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The analytical modelling and dynamic behaviour of tray-type binary distillation columns

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Summary

From the ordinary differential equations describing the individual trays of a multitray binary column a partial differential equation (p.d.e.) representation is derived together with the necessary boundary conditions. Large-signal steady-state solutions are derived to provide parametric data for a small-signal p.d.e. model obtained by linearisation. The small-signal p.d.e.'s are solved analytically for sinusoidal inputs to produce a parametric transfer-function matrix (T.F.M.) model for the system.

The plant parameters and operating conditions are chosen to produce a physically symmetrical system which, as a result, yields a completely diagonal T.F.M. choosing identical input and output vectors to those used in the author's earlier analysis of symmetrical spatially-continuous columns, (packed-columns). It is demonstrated that, even for identical operating conditions, important differences occur in the dynamics of the two types of column. These differences occur as a result of vapour capacitance and inter-phase resistance causing nonminimum-phase effects in packed-columns which do not occur in the tray-type. The two types of column can also call for opposite signs of controller gain.

It is argued that, whereas restrictions apply to the approximation of packed-columns by multivariable first-order lag models, such representations are much more widely applicable to the tray-type.

Introduction

Earlier research by the author into tray-type distillation columns[1,2] has produced models which predict analytically and by simulation, dynamic behaviour of more-or-less first-order lag nature. Some approximation was involved in the final stages of the analytical approach however because of complexity resulting largely from small degrees of asymmetry in the physical system studied. This complexity also precluded any analytical search for significant travelling-wave effects on system transfer-functions noted in other simple counterflow processes such as liquid heat-exchangers[2], also of interest to the author. As a result of this latter difficulty attention was transferred to packed-columns, these being rather more analogous to the heat-exchanger, and a successful and non-approximate analysis was completed[3]. This demonstrated not only travelling-waves but, more important from the distillation point of view, important differences in the dynamic behaviour of packed-columns, accurately predicted, and tray-columns, hitherto only approximately predicted.

In particular, packed-columns were shown to have high-frequency gains of sign opposite to those of the tray-type, and, under certain circumstances, reversed low-frequency behaviour as well, with serious implications for controller design. It is therefore important to investigate the source of these differences, particularly as real columns are likely to fall somewhere between the ideal conceptions of tray- and packed-type.

In this report therefore a highly symmetrical column, identical to the packed-column solved previously, (apart from its spatial segmentation), is analysed through to the production of accurate parametric transfer-function matrices, the only approximation now involved being that associated with the representation of the discrete spatial functions by their continuous equivalents. The broad predictions of the original approximate analyses are upheld by this rigorous analysis, confirming the predicted differences between packed- and tray-columns.

2. Large-signal model

The column as a whole and an individual rectifier cell are illustrated diagramatically in Figs. 1 and 2 respectively.

2.1 Equilibrium considerations

The vapour above and liquid on each tray are assumed to be in equilibrium with one another at all times so that, if Y(n) and X(n) denote the vapour and liquid mol-fractions of the more-volatile component on tray n of the rectifier then, for an ideal mixture,

$$Y(n)\{1-X(n)\}/\left[X(n)\{1-Y(n)\}\right] = \beta \qquad \ldots(1)$$

where β is the constant relative volatility of the mixture. Similarly, at the nth tray in the stripper:

$$Y'(n)\{1-X'(n)\}/\left[X'(n)\{1-Y'(n)\}\right] = \beta \qquad \ldots(2)$$

the primes distinguishing variables in the stripping section. Linearising this relationship whilst retaining symmetry about the line $Y = 1-X$ ($Y' = 1-X'$) we get:

$$Y(n) = X(n)/\alpha + (\alpha-1)/\alpha$$
$$\text{and} \quad Y'(n) = \alpha X'(n) \qquad \ldots(3)$$

where α is a constant and $\alpha = 1+\varepsilon$, $\varepsilon>0$ $\qquad \ldots(4)$

2.2 General tray equations

The assumption of continuous equilibrium effectively implies zero vapour capacitance (or infinite mass transfer rates between liquid and vapour phases) so that the material balance for tray n of the rectifier may be written:

$$H_\ell \delta h' dX(n)/dt = L_r\{X(n+1)-X(n)\}+V_r\{Y(n-1)-Y(n)\}$$

where H_ℓ = rectifier liquid capacitance per unit length of column, $\delta h'$ = total length of cell n, while L_r and V_r denote molar liquid and vapour flow rates within the rectifier.* Eliminating Y(n) in terms of X(n) using equation (3) therefore

$$H_\ell \delta h' dX(n)dt = L_r\{X(n+1)-X(n)\}+(V_r/\alpha)\{X(n-1)-X(n)\} \qquad \ldots(5)$$

Now, if X were a continuous function of height h', by Taylor's theorem:

$$\left.\begin{aligned} X(n+1)-X(n) &= \frac{\partial X(n)}{\partial h'}\delta h' + \frac{1}{2}\frac{\partial^2 X(n)}{(\partial h')^2}(\delta h')^2 + \text{higher powers of } \delta h' \\ \text{and} \quad X(n-1)-X(n) &= -\frac{\partial X(n)}{\partial h'}\delta h' + \frac{1}{2}\frac{\partial^2 X(n)}{(\partial h')^2}(\delta h')^2 + \text{higher power of } \delta h' \end{aligned}\right\} \ldots(6)$$

and if n is sufficiently large therefore the truly discrete function X(n) may from (5) and (6) be closely approximated by a continuous function X(h',t) described by

$$H_\ell \delta h' \frac{\partial X}{\partial t} = (L_r - V_r/\alpha)\frac{\partial X}{\partial h'}\delta h' + \frac{1}{2}(L_r + V/\alpha)\frac{\partial^2 X}{(\partial h')^2}(\delta h')^2 + \text{higher powers of } \delta h' \qquad \ldots(7)$$

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*Liquid and vapour flow rates will be spatially independent if operating conditions are adiabatic and the two components have equal latent heats per mol.

If the column is operated under the particular working condition

$$V_r = L_r \alpha \qquad \ldots(8)$$

then the odd powers of $\delta h'$ disappear, and if $h' >> \delta h'$ such that all powers of $\delta h'$ above the second may be ignored then in steady state

$$\partial^2 X/(\partial h')^2 = 0 \qquad \ldots(9)$$

so yielding a constant composition gradient through the rectifier (i.e. all trays perform the same duty: a good design criterion).

Similarly for the stripper, if H_ℓ', L_s and V_s denote liquid capacitance p.u. length, liquid and molar flow rates (see Fig. 1) then mass balance considerations together with equation (4) give:

$$H_\ell' \delta h' \frac{dX'(n)}{dt} = L_s\{X'(n+1)-X'(n)\}+V_s\alpha\{X'(n+1)-X'(n)\}$$

so that again using a continuous approximation $X'(h',t)$ to $X'(n,t)$ we obtain

$$H_\ell' \delta h' \frac{\partial X'}{\partial t} = (L_s - V_s\alpha)\frac{\partial X'}{\partial h'}\delta h' + \frac{1}{2}(L_s + V_s\alpha)\frac{\partial^2 X'}{(\partial h')^2}(\delta h')^2$$

$$+ \text{ higher powers of } \delta h' \qquad \ldots(10)$$

and if for even "loading" the column is operated such that

$$L_s = V_s \alpha \qquad \ldots(11)$$

then $\partial^2 X'/(\partial h')^2 = 0$ ...(12)

in steady state.

To solve general p.d.e.'s 7 and 10 or their special cases (9) and (12) it is necessary to consider the four boundaries of the system, i.e. the terminating vessels and the feed trays.

2.3 Feed boundary conditions

Fig. 3 illustrates the feed section of the column in more detail. F denotes the feed rate of vapour and also of liquid, their compositions being z and Z respectively. Considering first the tray above the feed point, the mass balance is

$$H_\ell \delta h' \frac{dX(0)}{dt} = Fz + V_s Y'(0) - V_r Y(0) + L_r\{X(1) - X(0)\} \qquad \ldots(13)$$

If, for symmetry, the column is operated such that

$$V_r = L_s \quad \text{and} \quad L_r = V_s \qquad \ldots(14)$$

then, using these relationships together with (3), (8) and (11) and replacing finite difference X(1)-X(0) by $\{\partial X(0)/\partial h'\}\delta h'$ we obtain

$$H_\ell \delta h' \partial X(0)/\partial t = Fz + V_r\{X'(0) - Y(0)\} + V_r\{\partial Y(0)/\partial h'\}\delta h'$$

Now $$F = V_r - V_s \qquad \ldots(15)$$

and in our special case therefore

$$F = V_r - L_r = V_r(1-1/\alpha) = V_r \varepsilon/\alpha \qquad \ldots(16)$$

Furthermore, if the feed mixture is in equilibrium so that

$$z = \alpha Z \qquad \ldots(17)$$

and the compositions such that the point z, Z lies on the minus 45^o line of the vapour / liquid diagram (for symmetry)

i.e. $$z = 1 - Z \qquad \ldots(18)$$

then it follows that

$$z = \alpha/(1+\alpha) \quad \text{and} \quad Z = 1/(1+\alpha) \qquad \ldots(19)$$

Substituting for F and z in our special case boundary equation and eliminating X(0) in favour of Y(0) therefore yields

$$\frac{\alpha H_\ell \delta h'}{V_r} \frac{\partial Y(0)}{\partial t} = -\frac{2}{\alpha+1} + X'(0) + \{1-Y(0)\} + \frac{\partial Y(0)}{\partial h'} \delta h' \quad \ldots(20)$$

