Identification of systems composed of linear dynamic and static nonlinear elements

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IDENTIFICATION OF SYSTEMS COMPOSED OF LINEAR DYNAMIC AND STATIC NONLINEAR ELEMENTS

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Abstract

Identification of nonlinear systems which can be represented by combinations of linear dynamic and static nonlinear elements are considered. Previous results by the authors based on correlation analysis are combined to provide a unified treatment for this class of systems. It is shown that systems composed of cascade, feedforward, feedback and multiplicative connections of linear dynamic and zero memory nonlinear elements can be identified in terms of the individual component subsystems from measurements of the system input and output only.

1. INTRODUCTION

Various authors have studied the class of systems which can be represented by interconnections of linear dynamic and zero-memory nonlinear subsystems. A transform representation and rules for algebraic manipulation were developed by George (1959). The analysis and synthesis of cascade systems has been studied by Smets (1960) and Shanmugam and Lal (1976), and a structure theory was developed by Smith and Rugh (1974). Zames (1963) and Narayanan (1970) studied nonlinear feedback systems and numerous authors (Narendra and Gallman (1966);

Economakos (1971): Goldberg and Durling (1971); Gardner (1973); Korenberg (1973a,b); Webb (1974); Cooper and Falkner (1975); Haber and Keviczky (1976); Douce (1976); Sandor & Williamson (1978); De Boer (1979)), have considered the identification of systems within this class. A review of these algorithms and alternative approaches to nonlinear systems identification are contained in a recent survey paper (Billings 1980).

In the present study previous results (Billings and Fakhouri, 1978a,b) derived for the general model defined as a linear system in cascade with a static nonlinear element followed by another linear system are briefly reviewed. By considering the separable class of random processes it is shown that computation of the first and second order cross-correlation functions when the input is white Gaussian effectively decouples the identification of this class of non-linear systems into two distinct steps; identification of the linear sub-systems and characterisation of the non-linear element. The relationship between the first and second order correlation functions also provides valuable information regarding the system structure; notably the position of the nonlinear element with respect to the linear systems. Although the algorithms cannot be directly applied for pseudorandom inputs an alternative procedure (Billings and Fakhouri, 1980) based on compound pseudorandom excitation is presented and the selection of inputs is discussed.

The results are extended to include the identification of the component subsystems in nonlinear feedback systems (Billings and Fakhouri, 1979a), feedforward systems (Billings and Fakhouri, 1979b), systems containing multiplicative connections of linear dynamic elements (Billings and Fakhouri, 1979d) and other common system structures

(Billings and Fakhouri, 1979c). In all these cases the identification procedure provides estimates of the individual elements of the system such that the components can be synthesised in a manner which preserves the system structure and provides valuable information for control.

2. THE OPEN-LOOP GENERAL MODEL

The general model illustrated in Fig.1 consists of a linear system $h_1(t)$ in cascade with a zero memory nonlinear element $F[\cdot]$ followed by a second linear system $h_2(t)$. For generality it is assumed that the measured output contains an unknown additive noise component $v(t)$ and that the nonlinear element can, in theory, be represented by a polynomial $y(t) = \sum_{i=1}^{k} \gamma_i x^i(t)$. The identification problem can now be defined as identification of the individual components $h_1(t)$, $h_2(t)$ and a suitable representation of the static nonlinear element $F[\cdot]$ from measurements of the input $u(t)$ and noise corrupted output $z_2(t)$.

2.1 Separable Process Inputs

The identification algorithm for the general model is based on the theory of separable processes which is briefly reviewed below (Nuttall 1958, Billings and Fakhouri 1978b).

Let $p(\alpha,\beta;\tau)$ be the joint probability density function for the stationary processes $\alpha(t)$ and $\beta(t)$ and define

$$g(\beta,\tau) = \int_{-\infty}^{\infty} \alpha p(\alpha,\beta;\tau)d\alpha \qquad (1)$$

If the g-function in eqn (1) separates as

$$g(\beta,\tau) = g_1(\beta)g_2(\tau) \quad \forall \ \beta,\tau \qquad (2)$$

then $\alpha(t)$ is said to be separable with respect to $\beta(t)$. The separable class of random processes is fairly wide and includes the Gaussian process, sinewave process, phase or frequency modulated process, squared Gaussian process etc.

Previous research has shown that separability is not in general preserved under either linear or double non-linear transformation. However both properties hold for the special case where $\alpha(t)$ is Gaussian and separable with respect to $\beta(t)$.

The importance of separable processes in the identification of nonlinear systems becomes apparent when analysing the system illustrated in Fig.2 where $F[\cdot]$ is the transfer characteristic of an instantaneous nonlinear element. It can readily be proved that the separability of $x_1(t)$ with respect to $x_2(t)$ is a necessary and sufficient condition for the invariance property

$$\phi^{F}_{y_1 y_2}(\tau) = C_F \phi_{x_1 x_2}(\tau) \ \forall \ F \text{ and } \tau \qquad (3)$$

to hold where C_F is a constant. When $x_1(t) = x_2(t) = x(t)$ which is separable, eqn (3) relates the input/output cross-correlation function to the input autocorrelation function

$$\phi^{F}_{x_1 y_2}(\tau) = C_{F_x} \phi_{xx}(\tau) \ \forall \ F \text{ and } \tau \qquad (4)$$

where

$$C_{F_x} = \frac{1}{\phi_{xx}(0)} \int xF[x]p(x)\,dx \qquad (5)$$

is a constant.

Provided $x^2(t)$ is separable with respect to $x_1(t) = x_2(t) = x(t)$ this result can be extended to include the case when the top lead in Fig.2 contains a square law device,

$$\phi_{x_1^2 y_2}(\tau) = C_{FF_x} \phi_{x^2 x^2}(\tau) \ \forall \ F \text{ and } \tau \qquad (6)$$

where

$$C_{FF_x} = \frac{1}{\phi_{x^2x^2}(0)} \int x^2 F[x] p(x) dx \tag{7}$$

is a constant. Other important properties of separable processes and a rigorous derivation of the above results are discussed in a previous publication (Billings and Fakhouri 1978b).

The results of eqn's (4) and (6) indicate the relationships between correlation functions taken across a nonlinear element for separable input processes and provide a firm basis for the development of identification algorithms for block oriented nonlinear systems.

Consider the extension of these results to the cascade nonlinear system illustrated in Fig.1. It will be assumed throughout that all random signals are ergodic, so that ensemble averages may be replaced by time averages over one sample function. All the systems considered are assumed to be asymptotically stable.

The output of the general model, Fig.1 can be expressed as

$$z_2(t) = \iint h_2(\theta) Q(t-\theta, \tau_1, u_2, h_1) u_2(t-\theta-\tau_1) d\theta d\tau_1 + v(t) \tag{8}$$

where $Q(t-\theta,\tau_1,u_2,h_1)$ is a function of t,τ_1,u_2 and $h_1(t)$ and can be readily evaluated by considering the Volterra series expansion for $z_2(t)$. The output correlation function can then be defined as

$$\phi_{z_1z_2}(\varepsilon) = \iint h_2(\theta) \overline{Q(t-\theta,\tau_1,u_2,h_1) u_2(t-\theta-\tau_1) u_1(t-\varepsilon)} d\tau_1 d\theta + \overline{v(t) u_1(t-\varepsilon)} \tag{9}$$

If $u_1(t)$ is separable with respect to $x(t)$, then from (3) the invariance property

$$\phi_{u_1y}(\sigma) = C_{FG}\phi_{u_1x}(\sigma) \;\forall\; F \text{ and } \sigma \tag{10}$$

exists across the nonlinear element where C_{FG} is Booton's equivalent gain. Expanding eqn (10)

$$\int \overline{Q(t,\tau_1,u_2,h_1)u_2(t-\tau_1)u_1(t-\sigma)}d\tau_1 = C_{FG}\int h_1(\tau_1)\overline{u_2(t-\tau_1)u_1(t-\sigma)}d\tau_1 \tag{11}$$

and substituting in eqn (9) yields

$$\phi_{z_1 z_2}(\varepsilon) = C_{FG}\iint h_2(\theta)h_1(\tau_1)\phi_{u_1 u_2}(\varepsilon-\theta-\tau_1)d\theta d\tau_1+\phi_{u_1 v}(\varepsilon) \tag{12}$$

In a similar manner defining the second degree correlation function

$$\phi_{u_1^2 z_2}(\varepsilon) = \iint h_2(\theta)\overline{Q(t-\theta,\tau_1,u_2,h_1)u_2(t-\theta-\tau_1)u_1^2(t-\varepsilon)}d\theta d\tau_1+\overline{u_1^2(t-\varepsilon)v(t)} \tag{13}$$

and expanding the invariance property eqn (6) for double nonlinear transforms

$$\phi_{u_1^2 y}(\sigma) = C_{FFG}\phi_{u_1^2 x^2}(\sigma) \ \forall \ F \text{ and } \sigma \tag{14}$$

and substituting into eqn (13) gives

$$\phi_{u_1^2 z_2}(\varepsilon) = C_{FFG}\iiint h_2(\theta)h_1(\tau_1)h_1(\tau_2)\overline{u_2(t-\theta-\tau_1)u_2(t-\theta-\tau_2)}\,\overline{u_1^2(t-\varepsilon)}\,d\tau_1 d\tau_2 d\theta+\phi_{u^2 v}(\varepsilon) \tag{15}$$

The results of eqn's (12) and (15) hold for the general class of separable inputs providing separability is preserved under linear and double nonlinear transformation respectively (Billings and Fakhouri 1978b). Both these properties are preserved when the input is a Gaussian process.

