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Guo, Y., Billings, S.A., Coca, D. et al. (2 more authors) (2009) Characterising spatio-temporal dynamical systems in the frequency domain. Research Report. ACSE Research Report no. 1000 . Automatic Control and Systems Engineering, University of Sheffield

Characterising Spatio-Temporal Dynamical Systems in the Frequency Domain

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Research Report No. 1000

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12 August 2009

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Abstract

In this paper a new concept, spatio-temporal generalised frequency response functions (STGFRF), is introduced for the first time to characterise spatio-temporal dynamical systems in the frequency domain. A probing method is developed to calculate the STGFRFs for both continuous and discrete spatio-temporal systems.

Keywords: Spatio-temporal systems, spatio-temporal generalised frequency response function, probing method.

1 Introduction

Linear spectral analysis for temporal systems has greatly matured and is widely used in almost every branch of science and engineering. Frequency domain analysis has also been extended to nonlinear dynamical systems. Many important methods have been introduced such as the calculation of generalised frequency response functions (Billings and Tsang 1989a; Lang et al. 2007), determination of the output frequency range (Lang and Billings 1997), characteristics of non-linear generalised frequency response functions (Yue et al. 2005) and the introduction of energy transfer filters (Billings and Lang 2002). Frequency domain analysis has also been widely used in the analysis of spatial systems, especially in image processing, such as in the analysis of image filters.

One excellent method to identify the generalised frequency response functions (GFRF) was introduced by Billings and Tsang (1989a; 1989b), which consists of estimating a NARMAX description of the system and computing the generalised frequency response functions directly from the estimated model using a development of the probing method. In this paper a similar method will be developed to obtain analytic expressions for spatio-temporal generalised frequency response functions for a class of spatio-temporal systems.

The concept of spatio-temporal transfer function was introduced by Billings and Wei(2007) and is a natural extension of the ordinary transfer function for classical linear time-invariant control systems. A similar concept, the multidimensional transfer function was proposed by Rabenstein and Trautmann(2002) and a Sturm-Liouville transformation based method was developed to obtain the multi-dimensional transfer function from continuous initial-boundary value problems. In this paper a much simpler method will be introduced to directly calculate the analytic expression of the spatio-temporal generalised frequency response functions from continuous and discrete spatio-temporal models.

The frequency response is a measure of a system's response to a sinusoidal input of varying frequency. The frequency response is typically characterised by the magnitude and the phase of the system's response versus frequency, that is, the frequency response function. In order to calculate and analyse the frequency response function, spatio-temporal systems with external inputs should be considered. An example of a spatio-temporal system with an external input is introduced in section 2. Section 3 introduces the probing method for the calculation of generalised frequency response functions for continuous and discrete temporal systems. Section 4 extends this method to spatio-temporal dynamical systems. Boundary-value problems are considered in section 5. Two examples are considered in section 6 to graphically show the spatio-temporal generalised frequency response functions. Conclusions are finally given in section 7.

2 Spatio-Temporal Systems with External Inputs

In many cases, spatio-temporal systems are autonomous systems, that is, all the systems evolve from an initial condition and only depend on the initial conditions and the dynamic characters of the model. The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The magnitude and phase of the output signal is a function of the input frequency. In order to calculate and analyse the frequency response functions, in this paper spatio-temporal systems with external inputs are considered. It will be shown that spatio-temporal dynamical systems have many similar features which were observed in temporal dynamical systems.

Consider a linear one dimensional spatio-temporal system with an external input.

$$\frac{\partial^2 y}{\partial t^2} + \xi_1 \frac{\partial y}{\partial t} + \xi_2 \left(\frac{\partial y}{\partial t}\right)^2 + \omega_0^2 y = c\frac{\partial^2 y}{\partial x^2} + bu \tag{1}$$