This equation may be more simply expressed in terms of normalised time τ and distance h, where

$$\tau = tV_r/(\alpha H_\ell \delta h') \quad \ldots(21)$$

and $h = h'/\delta h'$

thus:

$$\frac{\partial Y(0)}{\partial \tau} = -\frac{2}{\alpha+1} + X'(0) + \{1-Y(0)\} + \frac{\partial Y(0)}{\partial h} \quad \ldots(22)$$

Turning attention now to the tray immediately below the feed point, taking a mass balance we obtain

$$H_\ell'\delta h' \frac{dX'(0)}{dt} = FZ+V_s\{Y'(-1)-Y'(0)\}+L_rX(0)-L_sX'(0) \quad \ldots(23)$$

and, again under symmetrical operation, this reduces to

$$\frac{H_\ell'\delta h'}{V_r} \frac{\partial X'(0)}{\partial t} = \frac{2}{\alpha+1} - \frac{\partial X'(0)}{\partial h'} \delta h' - \{1-Y(0)\}-X'(0) \quad \ldots(24)$$

If therefore the tray capacitances are such that

$$H_\ell' = \alpha H_\ell \quad \ldots(25)$$

then, in terms of normalised time and distance

$$\frac{\partial X'(0)}{\partial \tau} = \frac{2}{\alpha+1} - \frac{\partial X'(0)}{\partial h} - \{1-Y(0)\} - X'(0) \quad \ldots(26)$$

Equations 22 and 26 clearly share a marked degree of symmetry.

2.4 Terminal boundary conditions

Fig. 4 shows the variables associated with the accumulator and reboiler ends of the process, the top tray being the Nth and integer N+1 denoting accumulator quantities. If H_a is the molar capacitance of the accumulator, then on taking a mass balance on this vessel we obtain:

$$H_a \partial X(N+1)/\partial t = V_r\{Y(N) - X(N+1)\}$$

so that again regarding Y as a spatially continuous variable we may eliminate Y(N) in terms of Y(N+1) using a truncated Taylor expansion to give

$$H_a \frac{\partial X(N+1)}{\partial t} = V_r\{Y(N+1) - \frac{\partial Y(N+1)}{\partial h'} \delta h' - X(N+1)\}$$

so that, in terms of Y(N+1) only, we have

$$H_a \alpha \frac{\partial Y(N+1)}{\partial t} = V_r\left[Y(N+1)\{1-\alpha\} + \alpha - 1 - \frac{\partial Y(N+1)}{\partial h'} \delta h'\right] \qquad \ldots(27)$$

and if Y(N+1) is now replaced by $Y(L_1)$ where L_1 is the normalised length of the rectifier, then in terms of τ and h we finally obtain the general result:

$$T_a \partial Y(L_1)/\partial \tau = \varepsilon\{1 - Y(L_1)\} - \partial Y(L_1)/\partial h \qquad \ldots(28)$$

where T_a is the normalised time-constant of the accumulator and given by

$$T_a = H_a/H_\ell \delta h' \qquad \ldots(29)$$

A similar treatment of the reboiler mass balance equation, viz.

$$H_b \partial X'(-L_2)\partial t = L_s\{X'(-L_2+\delta h') - Y'(-L_2)\}$$

where H_b is the molar capacitance of the reboiler and L_2 the normalised length of the stripping-section, produces the general result

$$T_b \partial X'(-L_2)/\partial\tau = -\varepsilon X'(-L_2) + \partial X'(-L_2)/\partial h \qquad ...(30)$$

where T_b is the normalised reboiler time constant, given by

$$T_b = H_b/H_\ell'\delta h' \qquad ...(31)$$

2.5 Steady-state solution

General p.d.e.'s (7) and (10), under the particular operating conditions (8) and (11), yield the steady-state d.e.'s (9) and (12) and under the further operating conditions (14) and (19) these must be solved subject to special feed boundary conditions (22) and (26) together with the general terminal conditions (28) and (30). In our symmetrical steady-state situation therefore, the system for solution is as follows, if $L_2 = L_1 = L$ ...(32)

$$d^2Y/dh^2 = d^2X'/dh^2 = 0 \qquad ...(33)$$

$$X'(0) + 1-Y(0) + dY(0)/dh = 2/(\alpha+1) \qquad ...(34)$$

$$X'(0) + 1-Y(0) + dX'(0)/dh = 2/(\alpha+1) \qquad ...(35)$$

$$dY(L)/dh = \varepsilon\{1-Y(L)\} \qquad ...(36)$$

and

$$dX'(-L)/dh = \varepsilon X'(-L) \qquad ...(37)$$

As shown in Appendix 1 this system has the solution

$$X'(-h) = 1 - Y(h) \qquad ...(38)$$

and

$$Y(h) = Y(0) + Gh \qquad ...(39)$$

where profile gradient G is constant and given by

$$G = 2\varepsilon/(\alpha+1)(2\varepsilon L+\alpha+1) \qquad ...(40)$$

and initial values $X'(0)$ and $Y'(0)$ are given by

$$X'(0) = 1-Y(0) = 2(\varepsilon L+1)/(\alpha+1)(2\varepsilon L+\alpha+1) \qquad ...(41)$$

From (38) to (41) it is also readily shown that the terminal values $X'(-L)$ and $Y(L)$ are given by

$$X'(-L) \quad = \quad 1-Y(L) \quad = \quad 2/(\alpha+1)(2\varepsilon L+\alpha+1) \qquad \ldots(42)$$

2.6 Comparison of large signal behaviour of tray and packed columns in steady-state

It is interesting to note that the formula (40) for normalised composition gradient G is identical with that derived for packed columns in the earlier report[3]. It must be noted however that normalised distance, h, is here expressed thus

$$h \quad = \quad h'/\delta h' \qquad \ldots(43)$$

whereas for packed columns h was given by

$$h \quad = \quad h'/(V_r/k) \qquad \ldots(44)$$

where k was the evaporation rate per unit length. Packed and tray columns in which $\delta h' = V_r/k$ will therefore perform identical steady-state duties. Under these circumstances we should also note that the formulae for $X'(0)$, $Y(0)$ $X'(-L)$ and $Y(L)$ above are identical to those for the equilibrium values $X_e'(0)$, $Y_e(0)$, $X_e'(-L)$ and $Y_e(L)$ in the equivalent packed column. This is to be expected since the tray column model assumes continuous vapour/liquid equilibrium.

Despite the similarity in steady large-signal behaviour we shall however find important differences in the small signal behaviour of the two systems about the steady state.

3. Small perturbation model

3.1 Partial differential equations

If lower-case symbols denote small changes in the variables previously denoted by upper case symbols, then implicitly differentiating the system's general p.d.e.'s (7) and (10) we obtain, on substituting steady state solutions for the upper-case symbols,

$$H_\ell \delta h' \frac{\partial x}{\partial t} = (\ell - \frac{v}{\alpha})\alpha G + \frac{V_r}{\alpha} \frac{\partial^2 x}{(\partial h')^2} (\delta h')^2$$

which, in terms of normalised time and distance and y, rather than x, becomes

$$\partial y/\partial \tau = (\ell\alpha - v) G/V_r + \partial^2 y/\partial h^2 \qquad \ldots(45)$$

and

$$H_\ell' \delta h' \frac{\partial x'}{\partial t} = (\ell - v\alpha) G + L_s \frac{\partial^2 x'}{(\partial h')^2} (\delta h')^2$$

which, since $V_r = L_s$ in our special symmetrical case, reduces to the normalised form

$$\partial x'/\partial \tau = (\ell - v\alpha) G/V_r + \partial^2 x'/\partial h^2 \qquad \ldots(46)$$

If, as in the packed column analysis, we now introduce the vector

$$\underline{q} = \begin{pmatrix} y - x' \\ y + x' \end{pmatrix} \qquad \ldots(47)$$

then the small signal p.d.e.'s (45) and (46) may be grouped to form the simple matrix equation

$$(\partial^2/\partial h^2 - \partial/\partial\tau)\underline{q} = \begin{pmatrix} -(\alpha-1), & 0 \\ 0, & \alpha+1 \end{pmatrix} \underline{u} \qquad \ldots(48)$$

where input vector $\underline{u}$ is given by

$$\underline{u} = \frac{G}{V_r} \begin{pmatrix} v+\ell \\ v-\ell \end{pmatrix} \qquad \ldots(49)$$

Laplace transforming (48) in s with respect to h and in p with respect to τ yields

$$(s^2-p)\tilde{\tilde{\underline{q}}} - s\tilde{\underline{q}}(0) - \dot{\tilde{\underline{q}}}(0) = s^{-1} \begin{pmatrix} -(\alpha-1), & 0 \\ 0, & \alpha+1 \end{pmatrix} \underline{u} \qquad \ldots(50)$$

where superscript "$\tilde{\tilde{}}$" denotes transforms w.r.t. h and τ, "~" w.r.t. τ only and ".", the spatial derivative.