For the special case when $u_1(t) = u(t)$, $u_2(t) = u(t)+b$ where $u(t)$ is a zero mean white Gaussian process and 'b' is a nonzero mean level eqn's (12) and (15) reduce to (Billings and Fakhouri, 1978b)

$$\phi_{uz'}(\varepsilon) = C_{FG}\int h_1(\tau_1)h_2(\varepsilon-\tau_1)d\tau_1 \tag{16}$$

$$\phi_{u^2 z'}(\varepsilon) = C_{FFG}\int h_2(\tau_1)h_1^2(\varepsilon-\tau_1)d\tau_1 \tag{17}$$

$$C_{FG} = \gamma_1 + 2\gamma_2 b\int h_1(\theta)d\theta + 3\gamma_3\int h_1^{\,2}(\theta)d\ + 3\gamma_3 b^2\iint h_1(\tau_1)h_1(\tau_2)d\tau_1 d\tau_2 + \ldots \tag{18}$$

$$C_{FFG} = 2\gamma_2 + 6\gamma_2 b\int h_1(\theta)d\theta + \ldots \tag{19}$$

where provided $h_1(t)$ is stable bounded-inputs bounded outputs, C_{FG} and C_{FFG} are constants and the superscript ' is used throughout to indicate a zero mean process. The noise terms $\phi_{uv}(\varepsilon)$ and $\phi_{u^2v}(\varepsilon)$ in eqn's (12) and (15) tend to zero when v(t) is zero mean and independent of the input. These are the usual conditions assumed in the identification of linear systems.

Equations (16) and (17) represent a generalisation of Korenberg's algorithm (Korenberg 1973a,b). Korenberg did not consider separable processes but derived the first and second degree correlation functions by considering the time average of each term in a Volterra series expansion. Additional calculations were necessary (Korenberg 1973a,b) depending on whether the nonlinearity was even or odd and a Fourier transform procedure was used to estimate the linear subsystems.

The estimates of eqn's (16) and (17) which exist for all continuous single valued nonlinearities are quite independent of the nonlinear element $F[\cdot]$ except for the constant scale factors C_{FG} and C_{FFG}. Correlation analysis thus effectively decouples the identification problem into two distinct steps; identification of the linear subsystems and characterisation of the nonlinear element. Estimates of the individual linear subsystems $\mu_1 h_1(t)$, $\mu_2 h_2(t)$, where μ_1 and μ_2 are constants, can be obtained directly from the results of eqn's (16) and (17) using a least squares decomposition technique (Billings and Fakhouri 1978a). Once the linear subsystems have been identified the problem is reduced to fitting a polynomial, a series of straight line segments or any other appropriate function to the static nonlinearity by minimising the sum of squares. Because the system is identified in terms of individual linear and nonlinear elements and not as a Volterra series even systems containing very violent nonlinearities such as saturation and deadzone can be readily identified (Billings and Fakhouri 1978c).

The results of eqn's (16) and (17) are the estimates of the first two Volterra kernels for the cascade system in Fig.1. Thus although the Volterra expansion for this system may contain numerous higher order terms these all collapse under the theory of separable processes to the form of eqn's (16) and (17) for this model structure. The problem of isolating the individual Volterra kernels normally encountered in nonlinear systems identification (Billings, 1980) is therefore avoided.

The influence of record length, mean level b, power and bandwidth of the input excitation, the effects of input and output noise, and errors introduced by the decomposition techniques have been studied (Fakhouri, Billings and Wormold 1980).

Analysis of higher order cascade connections of linear dynamic and static nonlinear systems shows that the first and second order correlation functions do not fit into the pattern of results derived above. For example, a system consisting of a nonlinear element in cascade with a linear system h(t) followed by a second nonlinearity gives rise to first and second order correlation functions which are power series in h(t). This problem arises because in general separability does not hold under linear transformation.

A general model consisting of a linear system with pulse transfer function

$$H_1(z^{-1}) = \frac{n_{1,1}z^{-1}}{1+d_{1,1}z^{-1}} = \frac{0.6z^{-1}}{1-0.8z^{-1}}$$

in cascade with the nonlinear device

$$y(t) = x(t) + 12.0x^2(t) + 6.0x^3(t) + 4.0x^4(t)$$

and a linear system

$$H_2(z^{-1}) = \frac{n_{2,1}z^{-1}}{1+d_{2,1}z^{-1}} = \frac{1.5z^{-1}}{1-0.4z^{-1}}$$

was simulated by recording the response to a white Gaussian input N{0.067,0.267} with 10,000 data points. Inspection of the estimated system parameters illustrated in Table 1 for varying degrees of output noise clearly demonstrates the effectiveness of the algorithm.

2.2 Pseudo-random Inputs

Although it can readily be shown that a binary pseudorandom sequence is a separable process, it is not separable under linear and double nonlinear transformation and hence the results of section 2.1 are not valid for these inputs. An alternative procedure must therefore be developed for this class of inputs (Billings and Fakhouri, 1980).

When the input to the general model illustrated in Fig.1 is a compound input $u_2(t)$ defined as the sum $u_2(t) = x_1(t)+x_2(t)$ where $x_1(t)$ and $x_2(t)$ are pseudorandom sequences the output $z_2(t)$ can be expressed as

$$z_2(t) = \sum_{i=1}^{k} \{\gamma_i \int \ldots \int h_1(\tau_1)\ldots h_1(\tau_i)h_2(\theta)$$

$$\{\sum_{j=1}^{i} (x_1(t-\tau_j-\theta)+x_2(t-\tau_j-\theta))d\tau_j\}d\theta\}+v(t)$$

$$= \sum_{i=1}^{k} w_i(t)+v(t) \qquad (20)$$

where $w_i(t)$ can be interpreted as the output of the isolated i'th order Volterra kernel. If the correlation functions are computed directly with the measured system output eqn (20), anomalies associated with the multi-dimensional autocorrelations of the pseudo-random sequences (Barker and Pradisthayon 1970; Barker and Obidegwu, 1973; Billings, 1980), are introduced and the estimates do not reduce to the form of eqns (16) and (17). This problem can be overcome by isolating the first and second order correlation functions of the outputs of the first and second order Volterra kernels respectively.

2.2.1 Multilevel testing

Consider a series of experiments with multi-level compound inputs $\alpha_i u(t)$ where $\alpha_i \neq \alpha_\ell \forall\ i \neq \ell$, then the output correlation function $\phi_{x_1'z_{\alpha_i}'}(\varepsilon)$ can be expressed as

$$\phi_{x_1'z_{\alpha_i}'}(\varepsilon) = \sum_{j=1}^{n} \alpha_i^{\,j} \phi_{x_1'w_j'}(\varepsilon) \quad , \quad i = 1,2,\ldots n \tag{21}$$

assuming that the input signal $x_1(t)$ and noise process $v(t)$ are independent. Providing $\alpha_i \neq 0$, $\alpha_i \neq \alpha_\ell \forall\ i \neq \ell$ eqn (21) has a unique solution for $\phi_{x_1'w_j'}(\varepsilon) \forall\ \varepsilon$, $j = 1,2\ldots n$. If $x_1(t)$ and $x_2(t)$ are independent, $\phi_{x_1x_2}(\lambda) = 0 \forall\ \lambda$, zero mean, $\bar{x}_1 = \bar{x}_2 = 0$, pseudo-random sequences with autocorrelation functions

$$\phi_{x_ix_i}(\lambda) = \beta_i \delta_i(\lambda) \quad , \quad i = 1,2$$

$$\text{where} \quad \delta_i(\lambda) = 1/\Delta t_i \qquad \lambda = 0 \tag{22}$$

$$= 0 \qquad \lambda \neq 0$$

Δt_i is the clock interval and $\int \delta_i(\lambda)d\lambda = 1.0$ then $\phi_{x_1'w_1'}(\varepsilon)$ which can be isolated using the above procedure reduces to

$$\phi_{x_1'w_1'}(\varepsilon) = \beta_1 \gamma_1 \int h_1(\varepsilon-\theta) h_2(\theta) d\theta \tag{23}$$

