Limit Sets and Switching Strategies in Parameter-Optimal Iterative Learning Control

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Abstract

This paper characterizes the existence and form of the possible limit error signals in typical parameter-optimal Iterative Learning Control. The set of limit errors has attracting and repelling components and the behaviour of the algorithm in the vicinity of these sets can be associated with the undesirable properties of apparent (but in fact temporary) convergence or permanent slow convergence properties in practice. The avoidance of these behaviours in practice is investigated using novel switching strategies. Deterministic strategies are analysed to prove the feasibility of the concept by proving that each of a number of such strategies is guaranteed to produce global convergence of errors to zero independent of the details of plant dynamics. For practical applications a random switching strategy is proposed to replace these approaches and shown, by example, to produce substantial potential improvements when compared with the non-switching case. The work described in this paper is covered by pending patent applications in the UK and elsewhere.

Keywords: Iterative learning control, robust control, nonlinear control, positive-real systems, switching control, optimization

1 Introduction

Iterative Learning Control (ILC) is concerned with the performance of systems that operate in a repetitive manner and includes examples such as robot arm manipulators and chemical batch processes, where the task is to follow some specified output trajectory r in a fixed specified time interval with high precision. The use of conventional control algorithms with such systems will, in the absence of noise or disturbance, result in the same level of tracking error being repeated time and time again. Motivated by human learning, ILC uses information from previous executions of the task in an attempt to improve performance from repetition to repetition in the sense that the tracking error (between the output and a specified reference trajectory) is sequentially reduced to zero (see, for example, [1], [2] and the recent reviews [3], [6]. Note that repetitions are often called trials, passes or iterations in the literature.

This first part of this paper is concerned with the analysis of the limiting behaviour of Parameter-Optimal Iterative Learning Control POILC introduced in Owens and Feng [4] and its generalization [5]. This extends the results in [7] and forms a motivation for the second part. The second part presents a new switching-based approach to eliminate undesirable effects of such limiting behaviour.

In [4], the discrete time, linear case is considered and the behaviour of a simple feedforward ILC algorithm is analysed in detail. The algorithm operates with a learning gain updated from trial to trial through the minimization of a quadratic performance index. Despite its simplicity the algorithm has the powerful property of ensuring that the mean square error (Euclidean norm) of the error time series reduces from trial to trial . The important property of guaranteed convergence to a zero tracking error is seen to depend critically upon positivity of a plant matrix description (or, more generally, the positivity of the representation of a compensated plant). If the positivity condition is not satisfied, convergence to a non-zero limit error is always a realistic possibility and can be seen in practice.

For the Owens and Feng algorithm, the form of the set of limit errors (the so-called limit set S_∞) is characterized as the points (error time series) where a certain quadratic form vanishes. This divides the error time series space into three components - the limit set and two open sets (one of which is convex). Based upon properties observed during simulation studies, the authors initiated (in [7]) a theoretical study of the phenomenon reported. In that publication, the limit set is shown to have interesting implications for POILC algorithm performance, namely

1. It is shown that the limit set S_∞, in general, will have attracting S_∞^- and repelling components S_∞^+ (i.e. it has internal structure) and that

2. the behaviour of the algorithm in the vicinity of the repelling component S_∞^+ is associated with temporary slow convergence (which may tempt the user to believe, erroneously, that convergence has occurred) whilst

3. close to the attracting component S_∞^- the algorithm does indeed converge to a nonzero point of

S_∞^-.

4. Moreover, close to the repelling part of the limit set, the behaviour of the POILC algorithm is highly sensitive to small variations in the initial error time series.

Sufficient conditions to characterize the attracting and repelling components of the limits set are derived in [7]. This paper extends these results to provide precise necessary and sufficient conditions. The ideas are illustrated by graphical analysis of a simple example in $\mathcal{R}^2$ (representing times series with only two elements). Although not a practical example (as time series will, in general have hundreds or thousands of data points), this simple case provides an indication of the phenomena summarised above and are included to motivate the switching concepts used later in the paper. A simple example using more realistic data sizes is seen to exhibit the temporary slow convergence property so it is believed that it is an issue that truly has practical implications. The authors argue that it is hence important to understand these issues theoretically and to look for theoretically sound approaches to eliminate or reduce the problem in practice.

The paper then considers the fundamental question of whether or not the ILC algorithm can be modified to eliminate the problems and ensure that convergence to a non-zero element of the limit set is replaced by a guarantee of convergence to a zero limit error. The innovative approach suggested is to avoid these behaviours by incorporating a switching strategy into the algorithm. The intuition behind the approach is that appropriate switching will cause the limit set to change constantly. In such circumstances it is possible that a monotonically reducing error cannot then converge to a non-zero limit leaving zero as the only other possibility. The existence of deterministic switching strategies is shown firstly through a dense search argument and then by switching through a sufficiently large set of data filters. Analysis proves convergence of errors to zero in both cases but the approaches are seen to have a large number of degrees of freedom. As a consequence an empirical random switching strategy is proposed. No analysis is provided but computational studies are presented that indicate that the most probable outcome is a substantial improvements when compared with the non-switching case. The paper concludes with a brief discussion of the possibilities for further work in the area.

2 Problem definition

Consider a standard discrete-time,linear, time-invariant single-input, single-output state-space representation defined over a *finite*, *discrete* time interval, $t \in [0, N]$ (in order to simplify notation it is assumed that the sampling interval, t_s is unity). The system is assumed to be operating in a repetitive mode where at the end of each repetition, the state is reset to a specified initial condition x_0 for the next operation during which a new control input signal can be used. A reference signal $r(t)$ is specified and the ultimate control objective is to find an input function $u^*(t)$ so that the resultant output function $y(t)$ tracks this reference signal $r(t)$ *exactly* on $[0, N]$. The process model is written in the form:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

where the state $x(\cdot) \in \mathbb{R}^n$, output $y(\cdot) \in \mathbb{R}$ and input $u(\cdot) \in \mathbb{R}$. The matricesA, B and C have an appropriate dimensions and D is a scalar. It will be assumed that either $D \neq 0$ or $CA^{j-1}B \neq 0$ for some $j = k^* =\geq 1$ (trivially satisfied in practice) and that the system (1) is both controllable and observable. If $D \neq 0$, then take $k^* = 0$. By construction, k^* is the relative degree of the transfer function $G(z) = C(zI - A)^{-1}B + D$ of the system. For notational convenience $f_k(t)$ will denote the value of a signal at time t on iteration k.

The repetitive nature of the problem permits the iterative modification of the input function $u(t)$ so that, as the number of repetitions increases, the system asymptotically learns the input function that gives perfect tracking. More precisely, the control objective is to find a causal recursive control law

$$u_{k+1}(t) = f(u_k(\cdot), u_{k-1}(\cdot), \ldots u_{k-r}(\cdot), e_{k+1}(\cdot), e_k(\cdot), \ldots, e_{k-s}(\cdot)) \tag{2}$$

with the properties that, independent of the control input time series chosen for the first trial,

$$\lim_{k\to\infty} \|e_k(\cdot)\| = 0 \quad \lim_{k\to\infty} \|u_k(\cdot) - u^*(\cdot)\| = 0 \tag{3}$$

where $\|\cdot\|$ denotes any norm for the time series. In what follows, this norm is taken to be the mean square error of the time series which is just the Euclidean norm of the vector of the time series. Other design issues of rate of convergence, robustness etc are also important to practical application but are not relevant to the main results of this paper and will be pursued in future research.

3 Matrix Representations of Plant Dynamics

The state space model is a natural description for the process. For this paper, an equivalent matrix description is the preferred method of analysis as it allows a more compact description of the theoretical results and releases methods of matrix algebra for analysis. This idea has been used elsewhere ([4]-[7] and by other authors) but the key concepts and notation are included for completeness and readability. The key is to note that the dynamical system maps input time series into output time series. It follows that, on finite time intervals, there exists a matrix relating these time series. This matrix is an equivalent description of systems dynamics.

To construct this matrix model in $\mathcal{R}^{N+1}$, define the time series "super-vectors" on the k^{th} trial via

$$u_k = [u_k(0), u_k(1), \ldots, u_k(N)]^T \tag{4}$$

$$y_k = [y_k(0), y_k(1), \ldots, y_k(N)]^T \tag{5}$$

$$e_k = [e_k(0), e_k(1), \ldots, e_k(N)]^T. \tag{6}$$

where $e_k(j) = r(j) - y_k(j), 0 \leq j \leq N$. Furthermore, let u^* be the input sequence (in time series or supervector form) that gives $r(t) = [G_c u^*](t)$ and G_c is the convolution mapping corresponding to (1).