3.2 Terminal boundary conditions

Differentiating (27) implicitly to obtain the small-perturbation boundary equation for the reboiler we obtain:

$$H_a\, \alpha \frac{\partial y(L_1)}{\partial t} = v\left[Y(L_1)\varepsilon + \varepsilon - \frac{\partial Y(L_1)}{\partial h'}\,\delta h'\right]$$

$$+ V_r\{ -\varepsilon y(L_1) - \frac{\partial y(L_1)}{\partial h'} \}$$

Now from (27) it is clear the coefficient of v above is zero in steady state so that the above equation reduces to the normalised form:

$$T_a \partial y(L_1)/\partial\tau = -\{\varepsilon y(L_1) + \partial y(L_1)/\partial h\} \qquad \ldots(51)$$

A similar treatment applied to the reboiler yields the similar result:

$$T_b \partial x'(-L_2)/\partial\tau = -\varepsilon x'(-L_2) + \partial x'(-L_2)/\partial h \qquad \ldots(52)$$

3.3 Inverted U-tube model

Now in the analysis of the packed column it was found that, in the symmetrical case $L_1 = L_2 = L$, solution of the small-signal equations was considerably simplified by redefining h and h' from an origin at the ends of the column rather than the central feed point by first bending the column conceptually into an inverted U-tube so that coordinate h is replaced by L-h in the rectifier and h by L+h in the stripper. Odd-powered spatial-derivatives in the rectifier are consequently reversed in sign whilst even-powered derivatives in the rectifier and all derivatives in the stripping-section are unaffected in sign.

The terminal boundary conditions (51) and (52) thus become

$$T_a \partial y(0)/\partial\tau = -\varepsilon y(0) + \partial y(0)/\partial h$$

and

$$T_b \partial x'(0)/\partial\tau = -\varepsilon x'(0) + \partial x'(0)/\partial h$$

or, if

$$T_a = T_b = T \qquad \ldots(53)$$

$$T\, p\, \underline{\tilde{q}}(0) = -\varepsilon\underline{\tilde{q}}(0) + \underline{\dot{\tilde{q}}}(0) \qquad \ldots(54)$$

Now the p.d.e. (48) is unaffected by change of distance base since this involves only even powered spatial derivatives so that its transformed version (50) is also unaffected provided $\underline{\tilde{q}}(0)$ and $\underline{\dot{\tilde{q}}}(0)$ are now interpreted as terminal conditions rather than feed-point conditions as previously.

The unknown $\underline{\dot{\tilde{q}}}(0)$ may therefore be eliminated from (50) using boundary condition (54) to give

$$(s^2-p)\underline{\tilde{q}} - (s+\varepsilon+Tp)\underline{\tilde{q}}(0) = s^{-1}\begin{pmatrix} -(\alpha-1) , & 0 \\ 0 & , \alpha+1 \end{pmatrix} \underline{\tilde{u}} \qquad \ldots(55)$$

now involving only the one unkown vector $\underline{q}(0)$ which may be eliminated after inversion by substitution of the feed conditions not yet invoked in this small signal analysis.

3.4 Feed boundary conditions

For the small signal feed boundary conditions it is necessary to differentiate implicitly the general large signal equations (13) and (23) then substituting the special-case steady-state solutions for the upper case symbols. Retaining the original distance base for the moment we thus obtain, for the rectifier

$$H_\ell \delta h' \partial x(0)/\partial t = v\{Y'(0)-Y(0)\}+V_r\{y'(0)/\alpha - y(0)\}$$
$$+ \ell\delta h'\partial X(0)/\partial h' + (V_r/\alpha)\delta h'\partial x(0)/\partial h'$$

or, in terms of y(0) rather than x(0) and normalising:

$$\partial y(0)/\partial\tau = (v/V_r)\{Y'(0)-Y(0)\}+\ell\alpha G/V_r + \partial y(0)/\partial h + x'(0) - y(0) \quad \ldots(56)$$

Now Y'(0)-Y(0) may be eliminated using the large signal steady-state solution (Section 2.5) thus

$$Y'(0)-Y(0) = \alpha X(0)'+1-Y(0)-1 = (\alpha+1)X'(0)-1$$
$$= 2(\varepsilon L+1)/(2\varepsilon L+\alpha+1)-1 = -\varepsilon/(2\varepsilon L+\alpha+1)$$
$$= -\ G(\alpha+1)/2$$

so that (56) reduces to

$$\partial y(0)/\partial\tau = (G/V_r)\{-(\alpha+1)v/2+\alpha\ell\}+\partial y(0)/\partial h + x'(0) - y(0) \quad \ldots(57)$$

An identical treatment applied to the first tray of the stripping section produces the similar result

$$\partial x'(0)/\partial\tau = (G/V_r)\{-\alpha v+(\alpha+1)\ell/2\}-\partial x'(0)/\partial h + y(0) - x'(0) \quad \ldots(58)$$

Transforming (57) and (58) to the new distance base for the inverted U-tube therefore yields

$$\partial y(L)/\partial\tau = (G/V_r)\{-(\alpha+1)v/2+\alpha\ell\}-\partial y(L)/\partial h+x'(L)-y(L)$$

and ...(59)

$$\partial x'(L)/\partial\tau = (G/V_r)\{-\alpha v+(\alpha+1)\ell/2\}-\partial x'(L)/\partial h+y(L)-x'(L)$$

Adding and subtracting these final equations and grouping the results in matrix form therefore yields

$$\begin{pmatrix} \partial/\partial t+\partial/\partial h+2 & , & 0 \\ 0 & , & \partial/\partial t+\partial/\partial h \end{pmatrix} \underline{q}(L) = \begin{pmatrix} 0.5\varepsilon & , & 0 \\ 0 & , & -0.5(3\alpha+1) \end{pmatrix} \underline{u}$$

or, in terms of Laplace transforms

$$\begin{pmatrix} p+\partial/\partial h+2 & , & 0 \\ 0 & , & p+\partial/\partial h \end{pmatrix} \underline{\tilde{q}}(L) = \begin{pmatrix} 0.5\varepsilon & , & 0 \\ 0 & , & -0.5(3\alpha+1) \end{pmatrix} \underline{\tilde{u}} \qquad \ldots(60)$$

3.6 Solution

From transformed p.d.e. (55) we get

$$\underline{\tilde{\tilde{q}}} = \frac{s+\varepsilon+Tp}{s^2-p}\,\tilde{q}(0) + \frac{1}{s(s^2-p)} \begin{pmatrix} -(\alpha-1) & , & 0 \\ 0 & , & \alpha+1 \end{pmatrix} \underline{u}$$

so that, inverting from the s,p domain to the h,p domain gives

$$\underline{\tilde{q}}(h) = \{\sqrt{p}\cosh\sqrt{p}h + (\varepsilon+Tp)\sinh\sqrt{p}h\}\underline{\tilde{q}}(0)/\sqrt{p}$$

$$-\{(1-\cosh\sqrt{p}h)/p\} \begin{pmatrix} -(\alpha-1) & , & 0 \\ 0 & , & (\alpha+1) \end{pmatrix} \underline{\tilde{u}} \qquad \ldots(61)$$

Now to substitute boundary condition (60) we first require $\partial\underline{\tilde{q}}(L)/\partial h$ which may be obtained by differentiating (61) w.r.t. h and setting h = L, giving

$$\frac{\partial\tilde{q}(L)}{\partial h} = \frac{\{p\ \sinh\sqrt{p}L + \sqrt{p}(\varepsilon+Tp)\cosh\sqrt{p}L\}}{\sqrt{p}}\ \underline{\tilde{q}}(0)$$

$$+ \frac{\sinh\sqrt{p}L}{\sqrt{p}} \begin{pmatrix} -(\alpha-1) & , & 0 \\ 0 & , & (\alpha+1) \end{pmatrix} \underline{\tilde{u}} \qquad \ldots(62)$$

The element of

$$\underline{\tilde{q}}(0) = \begin{pmatrix} \tilde{q}_1(0) \\ \tilde{q}_2(0) \end{pmatrix} \qquad \ldots(63)$$

are now calculated individually from (60), (61) and (62).

For $q_1(0)$ we have

$$(p + 2 + \partial/\partial h)\tilde{q}_1(L) = 0.5\varepsilon\tilde{u}_1$$

and on substituting for $\tilde{q}_1(L)$ and $\partial\tilde{q}_1(L)/\partial h$ we get

$$\frac{(p+2)\{\sqrt{p}\cosh\sqrt{p}L+(\varepsilon+Tp)\sinh\sqrt{p}L\}}{\sqrt{p}}\ \tilde{q}_1(0) + \frac{\{p\sinh\sqrt{p}L+\sqrt{p}(\varepsilon+Tp)\cosh\sqrt{p}L\}}{\sqrt{p}}\ \tilde{q}_1(0)$$

$$+ \frac{(p+2)}{p}\ (1-\cosh\sqrt{p}L)\varepsilon\tilde{u}_1 - \frac{(\sinh\sqrt{p}L)}{\sqrt{p}}\ \varepsilon\tilde{u}_1 = 0.5\varepsilon\tilde{u}_1$$

yielding the final solution

$$\frac{\tilde{q}_1(0)}{\tilde{u}_1} = \frac{\varepsilon\{(p+2)(\cosh\sqrt{p}L-1)/p+(\sinh\sqrt{p}L)/\sqrt{p}+0.5\}}{\{(1+T)p+2+\varepsilon\}\cosh\sqrt{p}L+\{(p+2)(\varepsilon+Tp)+p\}(\sinh\sqrt{p}L)/\sqrt{p}} \qquad \ldots(64)$$

