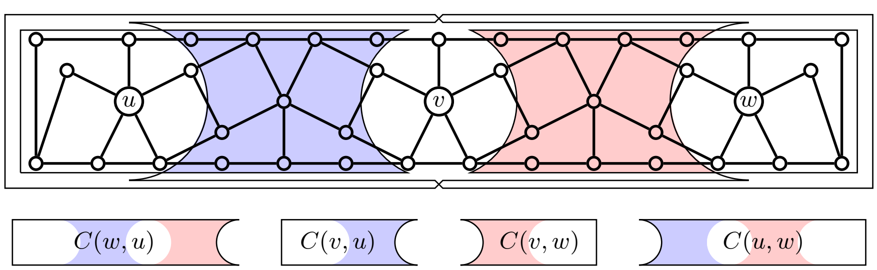

Fig. 2. An example illustrating Lemma 6.

Proof. Let x be a vertex in $C(v,u)$, which means there is a path $(x,\dots,u)$ in $G[C(v,u)]$. In G this path avoids $\mathrm{N}[w]$ since $C(v,u) \neq C(v,w)$. Therefore it exists in $G[C(w,u)]$ as well, and hence $C(v,u) \subseteq C(w,u)$ holds. By a symmetric argument we have $C(v,w) \subseteq C(u,w)$. □

Remember that an asteroidal subset A of an independent set D is a cell of D if the interior of A does not contain a vertex in D.

Lemma 7. *Let D be an independent set of $G=(V,E)$, $v \in V \setminus \mathrm{N}[D]$, and let B be the union of all subsets $A \subseteq D$ such that $A \cup \{v\}$ is a cell of $D \cup \{v\}$. Then B is a cell of D and $v \in I(B)$.*

Proof. Let $\mathcal{C}$ be the set of all subsets $A \subseteq D$ such that $A \cup \{v\}$ is a cell of $D \cup \{v\}$. For every $A \in \mathcal{C}$ and every $u \in A$ we have $A \setminus \{u\} \subseteq C(u,v)$ because $A \cup \{v\}$ is asteroidal. For vertices u and w in different sets in $\mathcal{C}$ we have $C(u,w) = C(u,v)$ and $C(w,u) = C(w,v)$ by Lemma 6. Hence B is an asteroidal set of G and $v \in I(B)$.

To show that B is a cell of D we assume a vertex $x \in I(B) \cap D$. Because $x \notin B$, the set $\{v,x\}$ is not asteroidal, and therefore we have $x \in \mathrm{N}[v]$, which contradicts $x \in D$ and $v \in V \setminus \mathrm{N}[D]$. □

Lemma 8. *Let B be a cell of an independent set D of $G=(V,E)$. Each vertex $v \in I(B)$ defines a partition $\mathcal{C}$ of B such that $A \cup \{v\}$ is a cell of $D \cup \{v\}$ for each $A \in \mathcal{C}$.*

Proof. Let $C_1, C_2, \dots, C_k$ induce the connected components of $G - \mathrm{N}[v]$. Then $\mathcal{C} = \{B \cap C_i \mid 1 \leqslant i \leqslant k\}$ is a partition of B, and for every set $A \in \mathcal{C}$, $A \cup \{v\}$ is asteroidal. By Lemma 7, $I(A \cup \{v\}) \subseteq I(B)$, which implies $I(A \cup \{v\}) \cap D = \varnothing$ because B is a cell of D. Since $v \notin I(A \cup \{v\})$ we conclude $I(A \cup \{v\}) \cap (D \cup \{v\}) = \varnothing$, and hence $A \cup \{v\}$ is a cell of $D \cup \{v\}$. □

Corollary 2. *1. Every vertex in an independent set D belongs to a cell of D.*
2. *For every independent set D of (V,E), $V \setminus \mathrm{N}[D]$ is the disjoint union of the interiors of the cells of D.*
3. *An independent set D is maximal independent if and only if $I(A) = \varnothing$ holds for every cell A of D.*