Note that if the mapping f in (2) is not a function of e_{k+1}, then it is typically said that the algorithm is of *feedforward* type. If it does not depend on any of the $e_j, 0 \leq j \leq k$, it is of feedback type. Otherwise it is of *feedback plus feedforward* type. In this paper, it is assumed that all feedback components of the control law have been incorporated into the underlying state space model and hence that the ILC law is a feedforward law.

With the above definitions, the usual formulae for the input-output response of the system can be written in the form, $k \geq 0$,

$$y_k = G_e u_k + d_0 \tag{7}$$

where G_e has dimension $(N+1) \times (N+1)$ and the lower triangular band structure $(G_e)_{ij} = G_e)_{(i+1)(j+1)}$ that is required by causality and time invariance of linear time-invariant convolution systems i.e.

$$G_e = \begin{bmatrix} D & 0 & 0 & \ldots & 0 \\ CB & D & 0 & \ldots & 0 \\ CAB & CB & D & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{N-1}B & CA^{N-2}B & \ldots & \ldots & D \end{bmatrix} \tag{8}$$

Also $d_0 = [Cx_0, CAx_0, ..., CA^N x_0]^T$.

The elements CA^jB of the matrix G_e are the Markov parameters of the plant (1). Assume from now on that the plant transfer function $G(z) = C(zI - A)^{-1}B + D$ has relative degree (pole-zero excess) $k^* \geq 0$. Assume also that the reference signal $r(t)$ satisfies $r(j) = CA^j x_0$ for $0 \leq j < k^*$ (or, alternatively, that tracking in this interval is not important). Then it is noted [6] that, for analysis, it is sufficient to analyse a 'lifted' plant equation that is just the above if $k^* = 0$ or, if $k^* \geq 1$,

$$y_{k,l} = G_{e,l} u_{k,l} + d_1 \tag{9}$$

where $u_{k,l} = [u_k(0), u_k(1), \ \ldots, \ u_k(N-k^*)]^T$, $y_{k,l} = [y_k(k^*) \ y_k(2) \ \ldots \ y_k(N)]^T$ etc and

$$G_{e,l} = \begin{bmatrix} CA^{k^*-1}B & 0 & 0 & \ldots & 0 \\ CA^{k^*}B & CA^{k^*-1}B & 0 & \ldots & 0 \\ CA^{k^*+1}B & CAk^*B & CA^{k^*-1}B & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{N-k^*}B & CA^{N-k^*-1}B & \ldots & \ldots & CA^{k^*-1}B \end{bmatrix} \tag{10}$$

and $d_1 = [CA^{k^*}x_0, ..., CA^N x_0]^T$. For notational convenience, the model is written in all cases $k^* \geq 0$ in the simplified notational form

$$y_k = Gu_k + d \tag{11}$$

which has the structure of discrete dynamics in $\mathcal{R}^{N+1-k^*}$. Note that G is invertible by construction which confirms that, for an arbitrary reference r on $0 \leq j \leq N$, there exists a time series u^* on $0 \leq j \leq (N+1-k^*)$ such that $r = Gu^* + d$ on $k^* \leq j \leq N$. From now on this lifted plant model will be used as a starting point for analysis.

This result is valuable for this paper which considers the basic algorithm described by the *feed-forward* ILC update rule

$$u_{k+1} = u_k + Ke_k, \quad K \in \mathcal{R}^{(N+1-k^*)\times(N+1-k^*)} \tag{12}$$

Note: in element by element form, this relation is simply

$$u_{k+1}(t) = u_k(t) + \Sigma_{j=1}^{N+1-k^*} K_{t+1,j} e_k(t+j-1+k^*), \quad 0 \leq t \leq N-k^* \tag{13}$$

For example, with $K = I$ the update law is just

$$u_{k+1}(t) = u_k(t) + e_k(t+k^*), \quad 0 \leq t \leq N-k^* \tag{14}$$

The matrix K can, in principle be arbitrary but, in practice, it is assumed that it will be connected with a dynamical system. As a consequence, it is assumed one of the following situations holds

1. $K \in \mathcal{R}^{N+1-k^*}$ generated from a linear, time invariant system model. Ke can then be computed as the time series generated by the response of the state space model of K from zero initial conditions to the time series e),

2. K is the transpose of the matrix description of a linear time invariant system i.e. K^T is derived from a linear time invariant model.

3. K is a obtained as the sum of series operations of the types discussed above.

The calculations associated with case one above is simple. The second case has a dynamical systems interpretation based on the following observations. Let F_0 be defined to be the matrix with elements $(F_0)_{ij} = \delta_{i,N-k^*-j}$ i.e.

$$F_0 = \begin{bmatrix} 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{bmatrix} \tag{15}$$

Note that $F_0 = F_0^T \quad , \quad F_0^2 = I$. Let K be obtained from a linear time invariant system and note, after a little manipulation, that K and K^T are related by the identity

$$F_0 K F_0 = K^T \tag{16}$$

If s is the column vector of a time series, then $F_0 s$ is a column vector of the same time series but reversed in time.

These definitions enable the interpretation of K^T as a dynamical system. More precisely it is easily proved that:

$$\{\tilde{y} = K^T \tilde{u}\} \Leftrightarrow \{(F_0 \tilde{y}) = K(F_0 \tilde{u})\} \tag{17}$$

That is, the time series $\tilde{y} = K^T \tilde{u}$ is simply the time reversed response of the linear system K to the time reversal of $\tilde{u}$.

Note: The precise structure of the matrix K is not central to the main results of this paper, as the results apply, in principle to any choice of K where $GK + (GK)^T$ is not sign-definite.

4 Limit Sets and Basic POILC Properties

The concepts of Parameter Optimal Iterative Learning Control (POILC) was introduced by Owens and Feng [4] and now has several different realisations (see [6] for a recent review). The basic idea is, given a plant description $y = Gu + d$ and the control law $u_{k+1} = u_k + \beta_{k+1} K e_k$ choose the free

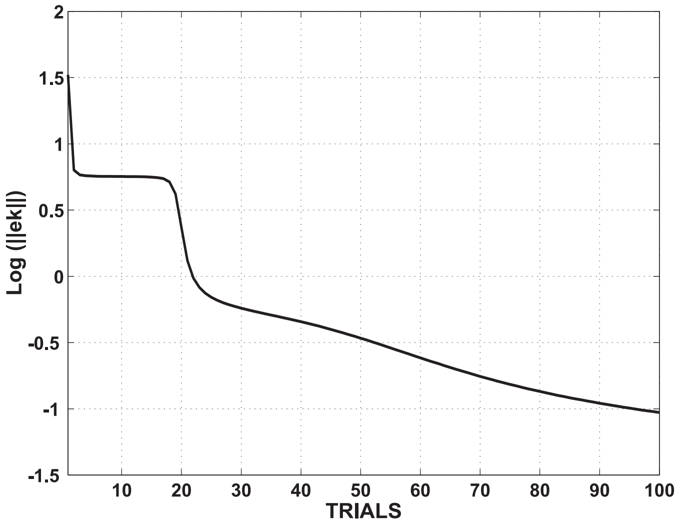

Figure 1: Log$||e_k||$ for a generic system

"learning gain" β_{k+1} to minimize the performance index

$$J(\beta_{k+1}) = ||e_{k+1}||^2 + w\beta_{k+1}^2, \quad w > 0 \tag{18}$$

where $w > 0$ is introduced to provide a degree of caution in the change in magnitude of the control signal from iteration to iteration. The results of Owens and Feng can be summarised as follows:

1. The ILC dynamics is described by the iterative matrix equation

$$e_{k+1} = (I - \beta_{k+1}GK)e_k, \quad k \geq 0 \tag{19}$$

2. The optimal value of β_{k+1} minimizing the performance index is given by

$$\beta_{k+1} = \frac{e_k^T GK e_k}{w + ||GKe_k||^2} \tag{20}$$

   where $|| \cdot ||$ is the Euclidean norm.

3. The sequence of learning gains converges to zero as $k \to \infty$ as

$$\sum_{k \geq 0} \beta_{k+1}^2 < \infty \tag{21}$$

4. The error sequence is monotonically improving in the sense that

$$||e_{k+1}|| \leq ||e_k|| \quad \forall k \geq 0 \tag{22}$$

with equality holding if, and only if, $\beta_{k+1} = 0$.