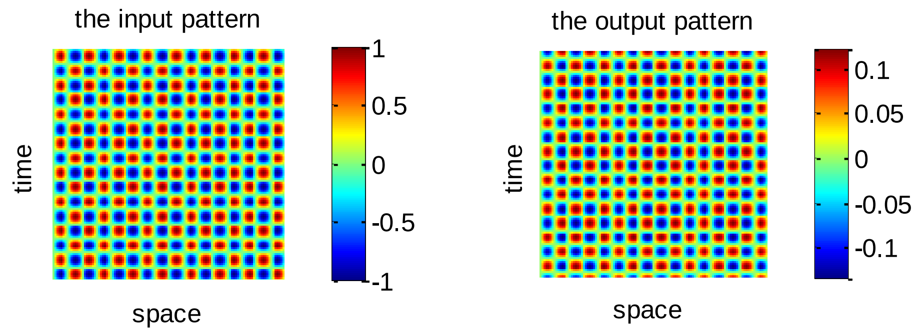
where y and u are functions of both the spatial coordinate x and the temporal coordinate t, that is, both the output and input signals are spatio-temporal patterns denoted as $y(t, x)$ and $u(t, x)$ respectively.

Define the input signal as

$$u(t, x) = \sin(\omega_{t0} t)\sin(\omega_{x0} x) \tag{2}$$

where $\omega_{t0} = 2\pi$ (rad/s) and $\omega_{x0} = 2\pi$ (rad/s) represent the spatial and temporal frequencies of the input separately. The input pattern is shown in Fig 1 (a).

Setting the parameters of system (1) as $\xi_1 = 4, \xi_2 = 0.5, \omega_0 = 1, c = 0.01, b = 1$ and simulating the system on a 1024×1024 lattice gave the steady-state output in Fig 1 (b).

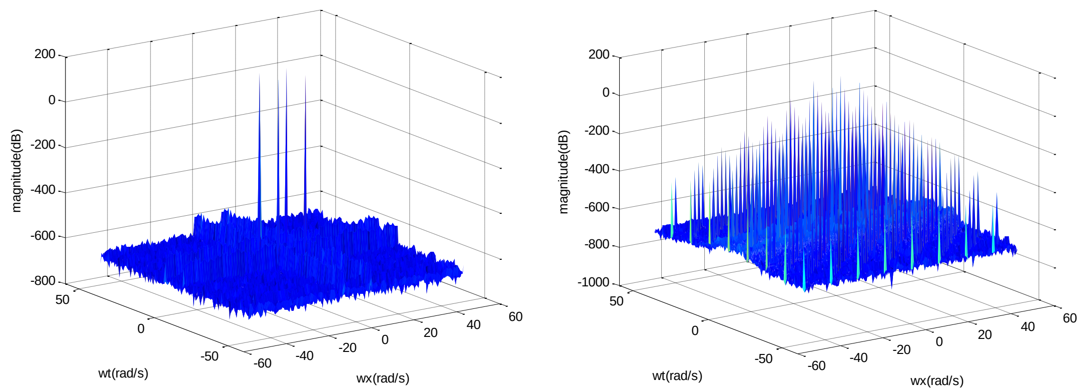


(a) (b)

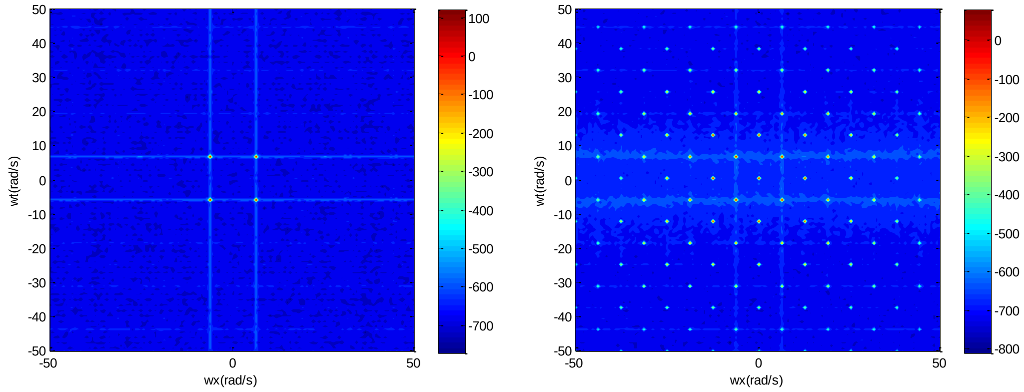
Fig 1 Simulation of a spatio-temporal system with input