Following a similar procedure as above and isolating the second order correlation function associated with the second Volterra kernel yields (Billings and Fakhouri 1980)

$$\phi_{x_1'x_2'w_2'}(\varepsilon) = \gamma_2\iiint h_1(\tau_1)h_1(\tau_2)h_2(\theta)$$

$$\{\overline{\prod_{j=1}^{2}(x_1(t-\tau_j-\theta)+x_2(t-\tau_j-\theta))d\tau_j} \tag{24}$$

$$-\overline{\prod_{j=1}^{2}(\bar{x}_1+\bar{x}_2)d\tau_j\}(x_1(t-\varepsilon)-\bar{x}_1)(x_2(t-\varepsilon)-\bar{x}_2)d\theta}$$

When $x_1(t)$ and $x_2(t)$ have the properties defined in eqn (22) this reduces to

$$\phi_{x_1'x_2'w_2'}(\varepsilon) = 2\beta_1\beta_2\gamma_2\int h_1^2(\varepsilon-\theta)h_2(\theta)d\theta \tag{25}$$

Although multilevel inputs must be employed only $\phi_{x_1'w_1'}(\varepsilon)$ and $\phi_{x_1'x_2'w_2'}(\varepsilon)$ and not the individual kernel outputs $w_i(t)$ must be computed. This considerably reduces the computational burden because for stable linear subsystems the correlation functions will tend to steady-state after a small number of values.

Providing $x_1(t)$ and $x_2(t)$ are pseudorandom sequences with properties defined in eqn (22) the results of eqns (23) and (25) are exact and the errors normally associated with the identification of this class of systems using pseudorandom inputs and correlation analysis are avoided.

Since the results of eqns (23) and (25) are dependent upon $x_1(t)$ and $x_2(t)$ having a zero mean value an obvious choice of input would be a compound ternary sequence. It would however be far more convenient if pseudorandom binary inputs could be employed in this application.

However, whilst the first order correlation function eqn (23) remains unbiased for a compound prbs input, the nonzero mean level of this input introduces a time varying bias (Billings and Fakhouri, 1980) $e(\varepsilon)$ into the estimate of eqn (25)

$$e(\varepsilon) = -2\gamma_2(a_1\bar{x}_1\beta_2+a_2\bar{x}_2\beta_1+\bar{x}_1^{\,2}\beta_2+\bar{x}_2^{\,2}\beta_1)\iint h_1(\tau_1)h_1(\varepsilon-\theta)h_2(\theta)d\tau_1 d\theta \qquad (26)$$

where $x_i = a_i/N_i$, a_i is the amplitude and N_i the sequence length. This bias tends to zero as N_1 and N_2 are increased and will be negligible in most applications. This assumption is supported by simulation results (Billings and Fakhouri, 1980).

Uncorrelated binary and ternary pseudorandom sequences with the same bit interval can be generated by correlating over the product of sequence lengths (Briggs and Godfrey 1966).

The results of eqns (23) and (25) are analogous to the results obtained for a separable white Gaussian input when $\gamma_1,\gamma_2 \neq 0$ and can be used directly to identify the individual linear and nonlinear elements of the system illustrated in Fig.1. The relationship between the first and second order correlation functions also provides valuable information regarding the system structure.

2.3 Structure Testing

Consider the identification of an unknown system which has been excited by a separable white Gaussian input with mean level b. Initially the experimenter must determine the structural form of the model which best describes the system under test. This information can be obtained by inspection of the first and second order correlation functions, eqns (16) and (17), for cascade connections of linear dynamic and static nonlinear subsystems.

If the system under test is linear then $\gamma_i = 0 \quad i \neq 1$ and eqns (16) and (17) reduce to

$$\phi_{uz'}(\varepsilon) = \gamma_1 \int h_1(\tau_1)h_2(\varepsilon-\tau_1)d\tau_1 \qquad (27)$$

$$\phi_{u^2z'}(\varepsilon) = 0 \; \forall \varepsilon \qquad (28)$$

Thus if $\phi_{u^2z'}(\varepsilon) = 0 \; \forall \varepsilon$ the system must be linear and once a pulse transfer function model has been fitted to $\phi_{uz'}(\varepsilon)$ the identification is complete. The second order correlation function $\phi_{u^2z'}(\varepsilon)$ is therefore a measure of nonlinearity.

If $h_1(t) = \delta(t)$ the general model reduces to the Hammerstein model (Billings and Fakhouri, 1979b) and

$$\phi_{uz'}(\varepsilon) = C_{FH}h_2(\varepsilon) \qquad (29)$$

$$\phi_{u^2z'}(\varepsilon) = C_{FFH}h_2(\varepsilon) \qquad (30)$$

If therefore $\phi_{uz'}(\varepsilon)$ and $\phi_{u^2z'}(\varepsilon)$ are equal except for a constant of proportionality the system must have the structure of a Hammerstein model.

When $h_2(t) = \delta(t)$ the general model reduces to the Wiener model (Billings and Fakhouri, 1978a,b) and

$$\phi_{uz'}(\varepsilon) = C_{FW}h_1(\varepsilon) \qquad (31)$$

$$\phi_{u^2z'}(\varepsilon) = C_{FFW}h_1^2(\varepsilon) \qquad (32)$$

Thus when $\{\phi_{uz'}(\varepsilon)\}^2$ is equal to $\phi_{u^2z'}(\varepsilon)$ except for a constant of proportionality the system must have the structure of a Wiener model.

Finally, if none of the above conditions hold the system may have the structure of the general model. However, this is a necessary and not a sufficient condition which must be confirmed by parameterising the linear systems and nonlinear element and examining the mean squared error. Alternatively, an algorithm by Douce (1976) provides a very convenient test for cascade systems in this class.

Identification of cascade connections of linear dynamic and static nonlinear systems using correlation analysis thus inherently provides information regarding the structure of these systems.

The relationship between $\phi_{x_1'w_1'}(\varepsilon)$ and $\phi_{x_1'x_2'w_2'}(\varepsilon)$ for pseudo-random inputs are analogous to the above results providing these correlation functions exist.

3. NONLINEAR FEEDBACK SYSTEMS

The identification algorithms derived in previous sections can be applied to nonlinear feedback systems (Billings and Fakhouri, 1979a, 1980) if the form of the Volterra kernels can be related to the component subsystems of the original process. As in the case of cascade systems the objective is to identify the individual elements of the system such that the structure of the process is preserved and truncation errors normally associated with a finite Volterra series description are avoided.

3.1 Unity Feedback Systems

Consider the unity feedback general model illustrated in Fig.3. Notice that in general the system output will be corrupted by noise and hence the feedback signal cannot be computed and the problem cannot be reduced to one of open-loop identification.

Applying the operator calculus developed by Brilliant (1958) and George (1959) it can readily be shown that the Volterra kernels of the equivalent open-loop system $\underline{G}$ can be expressed as

$$\underline{G}_1 = [\underline{I}+\gamma_1\underline{H}*\underline{C}]^{-1}*\gamma_1\underline{H}*\underline{C} \qquad (33)$$

$$\underline{G}_2 = [\underline{I}+\gamma_1\underline{H}*\underline{C}]^{-1}*[\gamma_2\underline{H}o(\underline{C}^2)o((\underline{I}-\underline{G}_1)^2)] \qquad (34)$$

$$\vdots$$

$$\underline{G}_\ell = [\underline{I}+\gamma_1\underline{H}*\underline{C}]^{-1}*[\sum_{n=2}^{\ell} \Sigma_Q \gamma_n(\underline{H}o(\underline{C}^n))o(\underline{K}_{i_1}..\underline{K}_{i_n})] \qquad (35)$$

where $\underline{K}_1 = \underline{I}-\underline{G}_1$, $\underline{K}_\ell = -\underline{G}_\ell$ for $\ell \geq 2$.

A schematic representation of this expansion is illustrated in Fig.4. Comparison of eqns (33)-(35) and Fig.4 indicates the definitions of the operators o and * (George 1959).