Note: this is an important benefit of POILC as it provides natural embedded improvement from iteration to iteration through learning gain changes.

5. The error sequence $\{e_k\}_{k\geq 0}$ converges to a point in the *Limit Set* S_∞ defined by the relation

$$S_\infty = \{e : \quad e^T GKe = 0 \quad \} \tag{23}$$

Note: In practice the intersection of this set with $\{e : ||e|| \leq ||e_o||\}$ is more precise but, for simplicity, this detail is omitted.

6. The error converges to zero for all initial choices of u_0 and hence e_0 if and only if $G + G^T$ is strictly sign definite.

7. All points in S_∞ are limits of the POLIC algorithm for some choice of e_0 (simply choose $e_0 \in S_\infty$.

Sign definiteness of $GK + (GK)^T$ is both necessary and sufficient for guaranteed convergence of the error to zero. In principle, the condition can be satisfied easily by, for example, the choice of $K = G^T$ or $K = G^{-1}$ but there is obviously a price to be paid in terms of complexity of the control computations. The choice of a simpler form of K may hence be preferred or be desirable for other reasons. Unfortunately, this condition is then often not satisfied and the consequent price of simplicity is that convergence to a *non-zero* limit error $e_\infty \in S_\infty$ will occur. A number of issues and questions can now be stated:

1. The magnitude $||e_\infty||$ of the limit error in both absolute terms and relative to the magnitude $||e_0||$ of the initial error are clear measures of algorithm success.

2. The form of e_∞ is also relevant and hence, in particular, information on those regions of S_∞ that attract solutions will be valuable.

3. The dependence and sensitivity of the limit error to the initial error e_0 is of interest.

4. The development of algorithms that reduce the impact or eliminate the impact of a limit set $S_\infty \neq \{0\}$ on algorithm performance should be considered.

The following sections provide useful insight into the dynamical effects of the existence of a limit set $S_\infty \neq \{0\}$ and the existence and form of simple *switching algorithms* that guarantee convergence of the error to zero. In these sections, for notational simplicity, we denote $M = GK$.

5 Attractivity and Sensitivity Properties of the Limit Set

In this section, limiting properties of iterations of the general form

$$e_{k+1} = (I - \beta_{k+1} M)e_k, \quad \beta_{k+1} = \beta(e_k) \tag{24}$$

where

$$\beta(e) = \frac{e^T Me}{w + ||Me||^2} \tag{25}$$

are considered with $w > 0$. The associated limit set is defined by

$$S_\infty = \{e : e^T Me = 0\} = \{e : \beta(e) = 0\} \tag{26}$$

which is clearly the set of error time series where $\beta = 0$. The general objective is to identify more detailed structural properties that effect algorithm performance. The work extends that in Owens et al [7] to provide a more complete description including necessary and sufficient conditions.

5.1 Attracting and Repelling Components

Close to the limit set, the fact that $\beta(e)$ is small indicates that the change in error $e_{k+1} - e_k = -\beta(e_k)Me_k$ is small in magnitude and hence the POILC algorithm is converging slowly. Slow convergence is an undesirable property so it is natural to ask whether or not the limit set is *attracting* or *repelling* such behaviours. Attraction can be interpreted as indicating that the ILC algorithm is indeed slowing down with little further improvement in error magnitudes possible. Repelling can be associated with a temporary slowing of the algorithm convergence, more rapid convergence being regained when the error time series has finally moved away from the limit set.

Whether or not the limit set repels or attracts local time series can be characterized in terms of sign properties of the sequence $\{\beta_{k+1}\}_{k\geq 0}$ close to S_∞. More precisely:

1. It is easily seen that a point $e \in S_\infty$ attracts trajectories in its vicinity (in the sense that $|\beta_{k+1}|$ is getting smaller) if

$$-\beta_{k+1}^2 < \beta_{k+1}\beta_{k+2} < \beta_{k+1}^2 \tag{27}$$

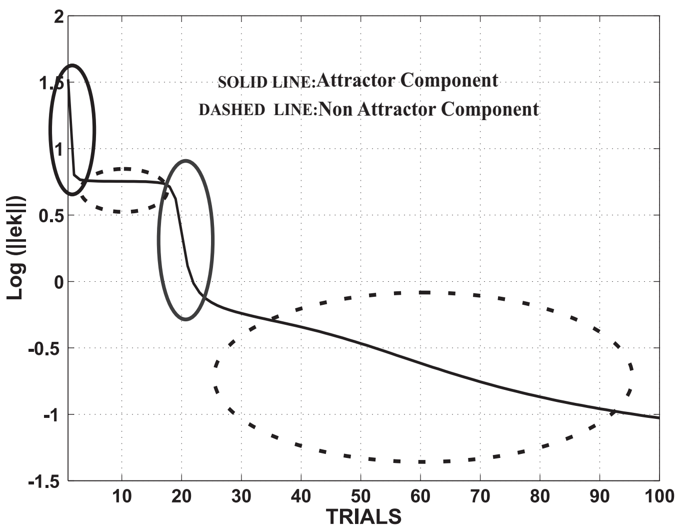

Figure 2: Different regions of attraction in the S_∞

for all trajectories entering a sufficiently small ball centred on the point $e \in S_\infty$. This expression has the simpler form

$$-2 < \beta_{k+1}^{-1}(\beta_{k+2} - \beta_{k+1}) < 0 \quad (28)$$

When β_{k+1} is small, this expression, together with the expression $e_{k+1} - e_k = -\beta_{k+1} M e_k$, can be analysed using the quantity

$$\mathcal{B}(e) = \frac{\partial \beta(\hat{e})}{\partial \hat{e}}|_{\hat{e}=e} M e \quad (29)$$

to produce the following sufficient condition for the point $e \in S_\infty$ to attract trajectories to S_∞

$$-2 < \lim_{e_k \to e} \beta_k^{-1} \frac{\partial \beta(\hat{e})}{\partial \hat{e}}|_{\hat{e}=e_k} \beta_k M e_k = -\mathcal{B}(e) < 0 \quad (30)$$

2. In a similar manner, if one of the following conditions holds

$$\mathcal{B}(e) > 2 \quad , \quad \mathcal{B}(e) < 0 \quad (31)$$

then the point $e \in S_\infty$ repels trajectories.

3. In the cases of $\mathcal{B}(e) = -2$ or $\mathcal{B}(e) = 0$ the situation is more complicated and is not considered here.

The quantity $\mathcal{B}(e)$ is clearly central to the behaviour of the algorithm close to S_∞ with the point $e \in S_\infty$ attracting trajectories being defined by the relations

$$\mathcal{B}(e) \in (0, 2), \quad e^T M e = 0 \quad (32)$$

The form of $\mathcal{B}(e)$ is therefore important and is computed below:

$$\mathcal{B}(e) = \frac{e^T(M+M^T)Me}{w+||Me||^2} - \beta(e)\frac{e^T 2M^T M^2 e}{w+||Me||^2} \quad (33)$$

which, when $e \in S_\infty$, is just

$$\mathcal{B}(e) = \frac{e^T(M+M^T)Me}{w+||Me||^2} \quad (34)$$

In summary, the following theorem has been proved and extends the results in [7] to provide necessary and sufficient condition:

Theorem 1 *1. A point $e \in S_\infty = \{e : e^T Me = 0\}$ attracts local trajectories of ILC algorithm if it lies in the attracting component $S_\infty^- \subset S_\infty$ defined by the equations*

$$e^T Me = 0 \quad , \quad \mathcal{B}(e) = \frac{e^T(M+M^T)Me}{w+||Me||^2} \in (0,2) \quad (35)$$

2. If, however, it satisfies $e^T Me = 0$ and one of the inequalities

$$\mathcal{B}(e) > 2 \quad , \quad \mathcal{B}(e) < 0 \quad (36)$$

then it repels local trajectories and is said to lie in the repelling component $S_\infty^+ \subset S_\infty$.

Note: The cases where $e^T Me = 0$ and $\mathcal{B}(e) = -2$ or $\mathcal{B}(e) = -2$ are not considered here. They lie on the boundary between the two sets defining attracting components and repelling components respectively and will require further analysis to resolve their characteristics.