(a) the input pattern (b) the output pattern

Except for a slightly smaller magnitude and phase delay, the output pattern looks almost the same as the input pattern. However the frequency domain analysis can discover more than these initial spatio-temporal appearances appear to show. Calculating the two dimensional Fast Fourier transforms of the input and output yields the approximate frequency spectra of the input and out patterns given in Fig 2. The spectrum of the input only has a peak at $(\omega_t, \omega_x) = (+2\pi, +2\pi)$, $(-2\pi, -2\pi)$, $(+2\pi, -2\pi)$ and $(-2\pi, +2\pi)$ separately, which corresponds to the temporal frequency ω_{t0} and the spatial frequency ω_{x0} of the input signal. However, the output spectrum is much richer than the input spectrum. The output pattern has peaks at all points $(\omega_t, \omega_x) = (p\omega_t, q\omega_x)$ where $p, q \in Z$. This is because of the effects of the nonlinear term $\xi_2 \left(\frac{\partial y}{\partial t}\right)^2$.



(a) (b)



(c) (d)

Fig 2 Frequency spectra of input and output patterns

(a) (c) input spectrum (b) (d) output spectrum

3 The Probing Method

Billings and Tsang(1989a) showed that the harmonic input or probing method can be used to determine the nth order generalised frequency response function (GFRF) of a nonlinear system by equating the coefficients of the system output for an input defined as $u(t)=\sum_{k=1}^{n} A_k e^{j\omega_k t}$. The response of the system is of the form $y(t)=\sum_{n=1}^{\infty} y_n(t)$ where $y_n(t)=\sum_{k_1=1}^{n}\cdots\sum_{k_n=1}^{n} H_n(\omega_{k_1},\cdots,\omega_{k_n}) e^{j(\omega_{k_1}+\cdots+\omega_{k_n})t}$ is the nth order output when $A_k=1$ for all $k=1,2,\cdots,n$. $H_n(\omega_{k_1},\cdots,\omega_{k_n})$ is the nth order GFRF. The procedure to calculate the generalised frequency response functions is briefly reviewed by considering a continuous and a discrete temporal example. For more details of the probing method, refer to the papers (Billings and Tsang 1989a; Billings and Tsang 1989b; Billings et al. 1990).

3.1 Calculation of Generalised Frequency Response Functions for Continuous Temporal Systems

Consider a purely temporal continuous dynamical system described by the differential equation

$$\frac{d^2y}{dt^2}+\xi_1\frac{dy}{dt}+\xi_2\left(\frac{dy}{dt}\right)^2+\omega_0^2 y(t)=bu(t) \tag{3}$$

The procedure begins by defining the probing input as

$$u(t)=e^{j\omega_t t} \tag{4}$$

and the corresponding output as

$$y(t)=H_1(j\omega_t)e^{j\omega_t t} \tag{5}$$

The derivative of $y(t)$ can then be written as

$$\frac{dy}{dt}=j\omega_t H_1(j\omega_t)e^{j\omega_t t} \tag{6}$$

Then the second derivative of $y(t)$ is

$$\frac{d^2y}{dt^2}=-\omega_t^2 H_1(j\omega_t)e^{j\omega_t t} \tag{7}$$

Substituting (4) ~ (7) into equation (3) yields

$$\begin{aligned}&-\omega_t^2 H_1(j\omega_t)e^{j\omega_t t}+\xi_1 j\omega_t H_1(j\omega_t)e^{j\omega_t t}\\&+\xi_2\left(j\omega_t\right)^2 H_1^2(j\omega_t)e^{j2\omega_t t}+\omega_0^2 H_1(j\omega_t)e^{j\omega_t t}=be^{j\omega_t t}\end{aligned} \tag{8}$$

Equating the coefficients of $e^{j\omega_t t}$ on both side yields

$$H_1(j\omega_t)=\frac{b}{(j\omega_t)^2+\xi_1 j\omega_t+\omega_0^2} \tag{9}$$

$H_1\left(j\omega_t\right)$ is the first order generalised frequency response function which characterises the linear portion of the response. In this example, the first order generalised frequency response is a typical second order linear system with the natural frequency ω_0 and the damping ratio $\xi_1/2\omega_0$.