Although the series is an infinite operator series the structural form of the first two kernels can be exploited to provide estimates of $\underline{C}$, $F[\cdot]$ and $\underline{H}$. Applying the multilevel testing algorithm of section 2.2.1 to isolate $\phi_{uw_1'}(\varepsilon)$ for a separable white Gaussian input process u(t) with mean level b, yields from eqn's (16) and (33) the estimate

$$\phi_{uw_1'}(\varepsilon) = \int G_1(\tau)\overline{\{u(t-\tau)+b\}u(t-\varepsilon)}d\tau = G_1(\varepsilon) \qquad (36)$$

Taking the Z-transform of eqn (36), a pulse transfer function model can be fitted to $\phi_{uw_1'}(\varepsilon)$ to yield

$$Z\{\phi_{uw_1'}(\varepsilon)\} = \frac{\hat{N}g_1(z^{-1})}{\hat{D}g_1(z^{-1})} = \frac{\gamma_1 H(z^{-1})C(z^{-1})}{1+\gamma_1 H(z^{-1})C(z^{-1})} \qquad (37)$$

and estimates of the numerator and denominator can be obtained from

$$\gamma_1 H(z^{-1})C(z^{-1}) = \frac{\hat{N}g_1(z^{-1})}{\hat{D}g_1(z^{-1})-\hat{N}g_1(z^{-1})} \qquad (38)$$

$$1+\gamma_1 H(z^{-1})C(z^{-1}) = \frac{\hat{D}g_1(z^{-1})}{\hat{D}g_1(z^{-1})-\hat{N}g_1(z^{-1})} \qquad (39)$$

The output data z(t) can now be filtered using the estimate of eqn (39), such that the kernels of the equivalent open loop system, eqn's (33) to (35), reduce to

$$\underline{G}_1 = \gamma_1 \underline{H}*\underline{C}$$
$$\underline{G}_2 = [\gamma_2 \underline{H} \circ (\underline{C}^2) \circ ((\underline{I}-\underline{G}_1)^2)] \qquad (40)$$

Inspection of eqn (40) and the schematic diagram in Fig.4 shows that the filtered output $w_{r2}(t)$ of the second order kernel is related to the input u(t) by a general model Fig.1 where $F[\cdot] = \gamma_2(\cdot)^2$. Following the procedure outlined in section 2 the second order correlation function can then be evaluated by combining the results of eqn's (17), (40) and section 2.2.1 to yield

$$\phi_{u^2 w_{r2}'}(\varepsilon) = 2\gamma_2 \int H(\theta) T_1^2(\varepsilon-\theta) d\theta \qquad (41)$$

where $\underline{T}_1 = \underline{C}*[\underline{I}-\underline{G}_1]$. Taking the Z-transform of eqn (41) a pulse transfer function can be fitted to $\phi_{u^2 w_{r2}'}(\varepsilon)$ to give

$$Z\{\phi_{u^2 w_{r2}'}(\varepsilon)\} = 2\gamma_2 H(z^{-1}) TT(z^{-1}) \qquad (42)$$

where $TT(z^{-1})$ is the Z-transform of T_1^2. The estimates of eqn's (38) and (42) (Billings and Fakhouri 1979a,1980) are analogous to the results for the cascade system eqn's (16) and (17) and can be solved in a similar manner using a least squares decomposition algorithm to provide estimates of the pulse transfer functions $\mu_1 H(z^{-1})$, $\mu_2 C(z^{-1})$ where μ_1 and μ_2 are constants. A suitable function can then be fitted to the nonlinear element by minimising the sum of squared errors using an algorithm by Peckham (1970).

Because the unity feedback Wiener and Hammerstein models are subclasses of the unity feedback general model the identification procedure is applicable to systems with these structures.

The identification procedure outlined above was used to identify a general model consisting of a linear system

$$C(z^{-1}) = \frac{n_{1,1}z^{-1}}{1+d_{1,1}z^{-1}} = \frac{0.2z^{-1}}{1-0.88z^{-1}}$$

in cascade with a nonlinear element

$$y(t) = q(t) + 0.4q^{2}(t) + 0.2q^{3}(t)$$

and another linear system

$$H(z^{-1}) = \frac{n_{2,1}z^{-1}}{1+d_{2,1}z^{-1}} = \frac{0.3z^{-1}}{1-0.7z^{-1}}$$

in a unity negative feedback loop. The system response was recorded for eight levels of input $\alpha_i u(t)$, $i = 1,2..8$ where $u(t)$ is a white Gaussian process $N\{0.4,0.8\}$ and $\alpha_j = \alpha_{j-1} - 0.04$, $\alpha_1 = 1.0$. The estimated parameters are summarised in Table 2.

3.2 Precascaded Feedback Systems

The first two Volterra kernels for the precascaded feedback system illustrated in Fig.5 can be expressed as

$$\underline{G}_1 = \underline{V}_1 * \underline{P} = \{[\underline{I}+\lambda_1\underline{A}_1]^{-1} * \underline{A}_1\} * \underline{P} \qquad (43)$$

$$\underline{G}_2 = \underline{V}_2 * \underline{P} = \{-\underline{V}_1 \circ \lambda_2(\underline{V}_1^{\,2})\} * \underline{P} \qquad (44)$$

Following the procedure of the previous section it can readily be shown that the first two Volterra kernels can be computed as

$$\phi_{uw_1'}(\varepsilon) = \int V_1(\varepsilon-\theta)P(\theta)d\theta \qquad (45)$$

$$\phi_{u^2w_2'}(\varepsilon) = 2\lambda_2\int G_1^{\,2}(\varepsilon-\theta)V_1(\theta)d\theta \qquad (46)$$

Estimates of $\mu_1 V_1(z^{-1})$ and $\mu_2 P(z^{-1})$ can be obtained by decomposing the pulse transfer functions $Z\{\phi_{uw_1'}(\varepsilon)\}$, $Z\{\phi_{u^2w_2'}(\varepsilon)\}$, and a suitable function can be fitted to the nonlinear element by minimising the sum of squared errors.

Although all the results for feedback systems have been derived for separable white Gaussian inputs analogous results can be obtained for a compound pseudorandom input by computing $\phi_{x_1'w_1'}(\varepsilon)$ and $\phi_{x_1'x_2'w_2'}(\varepsilon)$ (Billings and Fakhouri, 1980). The selection of pseudorandom inputs and the error analysis for binary sequences is exactly the same as the open-loop case section 2.2.

4. MULTIPLICATIVE SYSTEMS

Consider the multiplicative system illustrated in Fig.6 and commonly referred to as the factorable Volterra system where the factorable kernel of order k can be realised as a system composed of k linear dynamic subsystems connected in parallel with outputs multiplied in the time domain. Concepts of reachability and observability for this class of systems were studied by Harper and Rugh (1976) who developed an identification scheme based on the system response to two-tone sinusoidal inputs.

Identification algorithms based on both white Gaussian and pseudorandom excitation have been developed by the authors (Billings and Fakhouri, 1979d) but only pseudorandom inputs will be considered in the present analysis.

4.1 Identification of Factorable Kernels

Although the outputs of the factorable kernels $z_i(t)$, $i = 1,\ldots\ell\ell$ in Fig.6 can be isolated using multilevel testing, section 2.2.1, this may involve a long experimentation time and can be avoided by implementing the sequential algorithm outlined below.

Consider a factorable Volterra system which is composed of factorable kernels up to order $\ell\ell$. When the system is excited by the compound input $u(t) = \sum_{j=1}^{\ell\ell} x_j(t)$ the system response can be expressed as a Volterra series

$$y(t) = \sum_{j=1}^{\ell\ell} z_j(t) = \sum_{j=1}^{\ell\ell} \int \ldots \int h_{1,\ell\ell}(t_1) \ldots$$

$$h_{j,\ell\ell}(t_j)\,(\sum_{i=1}^{j} \sum_{k} x_k(t-t_i)\, dt_1 \ldots dt_j + v(t) \qquad (47)$$

If the individual inputs $x_j(t)$, $j = 1,2..\ell\ell$ are zero mean independent processes with autocorrelation functions $\phi_{x_j x_j}(\tau) = \beta_j \delta(\tau)$, $j = 1,2..\ell\ell$, then the system output correlation function defined as

$$\overline{y'(t)x_1(t-\sigma_1) \prod_{i=2}^{\ell\ell} x_i(t-\sigma)} = \sum_{j=1}^{\ell\ell} \int \ldots \int h_{1,j}(t_1) \ldots$$

$$h_{j,j}(t_j)\,\overline{(\prod_{i=1}^{j} \sum_{k=1}^{\ell\ell} x_k(t-t_j)) . x_1(t-\sigma_1) \prod_{m=2}^{\ell\ell}}$$

$$\overline{x_m(t-\sigma)}\, dt_1 \ldots dt_j + \overline{v'(t)x_1(t-\sigma) \prod_{j=2}^{\ell\ell} x_1(t-\sigma)} \qquad (48)$$

reduces to the output correlation function for the $\ell\ell$'th kernel

$$\overline{y'(t)x_1(t-\sigma_1) \prod_{i=2}^{\ell\ell} x_i(t-\sigma)} = \phi_{x_1 \ldots x_{\ell\ell} y'}(\sigma_1, \sigma \ldots \sigma)$$

$$= \phi_{x_1 \ldots x_{\ell\ell} z'_{\ell\ell}}(\sigma_1, \sigma .. \sigma) \qquad (49)$$

Thus by computing $\phi_{x_1 .. x_{\ell\ell} y'}(\sigma_1, \sigma .. \sigma)$ the correlation function associated with the $\ell\ell$'th kernel has been automatically isolated. This result holds exactly even for a compound prbs input. Notice that $\phi_{x_1..x_{\ell\ell}y'}(\sigma_1,\sigma,...\sigma)$ is a second order correlation function which does not increase in dimensionality with the order of the kernel being estimated as in the Lee and Schetzen algorithm (Lee and Schetzen, 1965).