The condition for the point e to lie in S_∞^- can be rewritten in the form of quadratic inequalities

$$e^T Me = 0 \quad , \quad 0 < e^T(M+M^T)Me < 2(w+||Me||^2) \quad (37)$$

The analysis of this expression as a function of the weight w can proceed as follows: firstly note that S_∞^- is defined by

$$e^T Me = 0 \quad , \quad e^T(M+M^T)Me > 0 \quad , \quad 0 < e^T(M-M^T)Me < 2w \quad (38)$$

which reduces in size as $w \to 0+$. Hence:

- Increasing w therefore tends to increase the size of the attracting component $S_\infty^- \subset S_\infty$ and hence increases the potential for convergence to non-zero limit errors.
- For sufficiently large values of w (relative to the range of $||e||$ of interest), the only active inequality is the simple quadratic inequality $e^T(M+M^T)Me > 0$. In particular, for any e satisfying this inequality, examination of $\mathcal{B}(e)$ indicates that there exists a $w^* \geq 0$ such that $e \in S_\infty^-$ for all $w > w^*$.
- Even if $w \to 0+$, the attractive component may still be nontrivial.

5.2 A 2-D Example

The purpose of this section is to illustrate the ideas using a simple example and reveal sensitivity properties of ILC algorithms close to S^+_∞. Consider the case of 2-dimensional matrix M:

$$M = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad M + M^T = \begin{pmatrix} 2 & \alpha \\ \alpha & 2 \end{pmatrix} \tag{39}$$

with eigenvalues of $M + M^T$ at $\lambda = 2 \pm \alpha$ and eigenvectors $v^+ = (1,1)^T, v^- = (-1,1)^T$.Note that $(M + M^T)$ is sign-indefinite iff $\alpha^2 > 4$

Solutions of $e^T(M + M^T)e = 0$ (represented in the orthogonal eigenbasis of $(M + M^T)$), can be written in the form:

$$e = \gamma\left(\sqrt{\alpha - 2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm \sqrt{\alpha + 2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) \tag{40}$$

where $\gamma \in R$ is arbitrary.

$S_\infty \neq \{0\}$ whenever $\alpha > 2$ and can be associated with he two subspaces S^+ and S^- shown in fig.(3) corresponding to the different values $\pm$ of (40). They are strongly connected to the components S^+_∞ and S^-_∞ of S_∞. More precisely, evaluating,

$$e^T(M^T + M)Me = \gamma^2\left(2\alpha(\alpha^2 - 4) \mp 2\alpha^2\sqrt{\alpha^2 - 4}\right)$$

it can be verified that it is negative on S^+ and positive on S^- if $\alpha > 2$. It follows that $S^+ \subset S^+_\infty$ and $S^-_\infty \subset S^-$. That part of S^- that can be identified with S^-_∞ is described by the solutions of the inequality $\mathcal{B}(e) < 2$ i.e. $e^T(M - M^T)Me < 2w$. That this leads to nontrivial solutions can be illustrated by looking at the case of $\alpha = 3$. In this case, a simple calculation indicates that this relation is satisfied in S^- for any choice of $\gamma^2 \in R$ and $w > 0$ i.e. $S^- = S^-_\infty$ and $S^+ = S^+_\infty$. The details are omitted for brevity.

The theory indicates (and computation supports) the notion that trajectories originating at e_0 close to the attracting component converge to a limit in that component with little decrease in error norm i.e algorithm performance is very poor. The situation is more complicated close to the repelling component. Trajectories originating close to S^+_∞ ultimately converge to limits in S^-_∞ but there is a surprising sensitivity issue. The above example can also be used to illustrate this sensitivity to the choice of initial error, e_0. More precisely, the behaviour of the error sequence following the initial error e_0 close to S^+_∞ depends critically on "which side of S^+_∞ e_0 lies". To illustrate this, let $w = 10^{-6}$ and consider the pair $e_0 = (1 - \sqrt{5} + \delta, 1 + \sqrt{5})^T$ where very small displacements $\delta = \pm 10^{-5}$ from

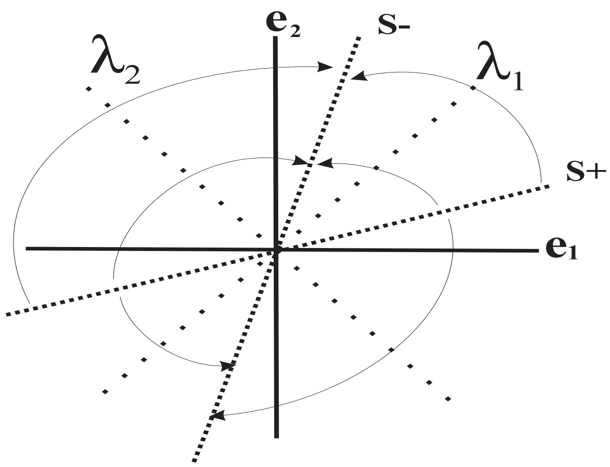

Figure 3: The components S^- and S^+ of S_∞

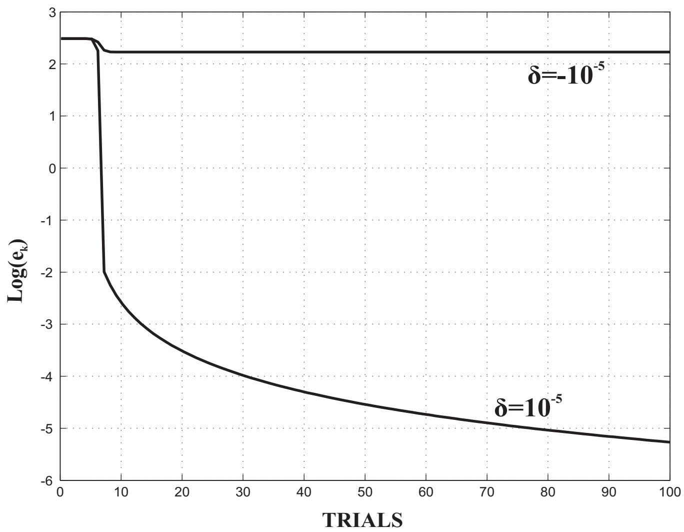

Figure 4: Log$(||e_k||^2)/k$ with 10^{-5} and -10^{-5} displacements from $e_0 \in S_\infty^-$

the point $(1-\sqrt{5}, 1+\sqrt{5})^T \in S_\infty^-$ are applied. In fig.(4) the different in performance of the sequence of error norms is shown over 100 iterations. The difference in performance is remarkably large; in the case of a negative perturbation $\delta = -10^{-5}$, very little improvement in error norm magnitude is achieved whereas, for the case of a positive perturbation $\delta = 10^{-5}$ a substantial and rapid decrease in magnitude is achieved after 10 iterations. In both cases, the first 10 iterations show little change in norm - a slow variation that is expected close to the limit set.

Using the same example, consider the cases when δ is varied using the positive and increasing values $\delta = 10^{-1}, 10^{-2}, 10^{-5}$. the results are given in fig. (5) with the expected conclusion that, the smaller the perturbation, the greater the tendency for the error norm to change slowly and hence stay close to S_∞^+ for more iterations. Combining this with the rapid movement later, the example illustrates the tendency of the error norm sequence $\{||e_k||\}_{k\geq 0}$ to exhibit *"stair-like"* properties close to S_∞^+ i.e a period of slow variation followed by a period of fast convergence followed by another period of slow convergence.

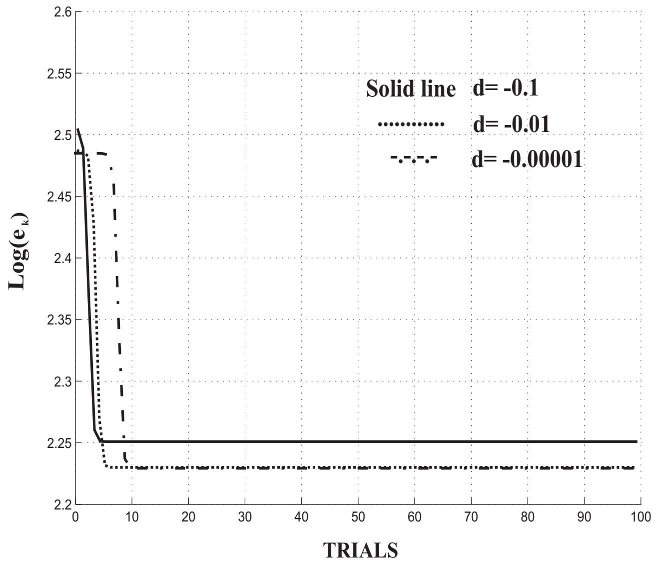

Figure 5: Log$(||e_k||^2)/k$ with different displacements from $e_0 \in S_\infty^-$

5.3 A Discussion of N-D Possibilities

The $2-D$ example illustrates the potential practical importance of the limit set to the ILC algorithm (seen as a dynamical system). In particular, the slow convergence close to the limit set may be a problem and lead to increased iterations (to achieve a required error tolerance) and stair-like behaviours where slow convergence is seen at least twice during the iteration sequence.