Let $G=(V,E)$ be a graph, A an asteroidal set of G, and let $B \subseteq I(A)$ induce some connected components of $G[I(A)]$. We define

$$s(A,B) = \min_{U} \max_{u \in U} \mathrm{d}_G(u)$$

where the minimum is taken over all maximal independent sets U of $G[B]$. Then $s(G) = s(\varnothing, V)$. The following recurrence allows us to compute the values of $s(A,B)$:

$$s(A,\varnothing) = 0$$
$$s(A,B) = \max_{D \in \mathcal{D}(B)} s(A,D) \qquad \text{if } G[B] \text{ is disconnected}$$
$$s(A,B) = \min_{v \in B} \max\left(\mathrm{d}_G(v), \max_{C \in \mathcal{C}(A \cup \{v\})} s(C, I(C))\right) \qquad \text{if } G[B] \text{ is connected}$$

where $\mathcal{C}(S)$ is the set of cells of the independent set S, and $\mathcal{D}(B)$ is the set of vertex sets that induce a connected component of $G[B]$. The recurrent equation above directly translates into an algorithm. The correctness follows from Lemmas 7 and 8.

A bottom-up dynamic programming algorithm first sorts all objects (A, B) by the cardinality of B and processes them in increasing order of $|B|$. It store the values $s(A, B)$ computed in a table that is an array indexed by B.

We bound the running time of the algorithm in terms of $k = \mathrm{an}(G)$, $n = |V|$ and $m = |E|$. The algorithm considers at most $\sum_{i=0}^{k} \binom{n}{i} = \mathcal{O}(n^k)$ asteroidal sets A of G. For each A it considers at most $n+1$ subsets $B \subseteq I(A)$, namely $B = I(A)$ and all set B that induce a connected component of $G[I(A)]$. For a fixed pair (A, B), it needs time $\mathcal{O}(n+m)$ to organise the look up of values computed before if $G[B]$ is disconnected. This is mainly for computing the connected components of $G[B]$. If $G[B]$ is connected, the algorithm minimises over $\mathcal{O}(n)$ vertices $v \in B$, and spends $\mathcal{O}(n + m)$ time per vertex v to organise the table look-up. That is, the algorithm runs in time $\mathcal{O}(n^{k+1}m)$.

Theorem 4. *For graphs G with $\mathrm{an}(G) \leq k$, $s(G)$ can be computed in time $\mathcal{O}(n^{k+1}m)$.*

5.5 Approximating treewidth

By Theorems 1 and 3 we have

$$\tfrac{1}{2} \cdot s(G) \leq \mathrm{tw}(G) < 2 \cdot s(G)$$

for AT-free graphs G, and in general Theorems 1 and 2 imply

$$\sqrt{s(G)/\mathrm{an}(G)} < \mathrm{tw}(G) < \mathrm{an}(G) \cdot s(G)\,.$$

These lower and upper bounds enable us to extend the algorithm from the previous subsection such that it approximates the treewidth of AT-free graphs by a factor of 4 in the worst case. In contrast, the algorithm developed in [3] guarantees an approximation factor of 2 for AT-free graphs. Theorem 1 and Conjecture 1 would imply

$$\frac{1}{\mathrm{an}(G)} \cdot s(G) \leq \mathrm{tw}(G) < \mathrm{an}(G) \cdot s(G)\,,$$

and the constant factor approximation would generalise to graphs G of bounded asteroidal number. Its ratio would be $\mathrm{an}(G)^2$ in the worst case, which would beat the $8\,\mathrm{an}(G)$ factor from [4] if the bound on the asteroidal number is less than eight. With Theorem 2 instead of Conjecture 1 we obtain an approximation ratio of $s(G)^{1/2}\,\mathrm{an}(G)^{3/2}$ via the stable degree.

6 Conclusions

Graph problems definable in MSOL can be solved in linear time when restricted to graphs of bounded treewidth [8]. We showed that inside AT-free graphs, bounded treewidth can be replaced by bounded stable degree. This allows us to concentrate on the hard cases when we consider problems on AT-free graphs for which the complexity status is still unknown, such as vertex colouring or Hamiltonicity.

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