When the inputs $x_j(t)$ have the properties defined above $\phi_{x_1..x_{\ell\ell}z'_{\ell\ell}}(\sigma_1,\sigma..\sigma)$ reduces to

$$\phi_{x_1..x_{\ell\ell}z'_{\ell\ell}}(\sigma_1,\sigma..\sigma) = (\ell\ell-1)!(\prod_{n=1}^{\ell\ell}\beta_n)\sum_{i=1}^{\ell\ell}\{h_{i,\ell\ell}(\sigma_i)\prod_{\substack{j=1\\j\neq i}}^{\ell\ell}h_{j,\ell\ell}(\sigma)\} \tag{50}$$

and the function $\psi_{\ell\ell}(\sigma_1,\sigma)$ can be defined as

$$\psi_{\ell\ell}(\sigma_1,\sigma) = \phi_{x_1..x_{\ell\ell}z'_{\ell\ell}}(\sigma_1,\sigma..\sigma).\frac{1}{(\ell\ell-1)!(\prod_{n=1}^{\ell\ell}\beta_n)}$$

$$= \sum_{i=1}^{\ell\ell}\{h_{i,\ell\ell}(\sigma_i)\prod_{\substack{j=1\\j\neq i}}^{\ell\ell}h_{j,\ell\ell}(\sigma)\} \tag{51}$$

The above results can be realised exactly using independent white Gaussian inputs $x_j(t)$ or independent ternary sequences. If prbs inputs are employed the errors introduced in the estimates of the first and second order factorable kernels have the same form as the errors for the cascade general model section 2.2.1 which tend to zero as the sequence lenghts become large. This is supported by simulation results (Billings and Fakhouri, 1979d).

Once $\psi_{\ell\ell}(\sigma_1,\sigma)$ has been computed estimates of the individual linear subsystems $h_{i,\ell\ell}(t)$ can be obtained by decomposing eqn (51).

If the estimate in eqn (51) is computed by fixing $\sigma = \delta_j$, $j = 1,2..m$ and in each case evaluating $\psi_{\ell\ell}(\sigma_1,\sigma)$ for all $\sigma_1 = \delta_1,\delta_2,..\delta_m$ this yields m^2 equations in $\ell\ell.m$ unknowns, $h_{j,\ell\ell}(\delta_i)$, $j = 1,2..\ell\ell$, $i = 1,2..m$

$$\psi_{\ell\ell}(\delta_p,\delta_q) = \sum_{i=1}^{\ell\ell} \{h_{i,\ell\ell}(\delta_p) \prod_{\substack{j=1\\j\neq i}}^{\ell\ell} h_{j,\ell\ell}(\delta_q)\} \tag{52}$$

where $q = 1,2..m$, $p = 1,2..m$ for each value of q, and $\psi_{\ell\ell}(\delta_p,\delta_q) \neq \psi_{\ell\ell}(\delta_\ell,\delta_f) p \neq \ell$, $q \neq f$.

Equation (52) can be solved for the $\ell\ell.m$ unknowns by minimising the cost function

$$J(h) = \sum_{i=1}^{m} \sum_{j=1}^{m} \{\psi_{\ell\ell}(\delta_j,\delta_i) - \hat{\psi}_{\ell\ell}(\delta_j,\delta_i)\}^2 \tag{53}$$

$$h = [h_{1,\ell\ell}(\delta_1)...h_{1,\ell\ell}(\delta_m), h_{2,\ell\ell}(\delta_1)..h_{2,\ell\ell}(\delta_m),...h_{\ell\ell,\ell\ell}(\delta_m)]^T$$

using a modified Marquardt algorithm (Marquardt 1963; Fletcher 1971). Whilst the linear subsystems $h_{j,\ell\ell}(\delta_i)$ can only be evaluated to within constant scale factors this does not jeopardize the final identification results.

Once estimates of $h_{j,\ell\ell}(\delta_i)$, $j = 1,2..\ell\ell$, $i = 1,2..m$ are available the following matrix equation can be formulated

$$\begin{pmatrix} \psi_{\ell\ell}(t,\delta_1) \\ \vdots \\ \psi_{\ell\ell}(t,\delta_m) \end{pmatrix} = \begin{bmatrix} \prod_{i=2}^{\ell\ell} \hat{h}_{i,\ell\ell}(\delta_1), & \dots & \prod_{\substack{i=1\\i\neq\ell\ell}}^{\ell\ell} \hat{h}_{i,\ell\ell}(\delta_1) \\ \vdots & & \vdots \\ \prod_{i=2}^{\ell\ell} \hat{h}_{i,\ell\ell}(\delta_m), & \dots & \prod_{\substack{i=1\\i\neq\ell\ell}}^{\ell\ell} h_{i,\ell\ell}(\delta_m) \end{bmatrix} \begin{bmatrix} \hat{h}_{1,\ell\ell}(t) \\ \vdots \\ \hat{h}_{\ell\ell,\ell\ell}(t) \end{bmatrix} \tag{54}$$

or $F_t = \theta h_t$

and estimates of the individual linear subsystems $h_{i,\ell\ell}(t)$, $i = 1,2..\ell\ell$ can be evaluated by solving

$$\hat{h}_t = \theta^{-1} F_t \qquad \text{for } m = \ell\ell$$

$$\text{or} \qquad \hat{h}_t = (\theta^T\theta)^{-1}\theta^T F_t \qquad \text{for } m>\ell\ell \tag{55}$$

for a range of t from zero to the system settling time.

When the linear subsystems associated with the $\ell\ell$'th kernel have been estimated using the algorithm outlined above the predicted output $\hat{z}_{\ell\ell}(t)$ can be computed

$$\hat{z}_{\ell\ell}(t) = \int..\int \hat{h}_{1,\ell\ell}(t_1)..\hat{h}_{\ell\ell,\ell\ell}(t_{\ell\ell})\left(\prod_{j=1}^{\ell\ell}\sum_{i=1}^{\ell\ell} x_i(t-t_j)dt_1..dt_{\ell\ell}\right. \tag{56}$$

and a reduced system output $\hat{yy}'_{\ell\ell-1}(t) = y'(t)-\hat{z}'_{\ell\ell}(t)$ can be defined.

Continuing the above procedure the $(\ell\ell-1)$'th kernel can be identified by computing the $(\ell\ell-1)$'th system output correlation function

$$\phi_{x_1..x_{\ell\ell-1}yy'_{(\ell\ell-1)}}(\sigma_1,\sigma..\sigma) = \phi_{x_1..x_{\ell\ell-1}z'_{(\ell\ell-1)}}(\sigma_1,\sigma..\sigma) \tag{57}$$

to provide an estimate of $\psi_{\ell\ell-1}(\sigma_1,\sigma)$ which can be decomposed using the Marquardt algorithm to yield $\hat{h}_{i,\ell\ell-1}(t)$, $i = 1,2..\ell\ell-1$. The linear systems associated with the remaining kernels can be identified by continuing this procedure.

It can readily be shown that providing any noise corrupting the system output y(t) is independent of the input process this tends to zero in the analysis and unbiased estimates are obtained.