The purpose of this section is to support the intuition that these properties will also be seen in more complex examples. In this section, the N-dimensional case will be considered in a similar manner to the 2-dimensional example. This is done using a suprisingly simple non-positive plant of the form;

$$G(s) = \frac{1}{(s+1)^2} \tag{41}$$

with zero initial conditions.

With sample interval $h = 0.1$, a discrete time representation of (41) is obtained:

$$\Phi = \begin{pmatrix} 0.81435 & -0.090484 \\ 0.090484 & 0.99532 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0.090484 \\ 0.0046788 \end{pmatrix}$$

$$C = (0, \quad 1) \tag{42}$$

The reference signal was chosen to be $r(t) = e^{\frac{t}{20}} sint$ over the time interval $t \in [0, 20]$. The value of the weight parameter w is $w = 10^{-6}$ and the initial control sequence was as chosen to be $u_0 = 0$. The control matrix was chosen to be $K = I$ for simplicity so that $M = G$.

The eigenvalues of the resulting 200x200 matrix $(M+M^T)$ are in the range of $(-0.2086, 1.8942)$, so it is not a positive definite matrix i.e. S_∞ contains non-zero points in R^{200}- a high dimensional space.

Fig.(6) shows the evolution of the norm of the error and the evolution of the quantity $Q = \frac{e_k^T(G+G^T)Ge_k}{e_k^T e_k}$. This second quantity is included to illustrate the general correlation between the rate of convergence and the sign of $\mathcal{B}(e)$. Clearly periods of slow convergence are associated with periods where $\mathcal{B}(e)$ is positive whilst periods of faster convergence are associated with periods where $\mathcal{B}(e)$ is negative. One of the remarkable aspects of this example is the fact that stair-like behaviours are present with *three* periods of slower convergence and two periods of faster convergence in the first 100 iterations. This suggests that stair-like behaviours may be a common feature of parameter optimal ILC and that they can occur several times within the iteration sequence.

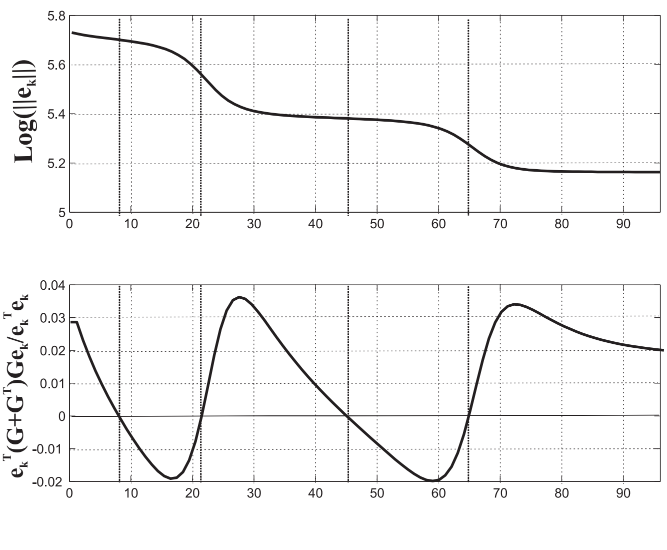

Figure 6: Error dynamics and the sign of $e_k^T(G + G^T)Ge_k$

In conclusion, the great potential of POILC to produce the very beneficial property of monotonic error norm sequences carries with it the need to consider the effects of non-positivity and the existence, in general, of a non-trivial limit set. The existence of this limit set can be associated with a number of algorithm properties that could reduce performance and, at worst, make the algorithm effectively impractical. Further work on this issue is desirable but the important question considered in the following sections is whether or not POILC algorithms can be simply modified to remove some of the undesirable features. The very positive answer obtained forms the motivation and basis for further work.

6 Global Stabilisation Using Switching Algorithms

The previous section has indicated that the limit set for a given algorithm introduces serious dynamical properties into the algorithm. In this section, the issue of generating simple algorithms that ensure convergence to zero is addressed. No assumption on the nature of the linear system G is required but the relative degree k^* is assumed known.

6.1 Switching using Dense Search

To motivate the analysis, note that the limit set is a *fixed* set in $\mathcal{R}^{N+1-k^*}$ part of which attracts the error sequence $\{e_k\}_{k\geq 0}$. The limit set depends crucially upon $M = GK$ through the choice of K. If K is changed, then the limit set will change (at least in part). The natural question to ask if whether or not variations in K (and hence the Limit Set) from iteration to iteration can ensure convergence of the tracking error to zero *without the need for any positivity assumptions*. The following analysis provides a positive answer to this question based on the idea of sweeping through a countably dense set of such K. That the variation of K has the potential to restrict the limit set is indicated by the following theorem:

Theorem 2 *Let $\mathcal{K} = \{K_{(i,j)}\}_{i,j\geq 1}$ be any countably dense set of matrices whose closure contains a compact annulus in $\mathcal{R}^{(N+1-k^*)\times(N+1-k^*)}$ defined by a relation of the form $0 < \underline{M}^2 \leq ||K||^2 \leq \overline{M}^2$. Then*

$$\bigcap_{K\in\mathcal{K}} \{e : M = GK, \quad e^T M e = 0 \quad \} = \{0\} \tag{43}$$

Proof: If incorrect, there exists a vector $e \neq 0$ such that, also using continuity, $e^T M e = 0$ for $M = GK$ and K *any* matrix. Independent variations of the diagonal elements of K then lead to the conclusion that $e = 0$ which is a contradiction. □

Intuitively, this result is interpreted to suggest that, if a sufficiently rich mechanism is introduced that changes the limit set from iteration to iteration, then, combining this with monotonicity of the error norm, the absence of a *fixed* limit set will perhaps ensure convergence to zero. This truth of this intuition and its development into a number of conceptual algorithms is the subject of this section.

Suppose that the previous POILC algorithm is replaced by the modified algorithm using the same performance index but the iteration-dependent control law

$$u_{k+1} = u_k + \beta_{k+1} K_{k+1} e_k \tag{44}$$

As a consequence, the POILC optimal parameter is defined by

$$\beta_{k+1} = \frac{e_k^T M_{k+1} e_k}{w + ||M_{k+1} e_k||^2}, \quad M_{k+1} = G K_{k+1} \tag{45}$$

The specification of the algorithm is completed by carefully choosing the sequence of the operations K_{k+1}. In what follows the k^{th} choice of $K_{i,j}$ is specified as the k^{th} element in the (i, j) index sequence

$$(1,1),(2,1),(1,2),(1,1),(2,1),(1,2),(3,1),(2,2),(1,3),(1,1),(2,1),.... \tag{46}$$

obtained using successive sweeping of an increasing triangle of data in the top left hand corner of the infinite matrix depiction of the elements of $\mathcal{K}$. The crucial aspect of the sequencing is that every element of $\mathcal{K}$ is used and repeatedly used an infinite number of times.

Using optimality, it is easily proved (as u_k is suboptimal) that

$$||e_{k+1}||^2 + w\beta_{k+1}^2 \leq ||e_k||^2 \tag{47}$$

and hence that the monotonicity of the error norm is still guaranteed. Also $\beta_{k+1} \to 0$.

From its construction, $\mathcal{K}$ contains a point $K = K_{p,q}$ sufficiently close to αG^T (for some $\alpha > 0$) to ensure that $GK+(GK)^T > 0$. Consider the subsequence with indices $\{k_j\}_{j\geq 0}$ such that $K_{k_j} = K_{p,q}$. As $\beta_{k+1} \to 0$, then $e_{k_j}^T G K_{p,q} e_{k_j} \to 0$ from which $\lim_{j\to\infty} ||e_{k_j}|| = 0$. Using the monotonicity of the error norm, the above completes the proof of the following theorem:

Theorem 3 *With the above construction, the defined "dense search" switching algorithm guarantees convergence to zero error for all linear systems of relative degree k^*. Moreover*

1. *monotonicity of the error norm is guaranteed from iteration to iteration*

$$||e_{k+1}|| \leq ||e_k|| \quad \forall k \geq 0 \tag{48}$$

and

2. *the learning gain sequence converges to zero as*

$$\sum_{k\geq 0} \beta_{k+1}^2 < +\infty \tag{49}$$

The result underlines the great potential of switching algorithms for POILC as convergence is guaranteed for all plants of relative degree k^* on any interval $0 \leq t \leq N$ and for any reference signal time series r. Its value is mostly conceptual as dense search is not a particularly practical option. The following section suggests that finite switching algorithms exist that retain the properties of dense search.