Probing with an input with two different frequencies which is given as

$$u(t)=e^{j\omega_{t1}t}+e^{j\omega_{t2}t} \tag{10}$$

the corresponding response of the system is

$$
\begin{aligned}
y(t) &= H_1(j\omega_{t1})e^{j\omega_{t1}t} + H_1(j\omega_{t2})e^{j\omega_{t2}t} \\
&+ H_2(j\omega_{t1}, j\omega_{t1})e^{j2\omega_{t1}t} + H_2(j\omega_{t2}, j\omega_{t2})e^{j2\omega_{t2}t} + 2H_2(j\omega_{t1}, j\omega_{t2})e^{j(\omega_{t1}+\omega_{t2})t}
\end{aligned} \tag{11}
$$

and the first and second order derivatives are

$$
\begin{aligned}
\frac{dy}{dt} &= j\omega_{t1}H_1(j\omega_{t1})e^{j\omega_{t1}t} + j\omega_{t2}H_1(j\omega_{t2})e^{j\omega_{t2}t} \\
&+ j2\omega_{t1}H_2(j\omega_{t1}, j\omega_{t1})e^{j2\omega_{t1}t} + j2\omega_{t2}H_2(j\omega_{t2}, j\omega_{t2})e^{j2\omega_{t2}t} \\
&+ 2j(\omega_{t1}+\omega_{t2})H_2(j\omega_{t1}, j\omega_{t2})e^{j(\omega_{t1}+\omega_{t2})t}
\end{aligned} \tag{12}
$$

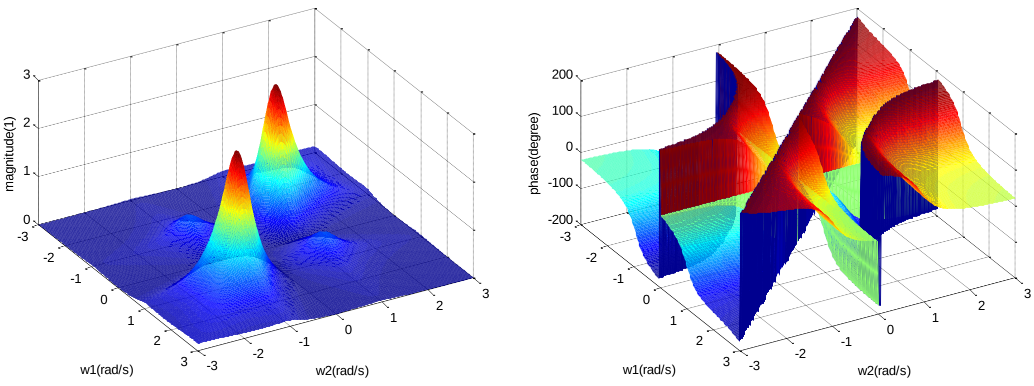
and

$$
\begin{aligned}
\frac{d^2y}{dt^2} &= -\omega_{t1}^2H_1(j\omega_{t1})e^{\omega_{t1}t} - \omega_{t2}^2H_1(j\omega_{t2})e^{j\omega_{t2}t} \\
&- 4\omega_{t1}^2H_2(j\omega_{t1}, j\omega_{t1})e^{j2\omega_{t1}t} - 4\omega_{t2}^2H_2(j\omega_{t2}, j\omega_{t2})e^{j2\omega_{t2}t} \\
&- 2(\omega_{t1}+\omega_{t2})^2H_2(j\omega_{t1}, j\omega_{t2})e^{j(\omega_{t1}+\omega_{t2})t}
\end{aligned} \tag{13}
$$

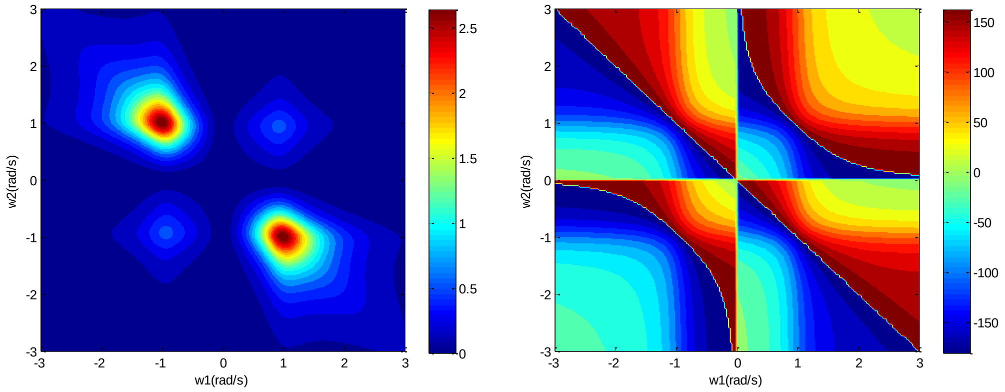
Substituting equation (10) ~ (13) into (3) and equating coefficients of $e^{j(\omega_{t1}+\omega_{t2})t}$ yields