To illustrate the above algorithm a second order factorable Volterra system, Fig.6, was simulated using a compound prbs input

$u(t) = x_1(t)+x_2(t)$ where x_1 and x_2 were defined by the difference equations

$$(I\oplus_2 D^3 \oplus_2 D^7)x_1 = 0$$
$$(I\oplus_2 D^1 \oplus_2 D^6)x_2 = 0$$

respectively, where D is the delay operator and $\oplus_2$ denotes modulo two addition. These two sequences are uncorrelated when correlation is performed over $N_1N_2 = (2^7-1)(2^6-1) = 8001$ points. The system included both 1st and 2nd order kernels and was defined as

$$H^3_{1,1}(z^{-1}) = \frac{0.2z^{-1}}{1-1.5z^{-1}+0.62z^{-2}}$$

$$H^3_{1,2}(z^{-1}) = \frac{0.2z^{-1}}{1-1.62z^{-1}+0.7z^{-2}}$$

$$H^3_{2,2}(z^{-1}) = \frac{0.2z^{-1}}{1-1.56z^{-1}+0.7z^{-2}}$$

The system was simulated with $m = 3$, $\delta_1 = 3$, $\delta_2 = 5$, $\delta_3 = 7$ in eqn's (52) and (54) and convergence of the Marquardt algorithm was achieved in seventeen iterations. The estimated parameters are summarised in Table 3. Although a slight bias can be detected in the estimated parameters due to the error term introduced when using a prbs input this is quite small and can be considered as negligible.

5. S_m SYSTEMS

The S_m model illustrated in Fig.7 consists of a series of general models with reduced nonlinear elements connected in parallel with outputs summated. This class of systems was originally studied by

Baumgartner and Rugh (1975) and later by Wysocki and Rugh (1976) and Sandor and Williamson (1978). Identification algorithms based on steady-state sinusoidal measurements were developed by these authors.

Complete identification of the component subsystems in the S_m model can be achieved (Billings and Fakhouri 1979c) by isolating the first and second degree correlation functions associated with each of the m branches. Since each branch has the structure of a general model the results of eqn's (16) and (17) are immediately applicable when the input is a Gaussian white process with mean b. Thus using the multilevel testing algorithm of section 2.2.1 the first and second degree correlation functions for the k'th branch can be expressed as

$$\phi_{uw_k'}(\varepsilon) = C_{Fk}\int h_{k1}(\tau_1)h_{k2}(\varepsilon-\tau_1)d\tau_1$$

$$\phi_{u^2w_k'}(\varepsilon) = C_{FFk}\int h_{k2}(\tau_1)h_{k1}^{2}(\varepsilon-\tau_1)d\tau_1 \qquad (58)$$

These estimates are identical, except for the constant scale factors C_{Fk}, C_{FFk}, to the results for the open-loop general model eqn's (16), (17) and estimates of the linear subsystems $h_{ki}(t)$, $i = 1,2$, $k = 2,...m$ and the nonlinear coefficient γ_i can be obtained directly using the procedure outlined in previous sections.

6. CONCLUSIONS

A unified approach to the identification of nonlinear systems which can be represented by interconnections of linear dynamic and static nonlinear elements has been presented. Although the algorithms utilize the structural properties of the first two kernels in the Volterra series expansion characterization in terms of these kernels is avoided and truncation errors are not incurred. Thus even systems with very violent nonlinearities can be identified.

Cascade and multiplicative systems prove to be particularly tractable and estimates of the individual component subsystems can be readily obtained from single test experiments. The information regarding system structure which is inherent in the results for cascade systems should be particularly valuable. Although multi-level testing is necessary in the identification of feedback and S_m systems this is often necessary in nonlinear systems identification although several authors avoid this problem by considering systems which are defined by a single kernel. This constraint can be avoided by using the technique of Lee and Schetzen (1965) but this involves the computation of multidimensional correlation functions even for simple systems. Whilst the algorithms presented are based upon the calculation of first and second degree correlation functions both these are defined as functions of a single argument and estimates of the component subsystems can be obtained by using simple extensions of established linear techniques.

All the algorithms can be implemented for Gaussian white inputs but the convenience of pseudorandom sequences suggests that the compound input method would be more appropriate in many applications.

The algorithms can be readily applied to the identification of other nonlinear systems within this class including feedforward systems (Billings and Fakhouri 1979b) and other common system structures. A complete analysis of the estimation errors associated with the identification of the general model has been compiled (Fakhouri, Billings and Wormold 1980) and recent results (Fakhouri 1980) have shown that the algorithms can be implemented using Gaussian non-white inputs with typically 4000 data input/output pairs.

Identification of nonlinear systems in terms of the individual elements preserves the system structure and provides valuable information for control. This approach overcomes many of the disadvantages associated with black-box identification and provides a very concise description of the process.

Further research is required to simplify the selection of inputs, reduce the data record length, develop simple to implement structure detection algorithms for feedback and factorable Volterra systems, and investigate alternative methods of identifying nonlinear systems (Billings 1980, Billings and Leontaritis 1981).

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Parameters		$n_{1,1}$	$d_{1,1}$	γ_1	γ_2	γ_3	γ_4	$n_{2,1}$	$d_{2,1}$	MSE
Theoretical values		0.6	-0.8	1.0	12.0	6.0	4.0	1.5	-0.4	-
Estimated values	noise-free	0.6145	-0.785	1.14	12.05	6.18	4.55	1.49	-0.406	0.0165
	S/N 2.5:1	0.684	-0.716	1.41	10.95	9.03	8.45	1.44	-0.466	0.3507
	S/N 1:1	0.722	-0.649	1.09	11.41	16.71	7.91	1.39	-0.51	0.8706

Table 1. Identification results for the general model

Parameter	$n_{1,1}$	$d_{1,1}$	γ_1	γ_2	γ_3	$n_{2,1}$	$d_{2,1}$
Theoretical value	0.2	-0.88	1.0	0.4	0.2	0.3	-0.7
Estimated value	0.1963	-0.8845	1.0145	0.4065	0.2038	0.303	-0.69

Table 2. Identification results for the unity feedback general model

Parameters		n_1	n_2	d_1	d_2
$H_{1,1}(z^{-1})$	Theoretical values	0.2	0.0	-1.5	0.62
	Estimates	0.197	0.004	-1.499	0.620
$H_{1,2}(z^{-1})$	Theoretical values	0.2	0.0	-1.62	0.7
	Estimates	0.201	-0.003	-1.618	0.699
$H_{2,2}(z^{-1})$	Theoretical values	0.2	0.0	-1.56	0.7
	Estimates	0.198	0.004	-1.558	0.701