6.2 Existence of Finite Switching Algorithms

Theorem 4 *There exists a (non-unique) finite set $\{\tilde{K}_j\}_{1\leq j\leq N+1-k^*}$ that retains the convergence properties of the POILC dense search algorithm if K_k is selected using the "circulation algorithm" $K_{(k-1)(N+1-k^*)+j} = \tilde{K}_j$ for all $k \geq 1$ and $1 \leq j \leq N+1-k^*$.*

Proof: The proof is by construction. By construction G is invertible so define

$$\tilde{K}_j = G^{-1} diag\{\delta_{i,j} + \epsilon\}_{1\leq i\leq N+1-k^*} \tag{50}$$

where $\delta_{i,j}$ is the Kronecker Delta and $\epsilon > 0$. The first part of the proof looks at the intersection of the solutions of a set of equations each of which will define possible values of cluster points of subsequences. More precisely, if, for any $e \in \mathcal{R}^{N+1-k^*}$,

$$e^T G\tilde{K}_j e = e_j^2 + \epsilon \sum_{i\geq 0} e_i^2 = 0, \quad 1 \leq j \leq N+1-k^* \tag{51}$$

then, for sufficiently small ϵ (e.g $0 < \epsilon < \frac{1}{N+1-k*}$), it follows that $e = 0$.

Turning now to the POILC algorithm, monotonicity of the error norm sequence and convergence of β_{k+1} to zero follow in the normal manner from optimality. Monotonicity ensures the existence of (possibly non-unique) cluster points $e_{\infty i}$ of every subsequence $\{e_{j(N+1-k^*)+i}\}_{j\geq 0}, 1 \leq i \leq N+1-k^*$. Clearly $e_{\infty i}$ satisfies $e_{\infty i}^T G\tilde{K}_i e_{\infty i} = 0$. Next note that the set of all such cluster points is independent of i as the convergence of the β_{k+1} to zero proves that, if $e_{\infty i}$ is a cluster point of subsequences with index i, it is also a cluster point of subsequences with index $i+1$. All such cluster points are the simultaneous solutions of the equations defined above and hence, if ϵ is sufficiently small, the only possible cluster point is the limit $e_\infty = 0$. □

The result indicates that the use of $N+1-k^*$ switching matrices is sufficient for convergence of the error to zero to be guaranteed. The choice of gain matrices used in the proof is purely for convenience and is not unique. The following discussion looks at a number of possibilities that have simple dynamical systems interpretations.

6.3 Switching Between Causal First Order Filters

The benefits of switching for convergence has been demonstrated above but the methods, as presented, are relatively complex. Ideally, the switching algorithm should have the desired properties of ensuring convergence and yet be conceptually simple for implementation e.g. the use of the inverse system above is not ideal and simpler, more "robust" schemes would be preferred. In what follows, the use of simple, first order filters is considered using the finite set

$$\tilde{K}_j = KF_j, \quad 1 \le j \le N_s, \quad N_s \ge N + 1 - k^*, \tag{52}$$

$$u_{k+1} = u_k + K_{k+1} e_k \tag{53}$$

and the "circulating assumption"

$$K_{kN_s+j} = \tilde{K}_j = KF_j, \quad k \ge 0, \quad 1 \le j \le N_s \tag{54}$$

Here K is a fixed element (included for generality) and each F_j is selected to be a simple, distinct dynamical system. In this section it is chosen to be a matrix representation of the transfer function

$$F_j(z) = \frac{1-\lambda_j}{1-\lambda_j z^{-1}} \tag{55}$$

(normalized such that $F_j(1) = 1$) which is a simple first order filter with pole at the position $z = \lambda_j$. All poles are assumed to be real and distinct and, for practical reasons, in the open interval $(-1, 1)$ to ensure stability of the filter. The matrix representation is precisely

$$F(\lambda) = (1-\lambda) \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \lambda & 1 & 0 & \dots & 0 \\ \lambda^2 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda^{N-k^*} & \lambda^{N-k^*-1} & \dots & \dots & 1 \end{bmatrix} \tag{56}$$

evaluated at $\lambda = \lambda_j$. For convenience, define the quadratic form

$$\rho(\lambda, e) = e^T GKF(\lambda) e (1-\lambda)^{-1} \tag{57}$$

which is also a polynomial in λ of degree less than or equal to $N - k^*$. The following theorem states the convergence properties of the above algorithm which is based on the analysis of limit sets and the fact that ρ is either identically zero or cannot have more than $N - k^*$ roots and hence cannot have N_s roots.

Theorem 5 *Suppose that $M = GK$ has the property that, for all $p \geq 1$, the $p \times p$ Principal Minor of M generated by rows and columns indexed by $N + 2 - k^* - p \leq i, j \leq N + 1 - k^*$ is nonzero. Then with the above notation, the POILC algorithm*

$$u_{k+1} = u_k + \beta_{k+1} K_{k+1} e_k, \quad y_{k+1} = G u_{k+1} + d \tag{58}$$

with

$$\beta_{k+1} = \arg\min\{J(\beta_{k+1}) = ||e_{k+1}||^2 + w\beta_{k+1}^2\} = \frac{e_k^T G K_{k+1} e_k}{w + ||GK_{k+1}e_k||^2}, \quad w > 0, \tag{59}$$

where K_{k+1} circulates through the finite set $\{KF_j\}_{1 \leq j \leq N_s}$, is convergent to zero error in the sense that

$$\lim_{k \to \infty} e_k = 0 \quad \& \quad \sum_{k \geq 0} \beta_{k+1}^2 < \infty \tag{60}$$

Notes:

1. The assumption on GK is generically satisfied and is satisfied for the following illustrative cases

   - K is a representation of any linear, proper time-invariant dynamical system,
   - K is obtained via the inverse algorithm $K = G_0^{-1}$ where G_0 is a sufficiently good approximation to G or
   - K is chosen to be $K = G_0^T$ where G_0 is a sufficiently good approximation to G^T (when the property of the Principal Minors follows from the fact that GG^T is positive definite.)

2. Note that no other assumption is made on the form of the system G (or GK) and, in this sense, the algorithm has universal stabilising properties.

3. For implementation purposes, the matrix computation $F_j e_k$ is best undertaken using the standard realisation of the transfer function $F_j(z)$.

Proof of the Theorem: The proof follow in a similar way to the previous result i.e monotonicity follows from the POILC paradigm and $\beta_{k+1} \to 0$ in the manner required. Also the intersection of the sets $\{e : \rho(\lambda_j, e) = 0\}, 1 \leq j \leq N_s$ defines the set of all cluster points of the algorithm. The proof is complete if it is shown that the only error vector e satisfying these equations is the single point $e = 0$.

Firstly note that it follows that $\rho(\lambda, e) = 0, \forall \lambda$ as ρ has degree less than $N + 1 - k^* \leq N_s$. As a consequence, all coefficients of $\lambda^j, 1 \leq j \leq N + 1 - k^*$ are zero. It is easily seen that these coefficients are given by

$$e^T GKF^{j-1} e = 0, \quad 1 \leq j \leq N + 1 - k^* \tag{61}$$

where F is a $(N+1-k^*) \times (N+1-k^*)$ matrix with elements $\delta_{i,j+1}$ i.e.

$$F = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix}, \quad F^{N+1-k^*} = 0 \tag{62}$$

and hence F^ℓ is a matrix with elements $\delta_{i,j+\ell}$.