$$
H_2(j\omega_{t1}, j\omega_{t2}) = \frac{\xi_2\omega_{t1}\omega_{t2}H_1(j\omega_{t1})H_1(j\omega_{t2})}{-(\omega_{t1}+\omega_{t2})^2 + \xi_1 j(\omega_{t1}+\omega_{t2}) + \omega_0^2} \tag{14}
$$

$H_2(j\omega_{t1}, j\omega_{t2})$ is the second order generalised frequency response function which characterises the quadratic contribution to the response. An example of $H_2(j\omega_{t1}, j\omega_{t2})$ is given in Fig 3.



(a) (b)



(c) (d)

Fig 3 The second order generalised frequency response function of system (3)

(a) (b) magnitude (c) (d) phase $\xi_1 = 0.6, \xi_2 = 1, \omega_0 = 1, b = 1$

Following this procedure all the higher order generalised frequency response functions can be calculated recursively.

3.2 Calculation of Generalised Frequency Response Functions for Discrete Temporal Systems

The probing method can also be applied to calculating the generalised frequency response functions from a discrete model.

System (3) can be discretised into a discrete model given in (15) by the forward finite difference method with spatial and temporal sample intervals Δx and Δt.

$$\begin{aligned} y(k+2) &= a_1 y(k+1) + a_2 y(k) + a_3 y^2(k+1) \\ &+ a_4 y(k) y(k+1) + a_5 y^2(k) + b_1 u(k) \end{aligned} \tag{15}$$

where

$$\begin{aligned} a_1 &= 2-\xi_1\Delta t \\ a_2 &= -1+\xi_1\Delta t-\omega_0^2\left(\Delta t\right)^2 \\ a_3 &= -\xi_2 \\ a_4 &= 2\xi_2 \\ a_5 &= -\xi_2 \\ b_1 &= b\left(\Delta t\right)^2 \end{aligned} \tag{16}$$

System (15) will now be used to illustrate the probing method for the calculation of the generalised frequency response function of discrete systems.

Define the discrete probing input

$$u(k)=e^{j\omega_t k\Delta t} \tag{17}$$

and the corresponding output as

$$y(k)=H_{d1}(j\omega_t)e^{j\omega_t k\Delta t} \tag{18}$$

The one step ahead and two step ahead outputs are

$$y(k+1)=H_{d1}(j\omega_t)e^{j\omega_t (k+1)\Delta t} \tag{19}$$

and

$$y(k+2)=H_{d1}(j\omega_t)e^{j\omega_t (k+2)\Delta t} \tag{20}$$

Substituting (17) ～ (20) into (15) yields

$$\begin{aligned} H_{d1}(j\omega_t)e^{j\omega_t(k+2)\Delta t} &= a_1H_{d1}(j\omega_t)e^{j\omega_t(k+1)\Delta t}+a_2H_{d1}(j\omega_t)e^{j\omega_t k\Delta t} \\ &+a_3\left(H_{d1}(j\omega_t)e^{j\omega_t(k+1)\Delta t}\right)^2+a_4H_{d1}(j\omega_t)e^{j\omega_t k\Delta t}H_{d1}(j\omega_t)e^{j\omega_t(k+1)\Delta t} \\ &+a_5\left(H_{d1}(j\omega_t)e^{j\omega_t k\Delta t}\right)^2+b_1e^{j\omega_t k\Delta t} \end{aligned} \tag{21}$$

Equating the coefficient of $e^{j\omega_t k\Delta t}$ yields

$$H_{d1}(j\omega_t)=\frac{b_1}{e^{j\omega_t 2\Delta t}-a_1e^{j\omega_t\Delta t}-a_2} \tag{22}$$

$H_{d1}(j\omega_t)$ is the first order generalised frequency response function which describes the same system characteristics as $H_d(j\omega_t)$ does.