Table 3. Identification results for the factorable Volterra system

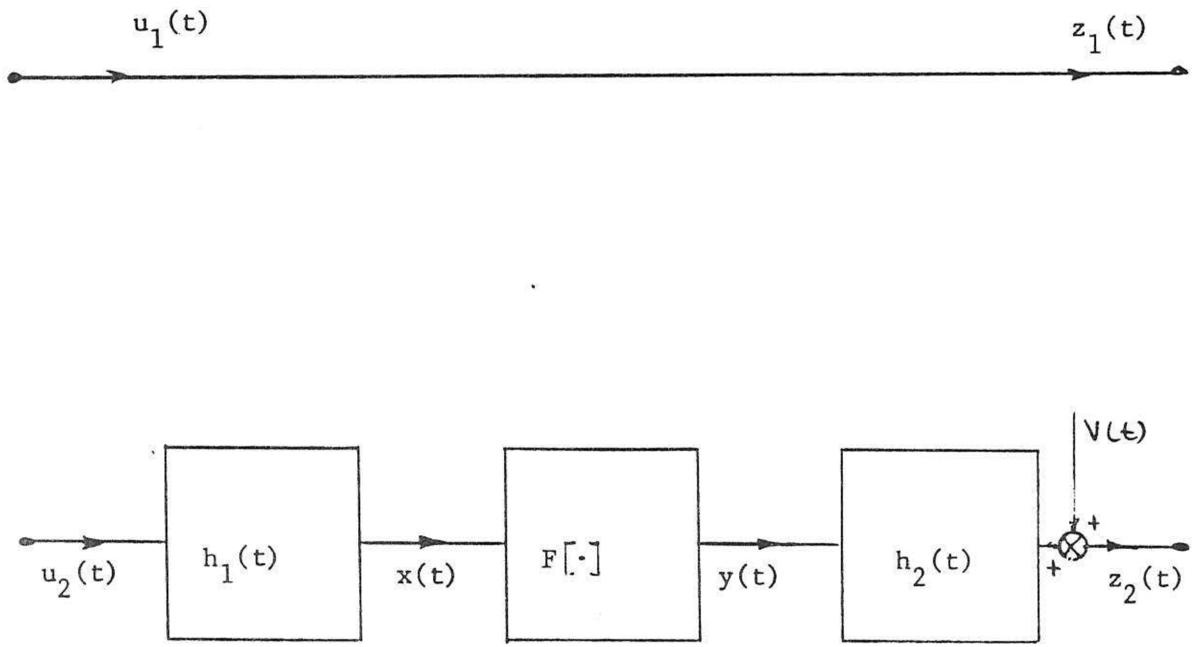

Fig. 1 The General Model

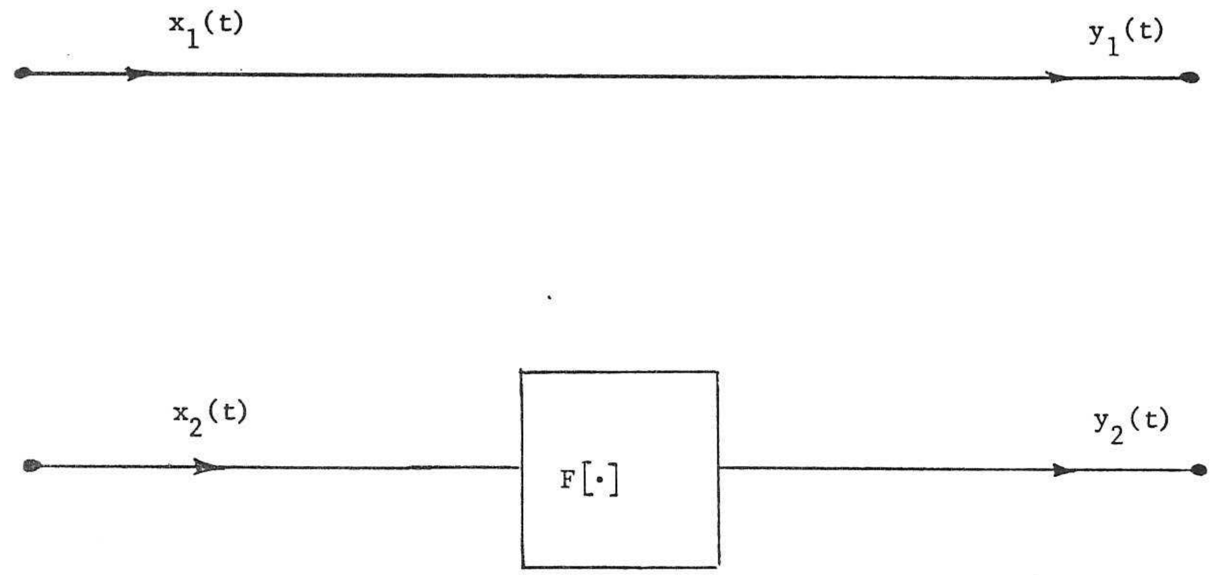

Fig. 2 Non-linear no-memory system

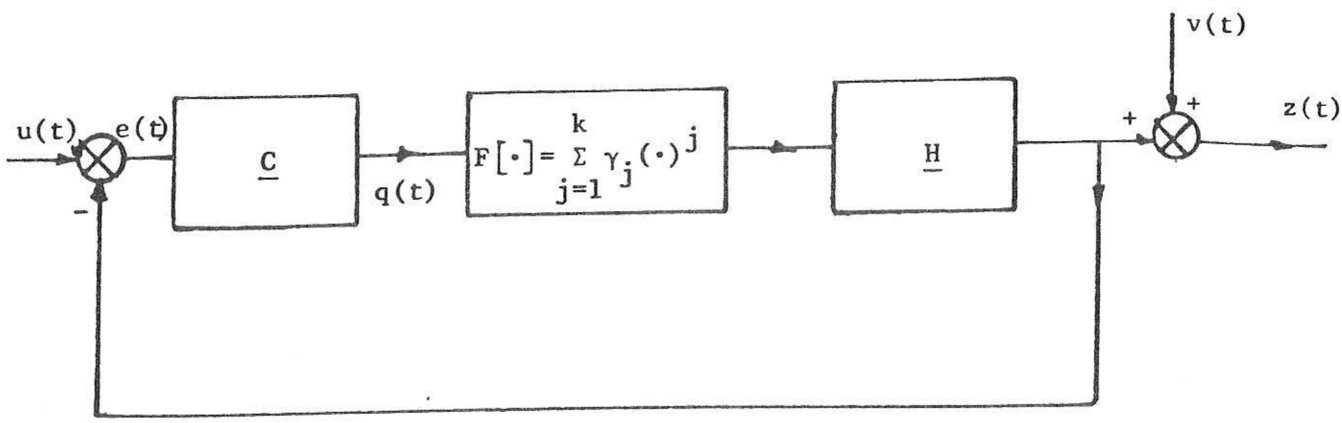

Fig. 3 The unity feedback general model

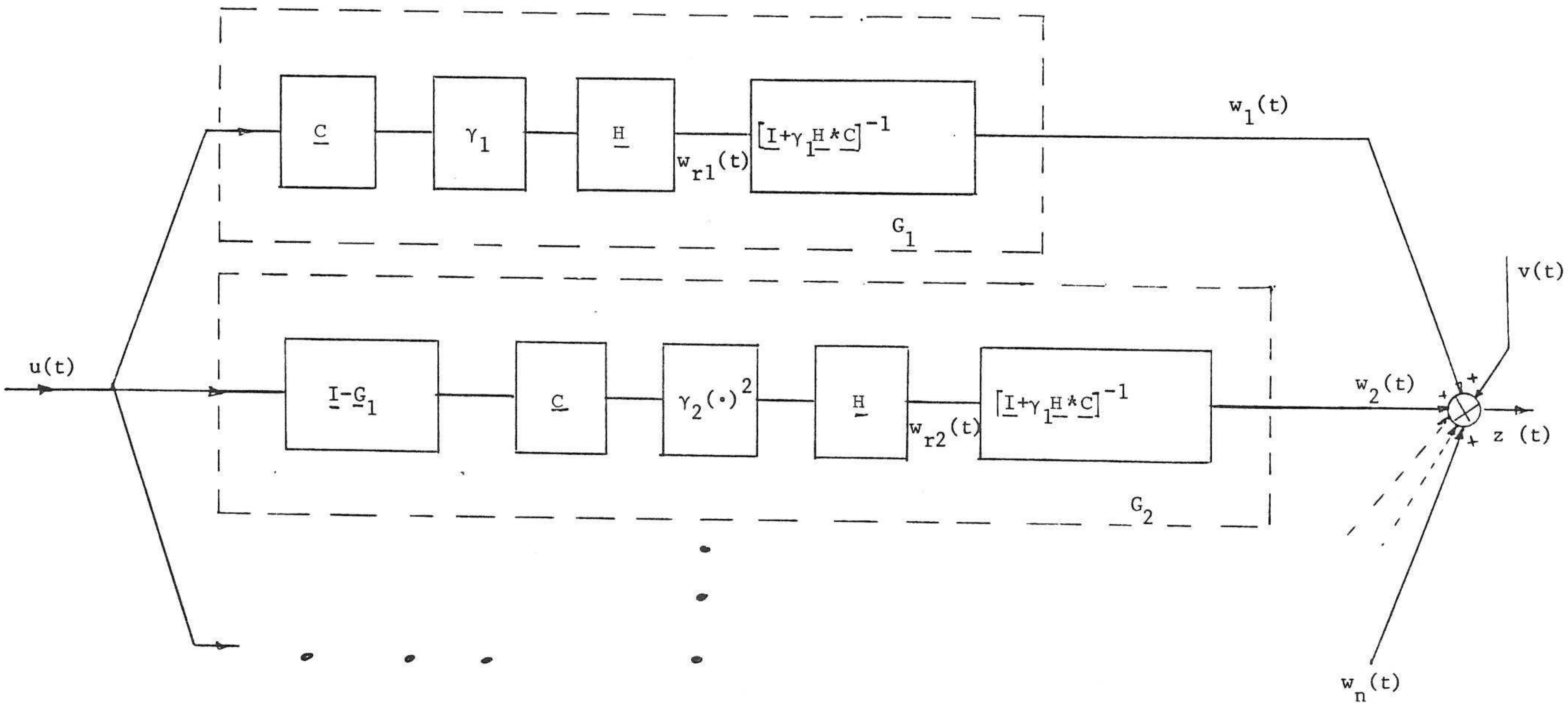