For notational convenience, let M have the column structure $M = [m_1, m_2, \dots, m_{N+1-k^*}]$ with each $m_j \in \mathcal{R}^{N+1-k^*}$. With this notation, the vanishing of the coefficients of ρ is represented precisely by the equations

$$\begin{aligned} 0 &= e_1 e^T m_{N+1-k^*} \\ 0 &= e_2 e^T m_{N+1-k^*} + e_1 e^T m_{N-k^*} \\ 0 &= e_3 e^T m_{N+1-k^*} + e_2 e^T m_{N-k^*} + e_1 e^T m_{N-1-k^*} \\ &\vdots \\ 0 &= e_{N+1-k^*} e^T m_{N+1-k^*} + e_{N-k^*} e^T m_{N-k^*} + \dots + e_1 e^T m_1 \end{aligned} \tag{63}$$

If $e \neq 0$, then suppose that e_j is the first non-zero element. It follows that $e^T m_{N+2-k^*-j} = 0$. Examination of the structure of the relationships and the use of an inductive argument then yields $e^T m_i = 0, j \leq i \leq N+1-k^*$ and the assumption on the Principal Minors then indicates that $e = 0$ which is a contradiction. Hence the only possible limit vector is $e = 0$. This completes the proof of the result. □

6.4 Switching Between Non-causal Filters

Similar results can be obtained for non-causal filtering. Although it is not yet known (conclusively) whether or not causal or no-causal filtering is the best practical option, the following brief discussion indicates that there is little theoretical difference between the two options in terms of convergence properties.

Using the notation of previous sections, consider the choice of

$$\tilde{K}_j = K F_j^T, \quad 1 \leq j \leq N_s, \quad N_s \geq N+1-k^* \tag{64}$$

As $F_j^T = F_0 F_j F_0$, the computations for $F_j^T e_k$ are approached by passing the time reversed error e_k through then filter $F_j(z)$ and time reversing the computed output.

For analysis purposes, the effect of the change of F_j to F_j^T is to replace ρ by

$$\rho(\lambda, e) = e^T GKF^T(\lambda)e(1-\lambda)^{-1} \tag{65}$$

from which the following theorem follows in a similar manner to the case of causal filters. The proof is omitted for brevity.

Theorem 6 *Suppose that $M = GK$ has the property that, for all $p \geq 1$, the $p \times p$ Principal Minor of M generated by rows and columns indexed by $1 \leq i, j \leq p$ is nonzero. Then with the above notation, the POILC algorithm*

$$u_{k+1} = u_k + \beta_{k+1} K_{k+1} e_k, \quad y_{k+1} = Gu_{k+1} + d \tag{66}$$

with

$$\beta_{k+1} = \arg\min\{J(\beta_{k+1}) = ||e_{k+1}||^2 + w\beta_{k+1}^2\} = \frac{e_k^T GK_{k+1}e_k}{w + ||GK_{k+1}e_k||^2}, \quad w > 0, \tag{67}$$

where K_{k+1} circulates through the finite set $\{KF_j^T\}_{1\leq j\leq N_s}$, is convergent to zero error in the sense that

$$\lim_{k\to\infty} e_k = 0 \quad \& \quad \sum_{k\geq 0} \beta_{k+1}^2 < \infty \tag{68}$$

6.5 Random Choice of Filters

The result for non-causal filters has exactly the same general form and interpretation as that for the causal filtering case. The obvious intuition is that there are many more suitable switching mechanisms with the set of possible finite switching mechanisms being very rich. Further research could identify improved options to the simple options introduced above. Four features do however seem worthy of emphasis and may apply more generally:

1. In both cases considered in this paper, the minimum number of filters required is $N + 1 - k^*$ (although more can be chosen). In general, this is a large number (because N is large) and the likely consequence of choosing fewer is to create a limit set $S_\infty \neq \{0\}$ with the consequent possibility for convergence to non-zero limit errors.

2. The theory provides no information on the ordering of a given set of filters, yet in practice, one could expect that the ordering is important to issues such as the rate of convergence. The number of such orderings is $N_s! \geq (N + 1 - k^*)!$ which is extremely large in practice.

3. The choice of distinct poles $\lambda_j \in (-1, 1)$ is arbitrary in the sense that the choice has no effect on the proof of the results. This leaves open the question of the choice of poles and, in particular, the effect of any choice on algorithm performance. At this time there seems to be no theoretical mechanism that could, for example, distinguish between the benefits of equally spaced distribution of poles λ_j within the open interval $(-1, 1)$ and a more randomly spaced set of poles on a (possibly much smaller) subinterval $[a, b] \subset (-1, 1)$. In both cases, large values of N_s will tend to make the density large. In this sense, the two cases are the same yet the spacing over $(-1, 1)$ will introduce a much richer set of filter dynamics as compared to the case when $|a - b| \ll 1$ i.e. when all filter poles are close together.

4. Stable filters have been used to reflect practical stability needs. More generally, there is no theoretical reason to prevent the choice of unstable filters!

As a preliminary practical approach to the resolution of these problems,

1. based on the intuition that a broader range of filter dynamic characteristics will increase the flexibility of the algorithm, this section assumes that variation of poles over the whole open interval $(-1, 1)$ is to be preferred.

2. In addition, it is proposed that the large number of poles (N_s) needed and the difficulty in choosing the positioning and ordering of such poles makes it reasonable to expect that a *random* choice of poles and hence filters *at each iteration* will be an effective practical approach.

3. It is also assumed that the use of standard uniform random number generators to create the sequence of poles $\{\lambda_j\}_{j\geq 0}$ is sufficient to provide the properties sought.

The concept has practical appeal. However, the theoretical difficulty with the concept is that it is not covered by the previous theorems. Its performance is, as a consequence, unknown. However, given the fact that, for each iteration of POILC, monotonicity of the norm is guaranteed plus the observation that random search is conceptually related to dense search in that all intervals of all sizes in the search space are visited an infinite number of times, convergence is expected on intuitive grounds.

7 A Numerical Study

The purpose of this section is to illustrate the possible performance of the switching algorithms proposed. Results are provided for the case using random, causal filters only. The simple algorithm introduced by Feng and Owens (where $K = I$) is used as the basis for the work. The example chosen for exploration is that examined in section 5.3. where the potentially poor convergence properties of the basic POILC method were illustrated. The poor convergence can be seen in Fig.(6) where the reduction of the logarithmic norm (squared) of the error from 5.7 to only 5.2 over 100 iterations is seen.

In what follows, it will be seen that the use of switching algorithms based on causal filters and the choice of a sequence of stable filter poles uniformly distributed in $(-1, 1)$ has the potential to greatly improve algorithm performance. Being random, the approach has no guarantees but, statistically, substantial benefits could be expected on average. Important insight into these statistical benefits is provided below for the given example through computational experiments using repeated runs of POILC from the same starting condition but using differing (uncorrelated) random pole sequences in each run. This is presented below with the conclusion that the most probable outcome in practice is a substantial improvement on the non-switching case.

The sequence of calculations undertaken was as follows:

1. 100 iterations from the initial control $u_0 = 0$ were undertaken using a pseudo random number generator to create a sequence of uniformly distributed poles $\{\lambda_k\}_{k\geq 0}$ in the open interval $(-1, 1)$. The experiment was repeated a number of times using different independent sequences but starting at the same starting point (i.e. the input to the first iteration $u_0 = 0$). The generated sequences of logarithmic squared error norms is given in fig.(7).

2. The results show the wide range of performance made possible (over 100 iterations) by the use of random switching. The improvement in error norm squared observed ranges from around three to five orders of magnitude (compared to the non-switching case) although the earlier rate of convergence varies more widely as is seen at 40 iterations where performance ranges from little improvement to three orders of magnitude. It is interesting to note that individual norm sequences consist of periods of modest reduction with sudden substantial improvements although these cannot be predicted theoretically.

3. A more detailed analysis of the statistical behaviour of the approach was obtained by repeating the algorithm (from the same initial error) for 350 independent random sequences $\{\lambda_{k+1}\}_{k\geq 0}$ uniformly distributed in $(-1, 1)$. These are illustrated using a frequency diagram in histogram form to identify the frequency with which the results generated logarithmic norms at iteration 100 within uniformly distributed logarithmic intervals (fig.(8)). The results suggest that the

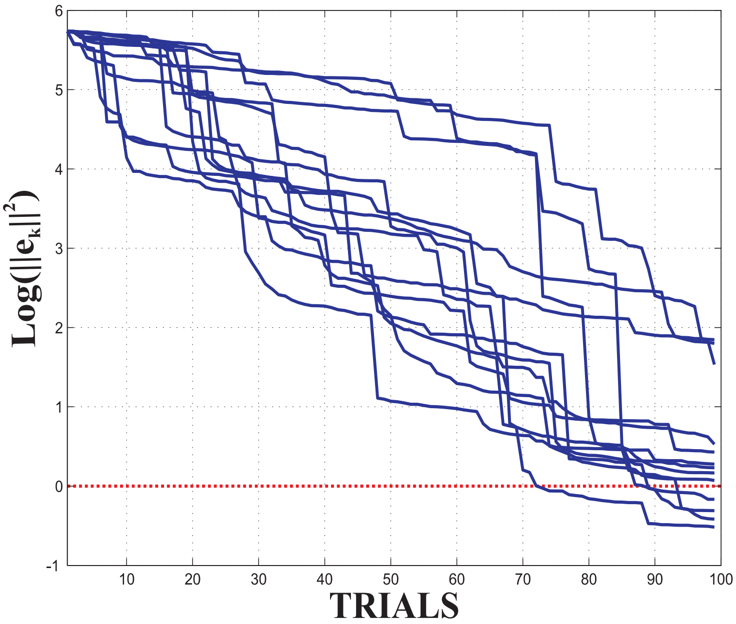

Figure 7: $Log(||e_k||^2)$ v k for several independent sequences of random filters