For $\tilde{q}_2(0)$, from the feed boundary condition, we have

$$(p+\partial/\partial h)\tilde{q}_2(L) = -0.5(3\alpha+1)\tilde{u}_2$$

so that on substituting for $\tilde{q}_2(L)$ and $\partial\tilde{q}_2(L)/\partial h$ we get

$$\{p\cosh\sqrt{p}L+\sqrt{p}(\varepsilon+Tp)\sinh\sqrt{p}L+\sqrt{p}\sinh\sqrt{p}L+(\varepsilon+Tp)\cosh\sqrt{p}L\}\tilde{q}_2(0)$$

$$+(\alpha+1)(\cosh\sqrt{p}L-1)\tilde{u}_2+(\alpha+1)(\sinh\sqrt{p}L)\tilde{u}_2/\sqrt{p} = -0.5(3\alpha+1)\tilde{u}_2$$

yielding the final solution:

$$\frac{\tilde{q}_2(0)}{\tilde{u}_2} = -\frac{(\alpha+1)(\cosh\sqrt{p}L-1)+(\alpha+1)(\sinh\sqrt{p}L)/\sqrt{p}+0.5(3\alpha+1)}{\{p(1+T)+\varepsilon\}\cosh\sqrt{p}L+\sqrt{p}(1+\varepsilon+Tp)\sinh\sqrt{p}L} \quad \ldots(65)$$

Setting $p = j\omega$ therefore, equations (64) and (65) may be used to compute the two Nyquist or inverse Nyquist loci for this diagonal system. Despite the complexity of the hyperbolic functions involved, the general form of the loci may be deduced with some precision analytically by considering the high- and low-frequency behaviour of these functions.

4. General form of inverse Nyquist Loci

The system may be described in transfer-function form

$$\underline{\tilde{q}}(h,p) = \underline{G}(h,p)\ \underline{\tilde{u}}(p) \quad \ldots(66)$$

where transfer-function matrix (T.F.M.) $\underline{G}(h,p)$ takes the diagonal form

$$\underline{G}(h,p) = \begin{pmatrix} g_1(h,p) & , & 0 \\ 0 & , & g_2(h,p) \end{pmatrix} \quad \ldots(67)$$

or alternatively

$$\underline{\tilde{u}}(p) = \underline{G}^*(h,p)\ \underline{\tilde{q}}(h,p) \quad \ldots(68)$$

where inverse T.F.M. $\underline{G}^*(h,p)$ is also diagonal, taking the form

$$\underline{G}^*(h,p) = \begin{pmatrix} g_1^*(h,p) & , & 0 \\ 0 & , & g_2^*(h,p) \end{pmatrix} \qquad ...(69)$$

where, of course,

$$g_1^*(h,p) = g_1^{-1}(h,p)$$
and
$$g_2^*(h,p) = g_2^{-1}(h,p) \qquad ...(70)$$

Attention will here be confined to the system outputs and therefore to $G^*(0,p)$, the elements of which may be calculated directly from (64) and (65).

4.1 Zero-frequency behaviour

By considering the limits, as $p \to 0$, of the right-hand sides of (64) and (65) it is quickly deduced that

$$g_1^*(0,0) = \frac{2\varepsilon L + \alpha + 1}{\varepsilon L^2 + \varepsilon L + 0.5\varepsilon} \qquad ...(71)$$

and
$$g_2^*(0,0) = -\frac{\varepsilon}{(\alpha+1)L + 0.5(3\alpha+1)} \qquad ...(72)$$

(These solutions may be obtained by alternatively solving the small signal p.d.e.'s (48), subject to the boundary conditions, with operator $\partial/\partial\tau$ set to zero).

It is interesting here to compare these results with the equivalent expressions for packed-columns[3], viz:

$$g_1^*(0,0) = \frac{2\varepsilon L + \alpha + 1}{\alpha\{\varepsilon L^2 - (\alpha+1)L - 0.5\varepsilon\}} \qquad ...(73)$$

and
$$g_2^*0,0) = -\frac{\varepsilon}{\alpha\{(\alpha+1)L + 0.5\varepsilon\}} \qquad ...(74)$$

the expressions approaching agreement as L becomes >>1 for which the tray-column model is only valid anyway since L = actual length of rectifier (or stripper)/$\delta h'$ = number of trays per section which, of course, must >>1 for the continuous spatial approximation to hold good. An important feature to note is that the static gain $g_1(0,0)$ for the long tray column can only be positive, unlike its packed column counterpart whose sign is parameter-dependent.

4.2 High-frequency behaviour

As ω is increased from zero it becomes necessary to consider the functions of p in expressions (64) and (65). Considering frequencies in the band

$$1.0 >> |p^{0.5}| >> 1/L \qquad \ldots(75)$$

which is a wide band, recalling that

$$L >> 1.0 \qquad \ldots(76)$$

it is clear that the hyperbolic functions $\cosh\sqrt{p}L$ and $\sinh\sqrt{p}L$ may be replaced by $0.5\exp(\sqrt{p}L)$ since

$$\sqrt{p} = (0.5\omega)^{0.5}(1+j) \qquad \ldots(77)$$

and therefore has a positive real part allowing the terms in $\exp(-\sqrt{p}L)$ to be neglected. Because of the upper bound on p set by (75) only the lowest powers of p outside the hyperbolic functions need be considered so that, from (64) we obtain

$$\frac{\tilde{q}_1(0,p)}{\tilde{u}_1(p)} \rightarrow \frac{\varepsilon\{2/p + 1/\sqrt{p}\}\exp(\sqrt{p}L)/2}{\{2 + \varepsilon + 2\varepsilon/\sqrt{p}\}\exp(\sqrt{p}L)/2} \qquad \ldots(78)$$

and

$$\frac{\tilde{q}_2(0,p)}{\tilde{u}_2(p)} \rightarrow -\frac{(\alpha+1)(1 + 1/\sqrt{p})\ \exp(\sqrt{p}L)/2}{\{p(1+T)+\varepsilon+\sqrt{p}(1+\varepsilon+Tp)\}\exp(\sqrt{p}L)/2} \qquad \ldots(79)$$

the exponential terms clearly cancelling. It is insufficient merely to now ignore all but the lowest powers of p however since

$$\varepsilon << 1.0 \qquad ...(80)$$

if (76) is to be satisfied as inspection of the static performance curves of Fig. 5 reveals. Terms having ε as coefficient are therefore non-dominant and (78) and (79) consequently reduce to

$$\left.\begin{aligned} \tilde{q}_1(0,p)/\tilde{u}_1(p) &= 2\varepsilon/(1+\alpha)p \\ \tilde{q}_2(0,p)/\tilde{u}_2(p) &= -(1+\alpha)/\alpha p \end{aligned}\right\}, \; L^{-1} << |p^{0.5}| << 1.0 \qquad ...(81)$$

or

$$\left.\begin{aligned} g_1^*(0,p) &= (1+\alpha)p/2\varepsilon \\ g_2^*(0,p) &= -\alpha p/(1+\alpha) \end{aligned}\right\}, \; L^{-1} << |p^{0.5}| << 1.0 \qquad ...(82)$$

The system thus approaches a purely integrating process above $\omega = L^{-1}$. Above $\omega = 1.0$ the dynamic response changes, but of course all real processes exhibit highly complex behaviour if the frequency of excitation is raised sufficiently. The band $L^{-1} << \omega << 1.0$ is however a wide band since the inverse gain moduli are of order

$$\left.\begin{aligned} |g_1^*(0,j\omega)| &\simeq \omega/\varepsilon \\ |g_2^*(0,j\omega)| &\simeq \omega/2 \end{aligned}\right\} \; L^{-1} << \omega^{0.5} << 1.0 \qquad ...(83)$$

compared to their approximate zero-frequency values from (71) to (72):

$$\left.\begin{aligned} |g_1^*(0,0)| &\simeq 2/L \\ |g_2^*(0,0)| &\simeq \varepsilon/2L \end{aligned}\right\} \qquad ...(84)$$

In approximating the high-frequency behaviour of the tray-column no wave phenomena have become apparent** which is not surprising since this model is confined to long columns which would be expected to drastically attenuate any reflected waves well before their arrival at the top and bottom of the column. Furthermore we note that the signs of the high-and low-frequency gains are identical in both cases so suggesting minimum-phase behaviour unlike packed-columns[3]. This would also seem to validate the use of a multivariable first-order lag approximation to tray column dynamics. Such a model is readily derived from the system p.d.e.'s without the necessity of tedious dynamic solution as is now demonstrated.

4.3 Multivariable first-order lag approximation

Owens[4,5] has proposed that a system of inverse T.F.M. $\underline{G}^*(p)$ whose high-and low-frequency behaviour is constrained as follows:

$$\lim_{p\to 0} \underline{G}^*(p) = \underline{A}_1 \qquad \ldots(85)$$

and

$$\lim_{|p|\to\infty} p^{-1}\underline{G}^*(p) = \underline{A}_o \qquad \ldots(86)$$

, where $\underline{A}_1$ and $\underline{A}_2$ are constant matrices, may under certain conditions, be closely approximated for controller design purposes by the multivariable first-order lag system

$$\underline{G}_A^*(p) = \underline{A}_1 + \underline{A}_o p$$

One restriction precluding the application of the approximation is nonminimum-phase behaviour of $\underline{G}^*(p)$ such as can be encountered in packed-columns[3] when the associated elements of $\underline{A}_o$ and $\underline{A}_1$ take

---

**No wave phenomena at the ends of the column have been revealed in this analysis. A consideration of $h \neq 0$ might however produce such effects.

opposite signs. No such phenomena have been revealed in the foregoing study of tray-columns however and, furthermore, the model reduction technique and associated control system design has been successfully applied to a simulated tray column in an earlier investigation[2] by Owens and the present author. The present investigation would suggest, on purely analytical grounds, that the technique is applicable to long tray-type columns in general.