Probing the discrete system with a input with two different frequencies ω_{t1} and ω_{t2}

$$u(k) = e^{j\omega_{t1}k\Delta t} + e^{j\omega_{t2}k\Delta t} \tag{23}$$

The corresponding output can be defined as

$$\begin{aligned} y(k) = {} & H_{d1}(j\omega_{t1})e^{j\omega_{t1}k\Delta t} + H_{d1}(j\omega_{t2})e^{j\omega_{t2}k\Delta t} + H_{d2}(j\omega_{t1}, j\omega_{t1})e^{j2\omega_{t2}k\Delta t} \\ & + H_{d2}(j\omega_{t2}, j\omega_{t2})e^{j2\omega_{t2}k\Delta t} + 2H_{d2}(j\omega_{t1}, j\omega_{t2})e^{j\ \omega_{t2}+\omega_{t2}\ k\Delta t} \end{aligned} \tag{24}$$

The time advanced outputs are

$$\begin{aligned} y(k+1) = {} & H_{d1}(j\omega_{t1})e^{j\omega_{t1}\ k+1\ \Delta t} + H_{d1}(j\omega_{t2})e^{j\omega_{t2}\ k+1\ \Delta t} \\ & + H_{d2}(j\omega_{t1}, j\omega_{t1})e^{j2\omega_{t2}\ k+1\ \Delta t} + H_{d2}(j\omega_{t2}, j\omega_{t2})e^{j2\omega_{t2}\ k+1\ \Delta t} \\ & + 2H_{d2}(j\omega_{t1}, j\omega_{t2})e^{j\ \omega_{t2}+\omega_{t2}\ \ k+1\ \Delta t} \end{aligned} \tag{25}$$

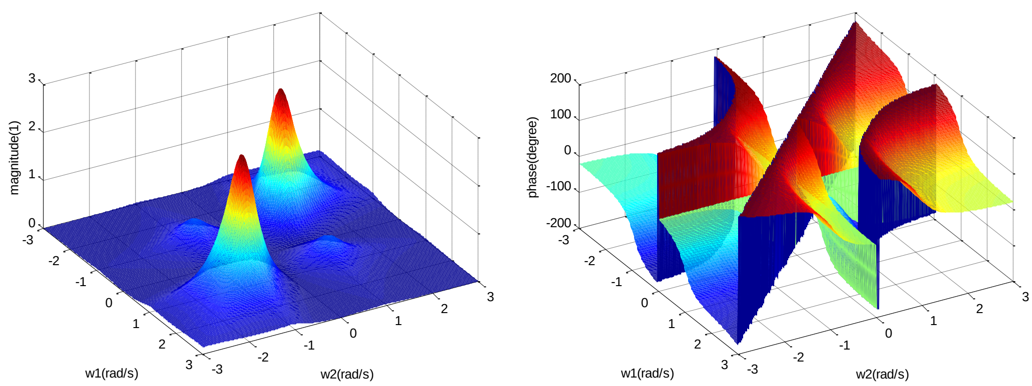
and

$$\begin{aligned} y(k+2) = {} & H_{d1}(j\omega_{t1})e^{j\omega_{t1}\ k+2\ \Delta t} + H_{d1}(j\omega_{t2})e^{j\omega_{t2}\ k+2\ \Delta t} \\ & + H_{d2}(j\omega_{t1}, j\omega_{t1})e^{j2\omega_{t2}\ k+2\ \Delta t} + H_{d2}(j\omega_{t2}, j\omega_{t2})e^{j2\omega_{t2}\ k+2\ \Delta t} \\ & + 2H_{d2}(j\omega_{t1}, j\omega_{t2})e^{j\ \omega_{t2}+\omega_{t2}\ \ k+2\ \Delta t} \end{aligned} \tag{26}$$

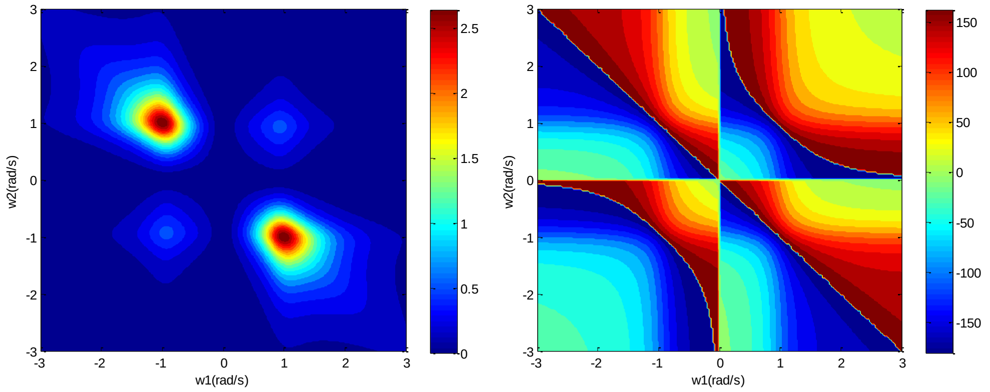
Substituting (23) ~ (26) into (15) and equating the coefficients of $e^{j\ \omega_{t2}+\omega_{t2}\ k\Delta t}$ on both sides yields