Fig. 4 Volterra series expansion of a nonlinear feedback system

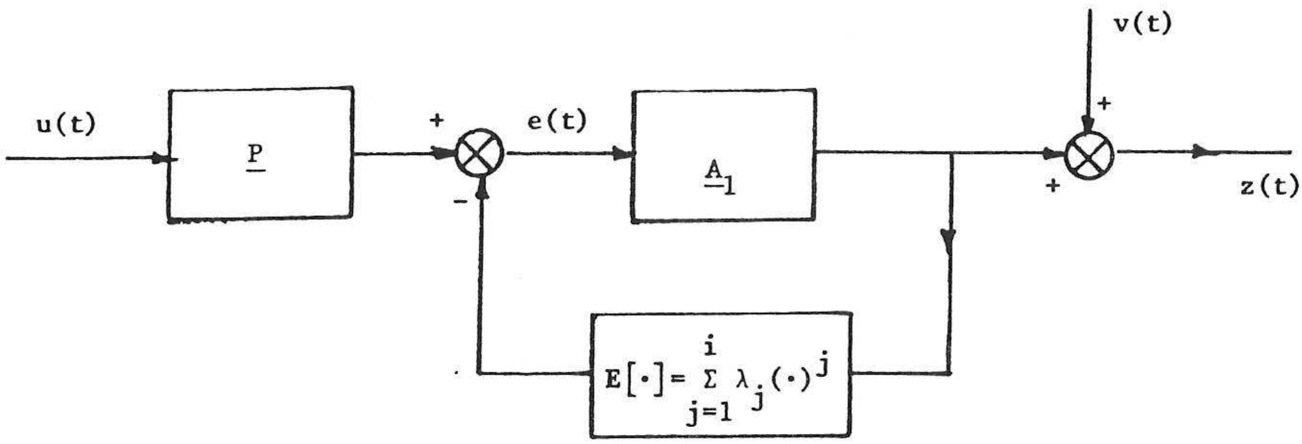

Fig. 5 Precascaded nonlinear feedback system

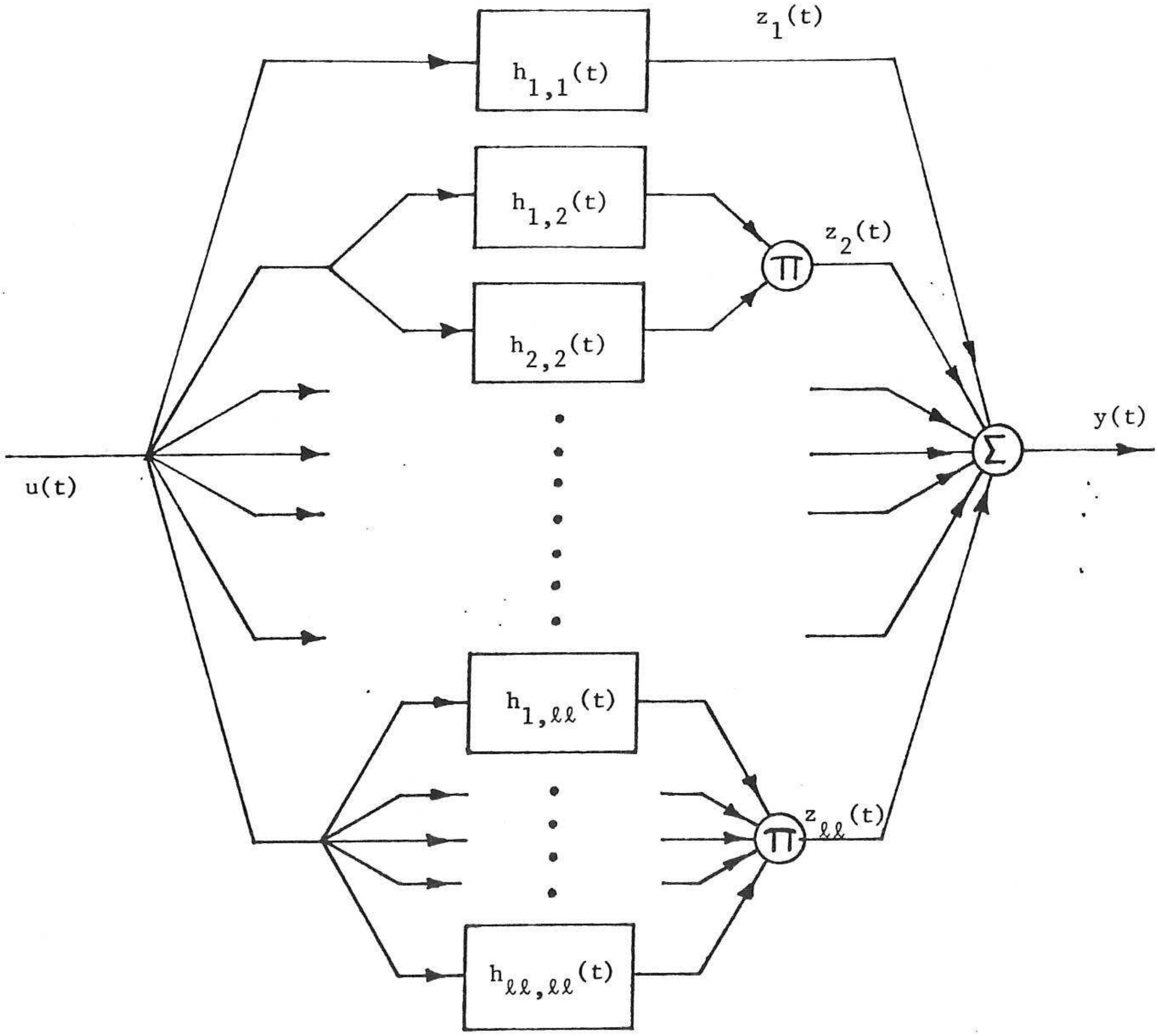

Fig. 6 A non-linear factorable Volterra system

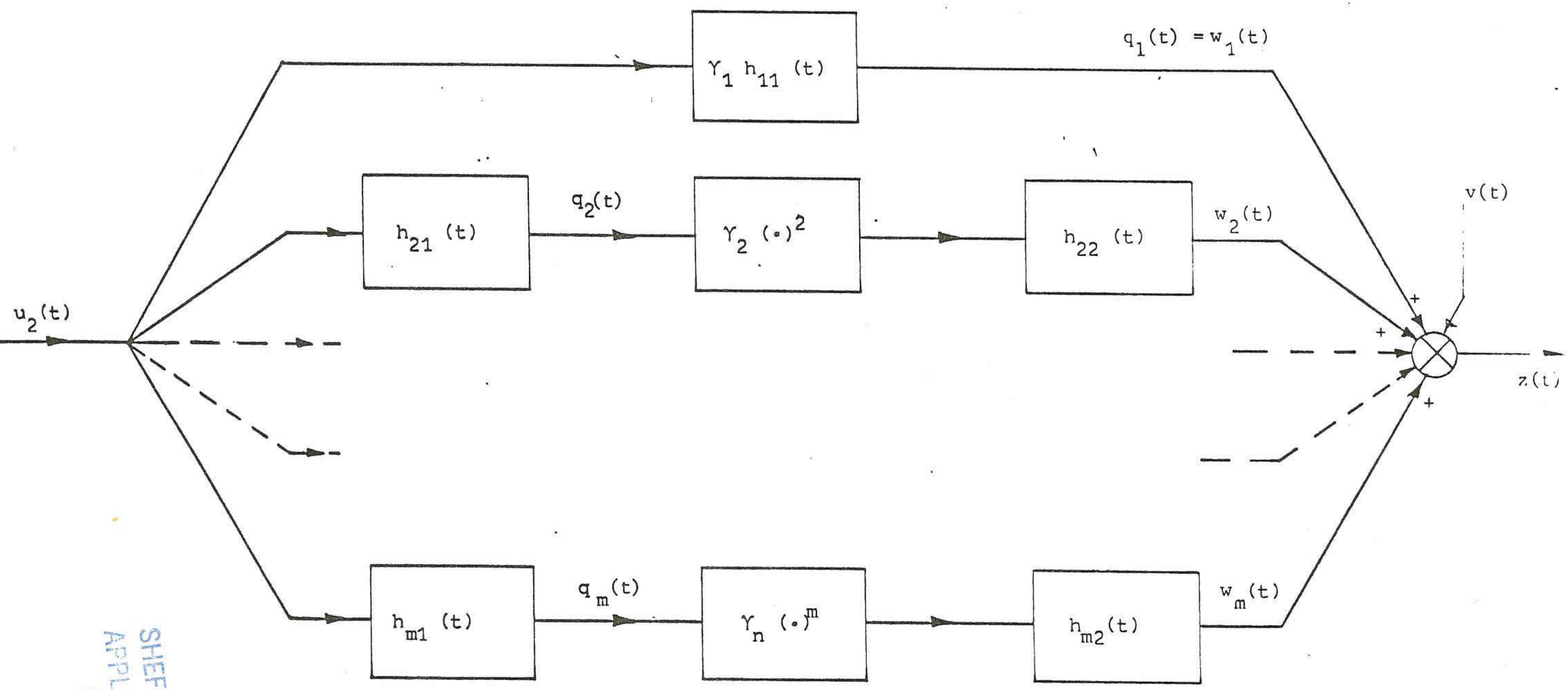

FIG 7 THE S_m MODEL