The matrices $\underline{A}_1$ and $\underline{A}_o$ are, furthermore, easily obtained without solving the system p.d.e.'s. $\underline{A}_1$ is, of course, given by

$$\underline{A}_1 = G^*(0,0) = \begin{pmatrix} g_1^*(0,0) , & 0 \\ 0 , & g_2^*(0,0) \end{pmatrix} \qquad \ldots(87)$$

the elements of which may be found by solution of merely the steady-state version of the system p.d.e. first putting $\partial/\partial\tau$ to zero. {The ease of this is demonstrated in Appendix 2 and the solutions will be seen to agree with equations (71) and (72)}.

$\underline{A}_o$ may be obtained by ignoring all but the p-dependent coefficients of the dependent variables in the system p.d.e. (48) transformed w.r.t. τ, giving

$$- p \, \underline{\tilde{q}} = \begin{pmatrix} -\varepsilon , & 0 \\ 0 , & \alpha+1 \end{pmatrix} \underline{\tilde{u}} \qquad \ldots(88)$$

from which we deduce that

$$\underline{A}_o = \begin{pmatrix} \varepsilon^{-1} , & 0 \\ 0 , & -(\alpha+1)^{-1} \end{pmatrix} \qquad \ldots(89)$$

which compares closely with

$$p^{-1}\underline{G}^*(0,p) = \begin{pmatrix} (1+\alpha)\varepsilon^{-1}/2 & , & 0 \\ 0 & , & -\alpha(\alpha+1)^{-1} \end{pmatrix} \qquad \ldots(90)$$

obtained from equation (82) in the high-frequency analysis of Section 4.2, since $0<\varepsilon<<1.0$ and therefore $\alpha \simeq 1.0$ for long tray-columns. Inverse Nyquist loci for the tray-column should therefore take the approximate form shown in Fig. 6.

5. Discussion and Conclusions

A parametric transfer-function matrix model has been derived completely analytically for long, symmetrically-operated, tray-type distillation columns separating binary mixtures. As with the earlier packed-column analysis[3], the system has been shown to enjoy a completely diagonal structure provided the selected outputs are $y(h,0)-x'(h,0)$ and $y(h,0)+x'(h,0)$ and the selected inputs are $v+\ell$ and $v-\ell$, where y denotes the change in top vapour composition, x' that in bottom liquid composition and v and ℓ are perturbations in the flow rates of vapour and reflux respectively.

Unlike the case of the packed-column, the gain of both elements of the T.F.M. are shown to have the same sign at high-and low-frequency so that nonminimum-phase effects are not anticipated in tray-column behaviour thus permitting confidence in the general application of a multivariable first-order lag approximation to long tray-type columns. The difference in the high-frequency behaviour of the two systems occurs in their responses to total flow change $v+\ell$ which, when suddenly increased in a packed-column, causes weaker vapour and richer liquid to be initially transported towards the top and bottom ends of the column respectively so producing a transient reduction in the overall separation y-x', even though the final response in y-x' may be positive. No such initial effect can occur in tray-columns because of the continuous equilibrium between vapour and liquid on each tray. The nonminimum-phase behaviour of packed-columns is therefore the result of there being distinct capacitances in the vapour and liquid streams separated by an interphase "resistance". The mere inclusion of vapour capacitance in the tray model would not produce similar effects while ever the continuous equilibrium-(i.e. zero resistance-) assumption is retained.

The analysis has indicated the absence of significant effects from the reflection of travelling waves in the tray-type column on the composition changes at the ends of the column. Such effects might nevertheless occur closer to the feed tray however and this possibility should be investigated since control measurements are frequently taken from points well away from the ends of distillation columns.

Since it has relied on the approximation of discretely-changing spatial functions by approximate continuous functions, the analysis here reported is, of course, restricted to columns of many trays separating mixtures of low relative-volatility. An analysis of short columns of both the tray- and packed-type would therefore complete the picture of general column behaviour.

6. References

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(2) Edwards, J.B., and Owens, D.H.: 'First-order type models for multivariable process control', Proc. I.E.E., 1977, 124, (11), pp.1083-1088.

(3) Edwards, J.B.: 'The analytical modelling and dynamic behaviour of a spatially continuous binary distillation column', University of Sheffield, Dept. of Control Eng., Research Report No. 86, April 1979, 45pp.

(4) Owens, D.H.: 'First and second-order-like structures in linear multivariable control system design', Proc. I.E.E., 1975, 122, (9), pp.935-941.

(5) Owens, D.H.: 'Feedback and Multivariable Systems Design', I.E.E. Control Engineering Series 7, Peter Peregrinus, 1978, 320pp.

7. List of Symbols

α	- initial slope of equilibrium curve approximation
β	- relative volatility of mixture
D	- molar distillate rate
ε	- $\alpha-1$
F	- molar feed rates of liquid and vapour
G	- normalised spatial composition gradient in steady-state
$\underline{G}$	- transfer function matrix (T.F.M.)

$\underline{G}^*$	-	inverse T.F.M.
$\underline{G}_A^*$	-	inverse T.F.M. of multivariable first-order lag approximation
g_1^*, g_2^*	-	diagonal elements of $\underline{G}^*$
g_{A1}^*, g_{A2}^*	-	diagonal elements of $\underline{G}_A^*$
H_ℓ, H_ℓ'	-	liquid capacitances p.u. length of rectifier and stripper
H_a, H_b	-	capacitances of accumulator and reboiler
h'	-	distance along column
$\delta h'$	-	length of column per tray
h	-	normalised distance $(h'/\delta h')$
$\underline{I}$	-	unit diagonal 2x2 matrix
L_1, L_2	-	normalised lengths of entire rectifier and stripper (= L where identical)
L_r, L_s, ℓ	-	molar flows of liquid in rectifier and stripper and small changes therein
n	-	tray number
N	-	number of top (bottom) tray in rectifier (stripper)
p	-	Laplace variable for transforms w.r.t. τ
$\underline{q}$	-	vector of difference and total of vapour and liquid composition changes
s	-	Laplace variable for transforms w.r.t. h
t	-	time
τ	-	normalised time $(= tV_r/\alpha H_\ell \delta h')$
T_a, T_b	-	normalised time-constants of accumulator and reboiler (= T where identical)
$\underline{u}$	-	vector of total and difference in vapour and reflux rate changes

$V_r(=V), V_s, v$	-	molar flows of vapour in rectifier and stripper and small changes therein
W	-	molar flow rate of bottom product
X,X'	-	liquid compositions (mol fractions) in rectifier and stripper
x,x'	-	small changes in X and X'
Y,Y'	-	vapour compositions in rectifier and stripper
y,y'	-	small changes in Y and Y'
Z	-	feed liquid composition
z	-	feed vapour composition
≈	-	superscript denoting Laplace transforms w.r.t. h and τ
~	-	superscript denoting Laplace transforms w.r.t. τ only
T.F.M.	-	transfer-function matrix
p.d.e.	-	partial differential equation
d.e.	-	ordinary differential equation

Appendix 1

Calculation of steady-state composition profiles

The large-signal steady-state system is described by the differential equations

$$d^2Y/dh^2 = d^2X'/dh^2 = 0 \qquad \text{...(A1.1)}$$

which are subject to the boundary conditions

$$X'(0)+1-Y(0)+dY(0)/dh = 2/(\alpha+1) \qquad \text{...(A1.2)}$$

$$X'(0)+1-Y(0)+dX'(0)/dh = 2/(\alpha+1) \qquad \text{...(A1.3)}$$

$$dY(L)/dh = \varepsilon\{1-Y(L)\} \qquad \text{...(A1.4)}$$

and

$$dX'(L)/dh = \varepsilon X'(-L) \qquad \text{...(A1.5)}$$

From the system symmetry it is clear that

$$X'(-h) = 1-Y(h) \qquad \text{...(A1.6)}$$

and from (A1.1) that

$$dY/dh = G = \text{constant}$$

$$\text{and} \quad dX'/dh = G' = \text{constant} \qquad \text{...(A1.7)}$$

and from (A1.6) it is obvious that

$$G = G' \qquad \text{...(A1.8)}$$

Hence

$$Y(h) = Y(0) + Gh$$

and

$$X'(h) = X'(0) + Gh \qquad \text{...(A1.9)}$$

Now from (A1.2) or (A1.3) and (A1.6) we get

$$2X'(0) + G = 2/(\alpha+1) \qquad \text{...(A1.10)}$$

and from (A1.4) or (A1.5) and (A1.9)

$$G = \varepsilon X'(-L) = \varepsilon\{X'(0)-GL\}$$

$$\therefore \quad X'(0) = G(\varepsilon L+1)/\varepsilon \qquad \text{...(A1.11)}$$

Hence, eliminating $X'(0)$ between (A1.10) and (A1.11), we get

$$2G(\varepsilon L+1)/\varepsilon + G = 2/(\alpha+1)$$

giving $G = 2\varepsilon/(\alpha+1)(2\varepsilon L+\alpha+1)$ ...(A1.12)

stated as equation (40) in the main text.