$$H_{d2}(j\omega_{t1}, j\omega_{t2}) = \frac{\left(2a_3 e^{j(\omega_{t1}+\omega_{t2})\Delta t} + a_4\left(e^{j\omega_{t1}\Delta t} + e^{j\omega_{t2}\Delta t}\right) + 2a_5\right) H_{d1}(j\omega_{t1}) H_{d1}(j\omega_{t2})}{2e^{j(\omega_{t2}+\omega_{t2})2\Delta t} - 2a_1 e^{j(\omega_{t2}+\omega_{t2})\Delta t} - 2a_2} \tag{27}$$

$H_{d2}(j\omega_{t1}, j\omega_{t2})$ is the second order generalised frequency response function which characterises the same features of the system as $H_2(j\omega_{t1}, j\omega_{t2})$ does. An example of $H_{d2}(j\omega_{t1}, j\omega_{t2})$ is given in Fig 4 which is almost exactly the same as the second order generalised frequency response function shown in Fig 3.



(a) (b)



(c) (d)

Fig 4 The second order generalised frequency response function of system (3)

$$a_1 = 1.994, a_2 = -0.9941, a_3 = -1, a_4 = 2, a_5 = -1, b_1 = 0.0001, \Delta t = \Delta x = 0.01$$

4 Calculation of STGFRF for Spatio-temporal Dynamical Systems

The probing method will now be developed to calculate the generalised frequency response functions of a class of spatio-temporal systems. For a bounded input bounded output stable spatio-temporal system, define the probing input as $u(t,x) = \sum_{k=1}^{n} e^{j\omega_{tk}t + j\omega_{xk}x}$, where ω_{tk} and ω_{xk} are the temporal and spatial frequencies separately. The steady-state output of a spatio-temporal system can then be defined as $y(t,x) = \sum_{n=1}^{\infty} y_n(t,x)$, where $y_n(t,x) = \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} H_n^{ST}\left(\omega_{tk_1}, \cdots, \omega_{tk_n}, \omega_{xk_1}, \cdots, \omega_{xk_n}\right)$ is the nth order output and $H_n^{ST}\left(\omega_{tk_1}, \cdots, \omega_{tk_n}, \omega_{xk_1}, \cdots, \omega_{xk_n}\right)$ is the nth order spatio-temporal generalised frequency response function.

4.1 Calculation of STGFRF for Continuous Nonlinear

Spatio-temporal systems

Now consider a nonlinear spatio-temporal system given as

$$\frac{\partial^2 y}{\partial t^2}+\xi_1\frac{\partial y}{\partial t}+\xi_2\left(\frac{\partial y}{\partial t}\right)^2+\omega_0^2 y(x,t)=c\frac{\partial^2 y}{\partial x^2}+bu(x,t) \tag{28}$$

Probing the system with an input defined as

$$u(x,t)=e^{j\omega_t t+j\omega_x x} \tag{29}$$

the output can then be defined as

$$y(t,x)=H_1^{ST}(j\omega_t,j\omega_x)e^{j\omega_t t+j\omega_x x} \tag{30}$$

Accordingly the temporal and spatial derivatives of the output are

$$\frac{\partial y}{\partial t}=j\omega_t H_1^{ST}(j\omega_t,j\omega_x)e^{j\omega_t t+j\omega_x x} \tag{31}$$

$$\frac{\partial^2 y}{\partial t^2}=-\omega_t^2 H_1^{ST}(j\omega_t,j\omega_x)e^{j\omega_t t+j\omega_x x} \tag{32}$$

and

$$\frac{\partial^2 y}{\partial x^2}=-\omega_x^2 H_1^{ST}(j\omega_t,j\omega_x)e^{j\omega_t t+j\omega_x x} \tag{33}$$