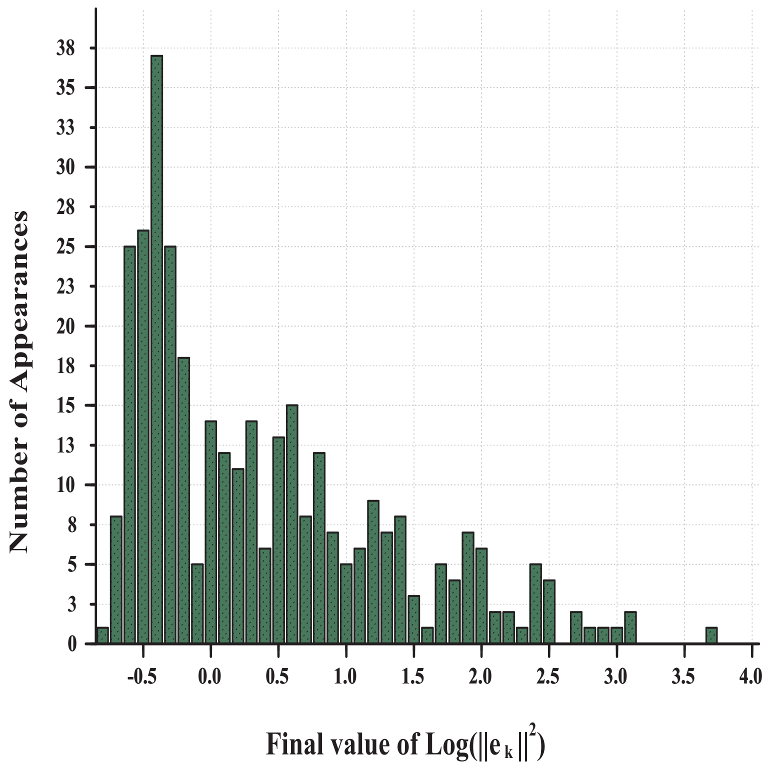

Figure 8: Frequency of $Log(||e_{100}||^2)$ for 350 independent random choices of filter sequences

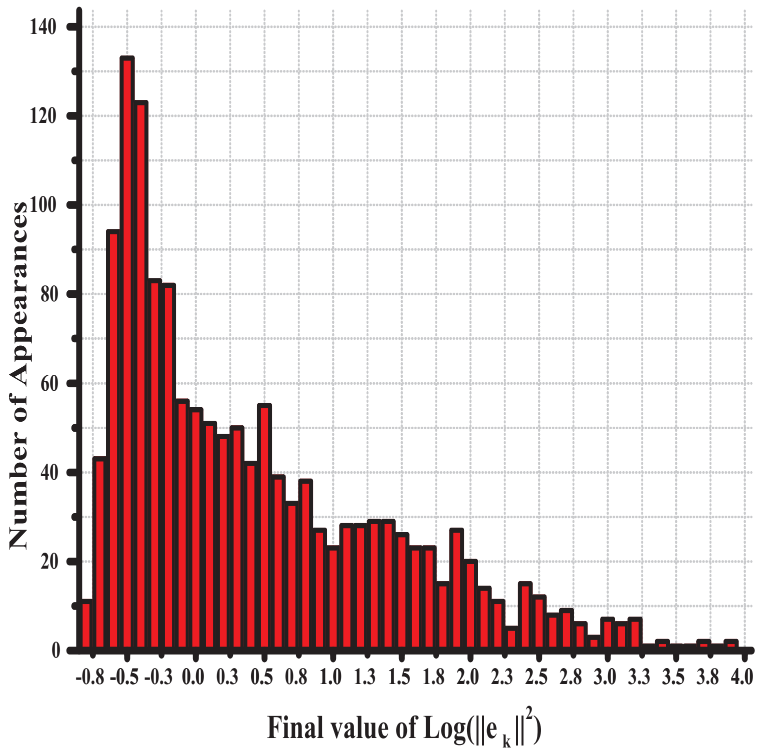

Figure 9: Frequency of $Log(||e_{100}||^2)$ for 1450 independent random choices of filter sequences

frequency distribution has a simple statistical form with the most probable outcome at iteration 100 improving the initial squared error norm $||e_0||^2$ by a factor of around 10^6 i.e. an improvement of in $||e_0||$ of a factor of 10^3.

4. Finally, the form of frequency distribution obtained above was refined by repeating the calculation again over 100 iterations but for the increased number of 1450 independent random sequences uniformly distributed in $(-1, 1)$. The results are shown in fig.(9) and reaffirm the conclusions reached above. Increasing the number of random sequences further supports these conclusions but details are omitted for brevity.

The conclusion reached from this example is that the use of randomly chosen filters in simple parameter optimal ILC is an effective tool in overcoming the problems of temporary and asymptotic poor convergence due to non-trivial stable components of the limit set. For the example chosen (where little error norm reduction is obtained without switching), the most probable benefits of switching (over 100 iterations) indicate improvements in norm reduction of several orders of magnitude. These benefits are expected more generally and are not specific to the system chosen or the choice of 100 iterations. A surprising observation is that these improvements are obtained over 100 iterations which is substantially less than $N + 1 - k^* = 200$.

8 Conclusions

The use of Parameter Optimal Iterative Learning Control has the benefit of ensuring monotonic convergence of the Euclidean norm (mean square value) of the error time series but will tend to produce non-zero limit errors lying within a well-defined but high-dimensional *limit set* $S_\infty \subset R^{N+1-k^*}$. The form of the limit set is critically dependent on the choice of algorithm and the form of plant dynamics. This phenomenon can lead to very poor improvements in performance and a number of other problems. These issues can be, in part, described by the properties of the attracting S_∞^- and repelling components S_∞^+ of S_∞. Both components lead to slow convergence when the error trajectory is close to them. In addition there are potential sensitivity properties close S_∞^+ and a tendency to generate *stair-like behaviours* where apparent convergence is, in fact, temporary slow convergence close to S_∞^+ which is ultimately replaced by a period of faster convergence. This can happen several times over many iterations leading to difficulties in practice in deciding when to terminate the algorithm.

An understanding of the limit set has successfully suggested that the use of switching algorithms can remove many of these problems and, in theory, provide an attractive guarantee of convergence to zero error independent of plant dynamics i.e., in a theoretical sense, switching algorithms are globally successful. The possible choices of switching sequence seems to be very rich. Dense search can be used but, in general it is sufficient to use "only" $N + 1 - k^*$ switching values. The use of causal and non-causal first-order filters (with distinct poles) has been shown to be a simple and effective way forward. This still leaves a large choice of switching elements and there remains problems of the choice of, for example, filter poles and the order in which they are to be used. Numerical computation based on the random choice of switching filters has demonstrated that a statistical approach has great potential. Computational examples show that it can considerably improve convergence properties with the "most-probable" error reduction being orders of magnitude better than that obtained in the non-switching case.

Although the paper has provided substantial theoretical support for the ideas introduced, there are many issues that arise from the developments. These include the development of a more rigorous general approach to the statistical method proposed (a difficult problem as the POILC method is nonlinear in the filter poles). Of particular value here would be the characterization of the frequency distribution function of the logarithmic error norm at a specified iteration. The simple form observed in the frequency plots computed in this paper suggest that these distributions could have quite simple forms. A simpler (but not necessarily simple) objective might be to compute the Expectation of $||e_\ell||$ for a give e_0 for a given probability distribution function used for the choice of filter poles.

This paper has not answered all relevant questions. Amongst the many issues that arise is the ultimate need to consider the robustness with respect to resetting of the initial condition and plant modelling error. These are non-trivial deterministic and stochastic questions that are left for future research and publications to resolve.

9 Final Comments

The second author was responsible for the computational results. The work described in this paper is covered by pending patent applications in the UK and elsewhere.

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