Hence from (A1.11) we get, eliminating G,

$$X'(0) = 2(\varepsilon L+1)/(\alpha+1)(2\varepsilon L+\alpha+1) = 1-Y(0) \qquad \text{...(A1.13)}$$

stated as equation (41) in the main text.

$X'(-L)$ and $Y(L)$ may now be calculated from general equations (A1.9) to give

$$X'(-L) = 1-Y(L) = 2/(\alpha+1)(2\varepsilon L+\alpha+1) \qquad \text{...(A1.14)}$$

stated as equation (42) in the main text.

Appendix 2

Calculation of static gains

From equation (55) the Laplace transformed p.d.e. of the system may be expressed:

$$\tilde{\tilde{\underline{q}}} = \left\{\frac{s}{s^2-p} + \frac{\varepsilon+Tp}{s^2-p}\right\} \tilde{\underline{q}}(0) + \frac{1}{s(s^2-p)} \begin{pmatrix} -(\alpha-1) & , 0 \\ 0 & , \alpha+1 \end{pmatrix} \tilde{\underline{u}} \qquad \ldots(A2.1)$$

so that, if $\underline{u}$ is a vector of step functions in time, and p set to zero for steady-state solutions, then we have:

$$\acute{\underline{q}} = (s^{-1}+\varepsilon s^{-2})\underline{q}(0) + s^{-3} \begin{pmatrix} -\varepsilon & , 0 \\ 0 & , \alpha+1 \end{pmatrix} \underline{u} \qquad \ldots(A2.2)$$

where $\acute{}$ here denotes Laplace transforms w.r.t. h only. Inverting back to the h-domain therefore we get

$$\underline{q} = (1+\varepsilon h)\underline{q}(0) + \begin{pmatrix} -\varepsilon h^2/2 & , 0 \\ 0 & , (\alpha+1)h^2/2 \end{pmatrix} \underline{u} \qquad \ldots(A2.3)$$

To find $\underline{q}(0)$ we now substitute the steady-state feedpoint boundary condition, (at h = L), viz:

$$\begin{pmatrix} d/dh+2 & , 0 \\ 0 & , d/dh \end{pmatrix} \underline{q}(L) = \begin{pmatrix} 0.5\varepsilon & , 0 \\ 0 & , -0.5(3\alpha+1) \end{pmatrix} u \qquad \ldots(A2.4)$$

$$\text{Now } q_1(h) = (1+\varepsilon h)q_1(0) - \varepsilon h^2 u_1/2 \qquad \ldots(A2.5)$$

$$\text{and hence } dq_1(h)/dh = \varepsilon\, q_1(0) - \varepsilon h u_1 \qquad \ldots(A2.6)$$

$$\text{so that} \quad (d/dh+2)q_1(L) = (2+2\varepsilon L+\varepsilon)q_1(0)-(\varepsilon L^2+\varepsilon L)u_1 = 0.5\varepsilon u_1$$

$$\text{giving} \quad q_1(0) = \frac{(\varepsilon L^2 + \varepsilon L + 0.5\varepsilon)}{2 + 2\varepsilon L + \varepsilon} u_1 \qquad \ldots(A2.7)$$

or since q_1 and u_1 are generally functions of τ and therefore their transforms w.r.t. τ functions of p, i.e. $\tilde{q}_1 = \tilde{q}_1(h,p)$ and $\tilde{u}_1 = \tilde{u}_1(p)$ we may write

$$\tilde{q}_1(0,0) = \frac{(\varepsilon L^2 + \varepsilon L + 0.5\varepsilon)}{2\varepsilon L + \alpha + 1}\,\tilde{u}_1(0) \qquad \text{...(A2.8)}$$

Now for $q_2(0)$ we note from (A2.4) that

$$dq_2(L)/dh = -0.5(3\alpha+1)u_2 \qquad \text{...(A2.9)}$$

and from general static solution A2.3 that

$$q_2(h) = (1+\varepsilon h)q_2(0) + (\alpha+1)h^2u_2/2 \qquad \text{...(A2.10)}$$

and therefore $dq_2(h)/dh = \varepsilon\, q_2(0) + (\alpha+1)hu_2$ ...(A2.11)

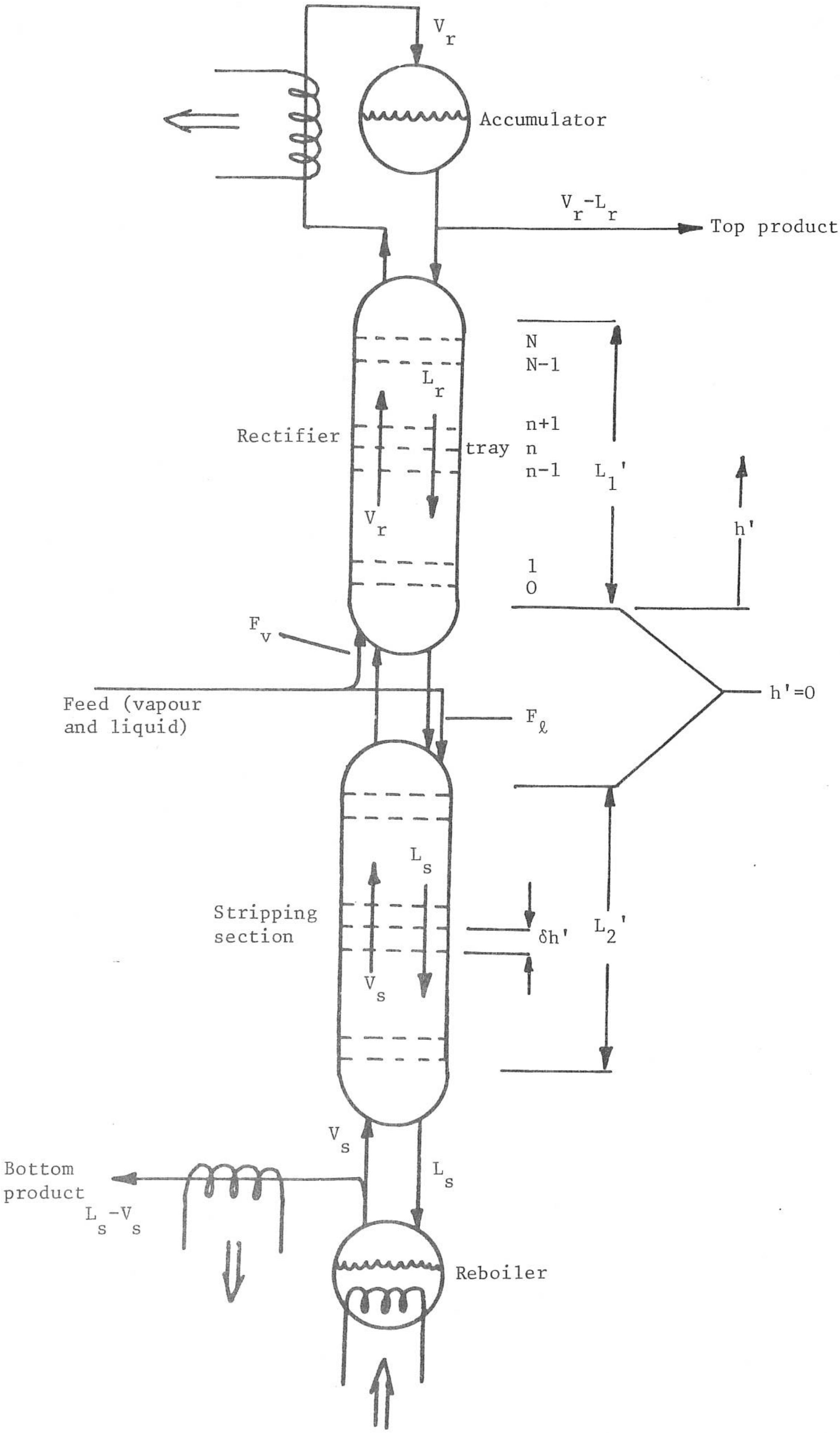
so that setting h = L in (A2.11) and substituting in (A2.9) we obtain

$$q_2(0) = -\varepsilon^{-1}\{(\alpha+1)L + 0.5(3\alpha+1)\}u_2$$

or in terms of Laplace transforms $\tilde{q}_2(0,p)$ and $\tilde{u}_2(p)$ in p w.r.t. τ, with p set to zero for static solutions we have

$$\tilde{q}_2(0,0) = -\varepsilon^{-1}\{(\alpha+1)L + 0.5(3\alpha+1)\}\tilde{u}_2(0)$$

Fig. 1. Illustrating the Complete System



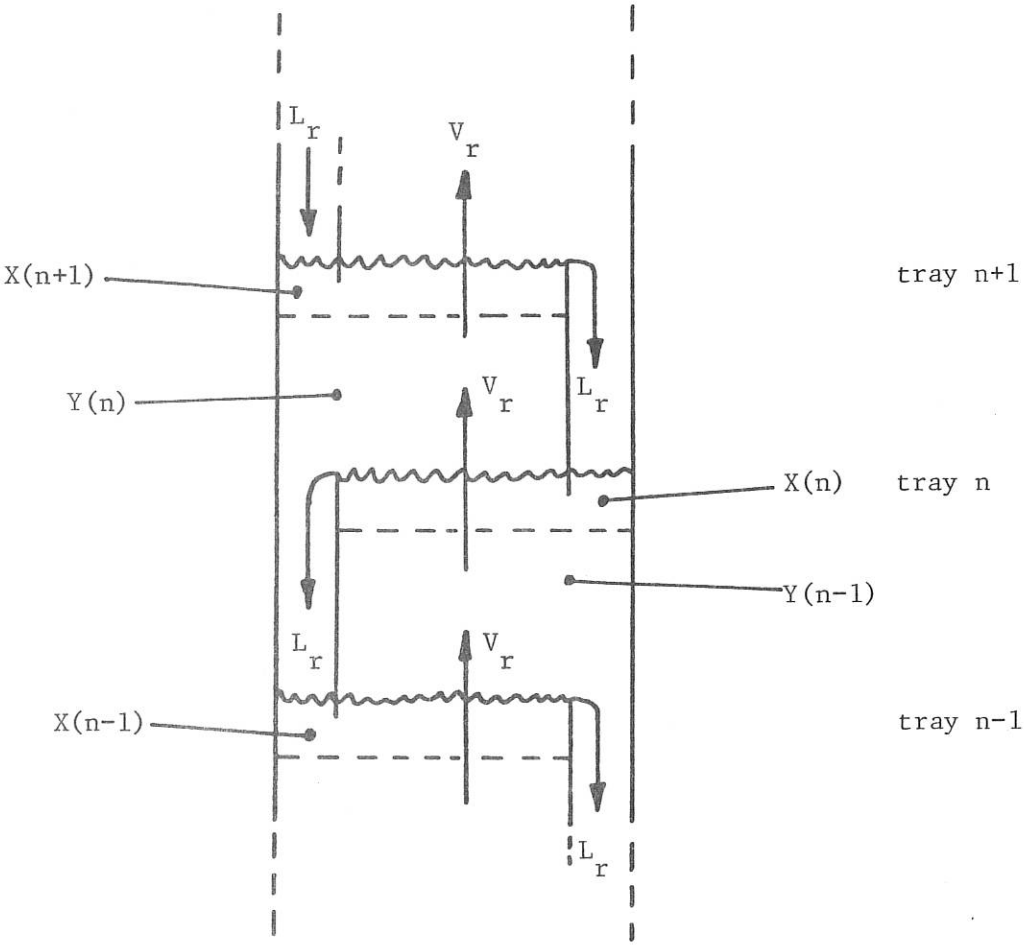


Fig. 2. Showing composition and flow variables associated with a general rectifier tray

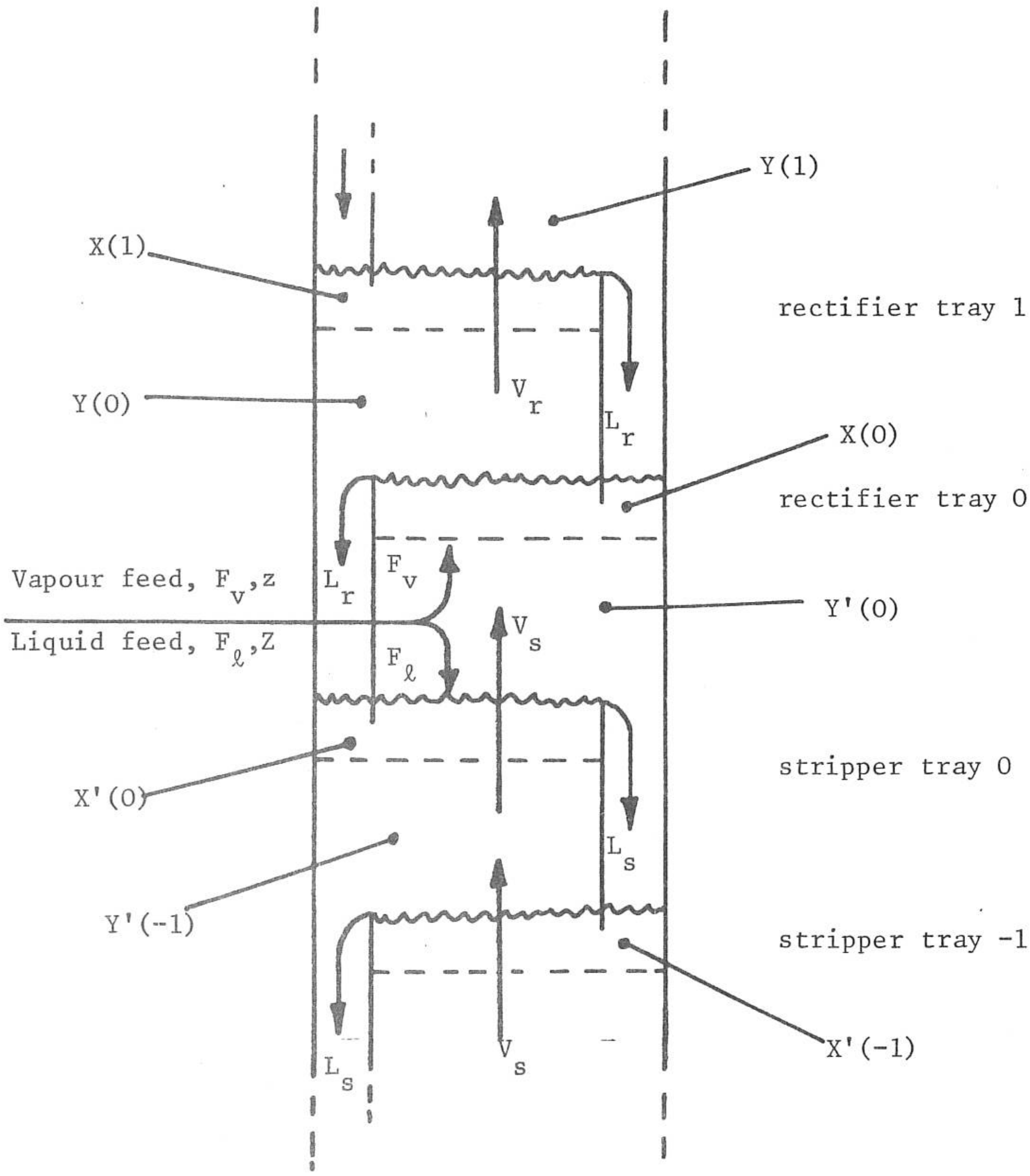


Fig. 3. Showing composition and flow variables associated with the feed section

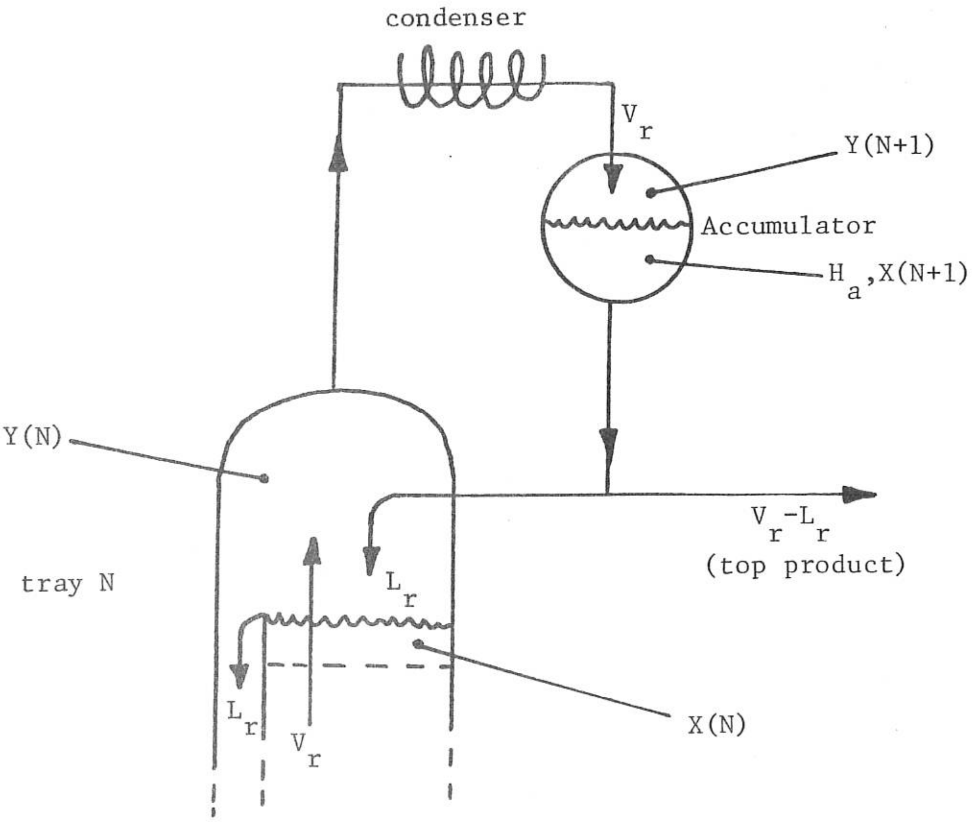


Fig. 4(a). Accumulator-end variables

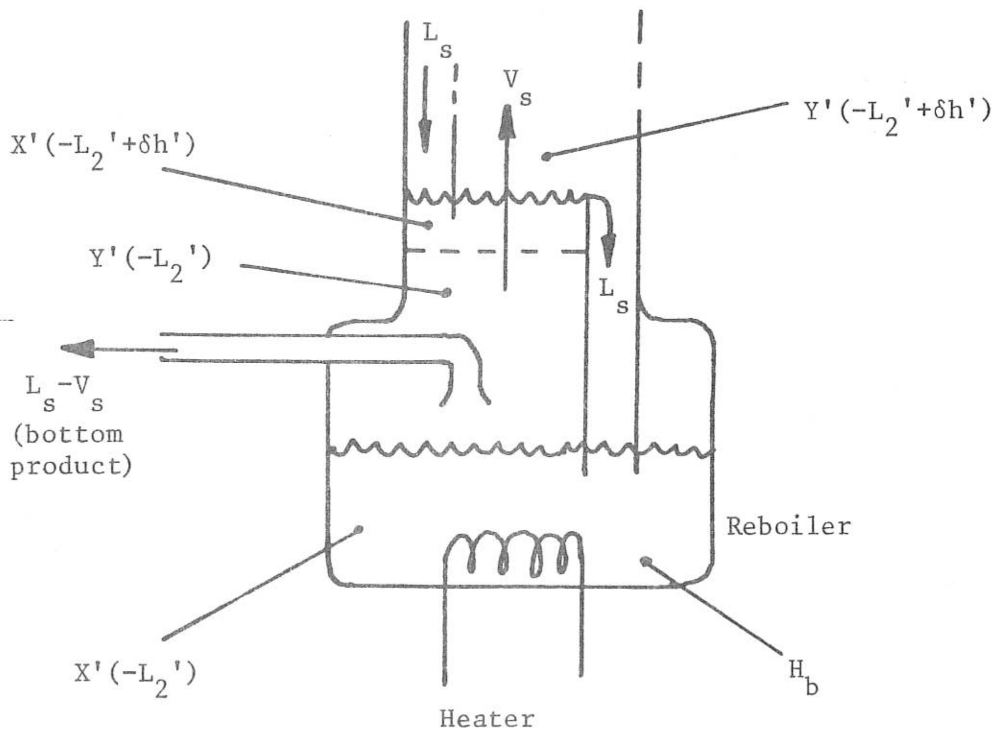


Fig. 4(b). Reboiler-end variables

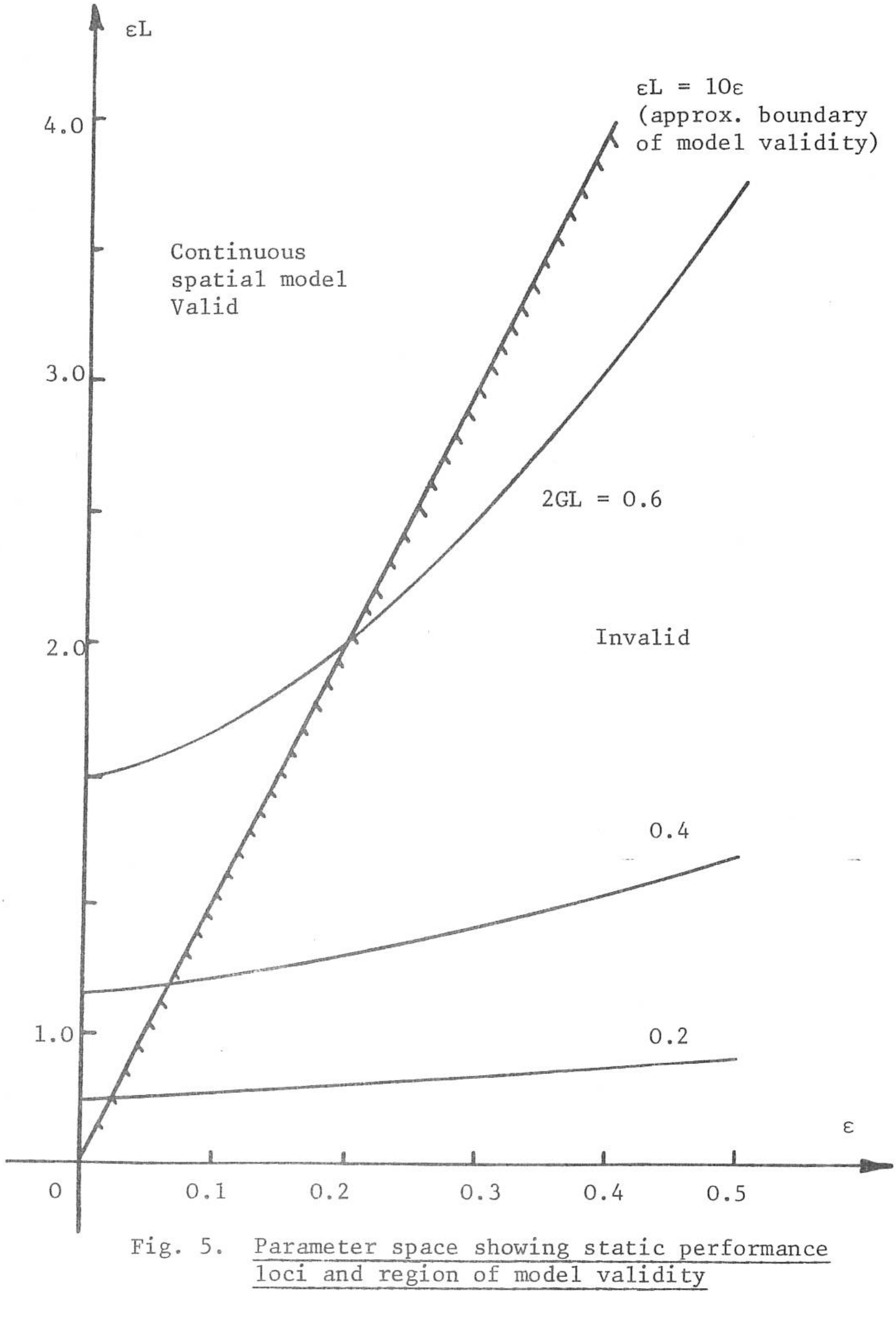


Fig. 5. Parameter space showing static performance loci and region of model validity

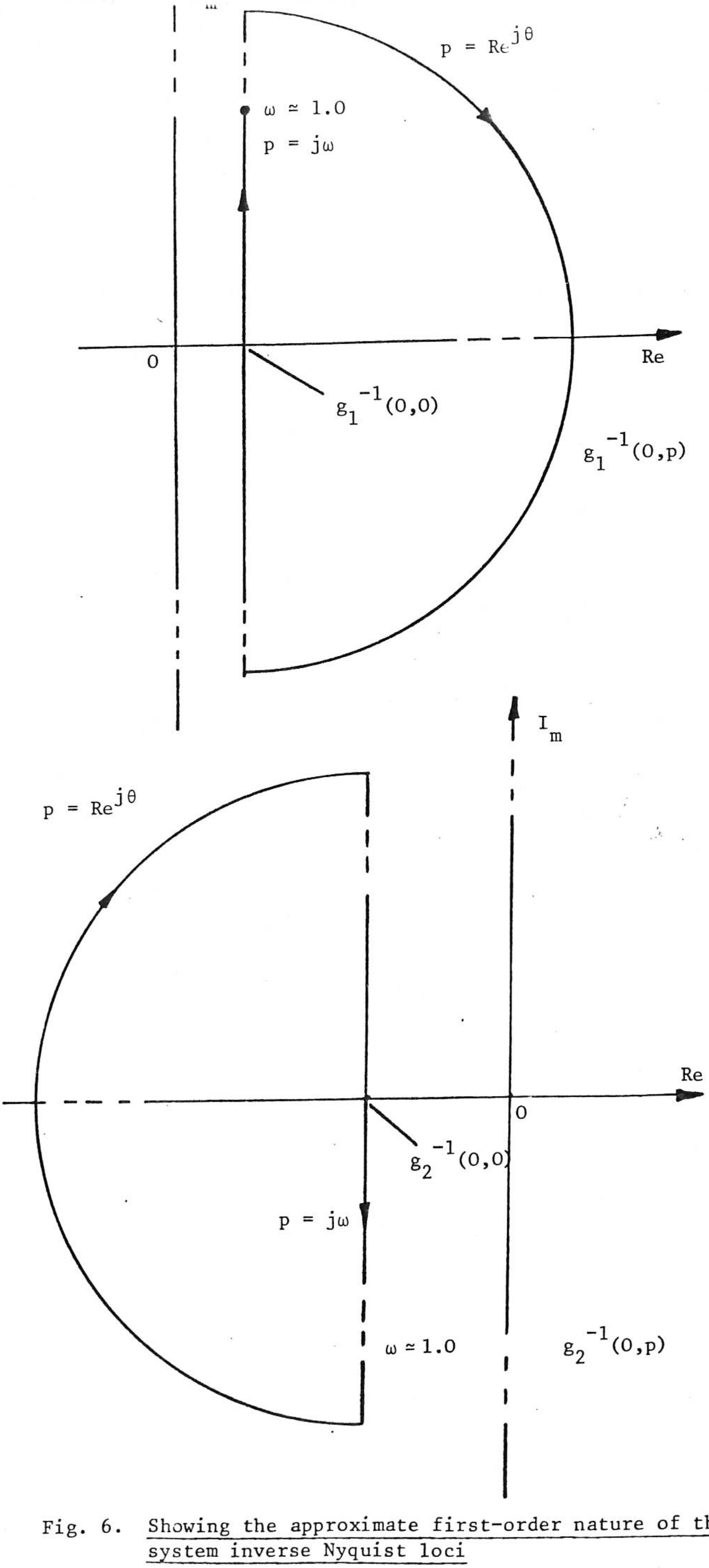


Fig. 6. Showing the approximate first-order nature of the system inverse Nyquist loci