Substituting equation (29) ~ (33) into system (28) yields

$$\begin{aligned}&-\omega_t^2 H_1^{ST}(j\omega_t,j\omega_x)e^{j\omega_t t+j\omega_x x}+\xi_1 j\omega_t H_1^{ST}(j\omega_t,j\omega_x)e^{j\omega_t t+j\omega_x x}\\&-\xi_2\omega_t^2\left(H_1^{ST}(j\omega_t,j\omega_x)\right)^2 e^{j2\omega_t t+j2\omega_x x}+\omega_0^2 H_1^{ST}(j\omega_t,j\omega_x)e^{j\omega_t t+j\omega_x x}\\&=-c\omega_x^2 H_1^{ST}(j\omega_t,j\omega_x)e^{j\omega_t t+j\omega_x x}+be^{j\omega_t t+j\omega_x x}\end{aligned} \tag{34}$$

Equating the coefficients of $e^{j\omega_t t+j\omega_x x}$ on both sides yields

$$H_1^{ST}(j\omega_t,j\omega_x)=\frac{b}{c\omega_x^2-\omega_t^2+j\xi_1\omega_t+\omega_0^2} \tag{35}$$

$H_1^{ST}(j\omega_t,j\omega_x)$ is the first order spatio-temporal generalised frequency response function which characterises the linear contribution to the output. Specially, for a linear spatio-temporal system, $H_1^{ST}(j\omega_t,j\omega_x)$ is the only STGFRF.

For the nonlinear spatio-temporal system in this case, define the input with two different spatial and temporal frequencies given as

$$u(x,t)=e^{j\omega_{t1}t+j\omega_{x1}x}+e^{j\omega_{t2}t+j\omega_{x2}x} \tag{36}$$

The corresponding response is defined as

$$\begin{aligned} y(t,x)=&H_1^{ST}(j\omega_{t1},j\omega_{x1})e^{j\omega_{t1}t+j\omega_{x1}x}+H_1^{ST}(j\omega_{t2},j\omega_{x2})e^{j\omega_{t2}t+j\omega_{x2}x}\\ &+2H_2^{ST}\left(j\omega_{t1},j\omega_{x1},j\omega_{t2},j\omega_{x2}\right)e^{j(\omega_{t1}+\omega_{t2})t+j(\omega_{x1}+\omega_{x2})x}\\ &+H_2^{ST}\left(j\omega_{t1},j\omega_{x1},j\omega_{t1},j\omega_{x1}\right)e^{j2\omega_{t1}t+j2\omega_{x1}x}\\ &+H_2^{ST}\left(j\omega_{t2},j\omega_{x2},j\omega_{t2},j\omega_{x2}\right)e^{j2\omega_{t2}t+j2\omega_{x2}x} \end{aligned} \tag{37}$$

and the spatial and temporal derivatives are

$$\begin{aligned} \frac{\partial y}{\partial t}=&\ j\omega_{t1}H_1^{ST}(j\omega_{t1},j\omega_{x1})e^{j\omega_{t1}t+j\omega_{x1}x}+j\omega_{t2}H_1^{ST}(j\omega_{t2},j\omega_{x2})e^{j\omega_{t2}t+j\omega_{x2}x}\\ &+j\left(\omega_{t1}+\omega_{t2}\right)2H_2^{ST}\left(j\omega_{t1},j\omega_{x1},j\omega_{t2},j\omega_{x2}\right)e^{j(\omega_{t1}+\omega_{t2})t+j(\omega_{x1}+\omega_{x2})x}\\ &+j2\omega_{t1}H_2^{ST}\left(j\omega_{t1},j\omega_{x1},j\omega_{t1},j\omega_{x1}\right)e^{j2\omega_{t1}t+j2\omega_{x1}x}\\ &+j2\omega_{t2}H_2^{ST}\left(j\omega_{t2},j\omega_{x2},j\omega_{t2},j\omega_{x2}\right)e^{j2\omega_{t2}t+j2\omega_{x2}x} \end{aligned} \tag{38}$$

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2}=&-\omega_{t1}^2H_1^{ST}(j\omega_{t1},j\omega_{x1})e^{j\omega_{t1}t+j\omega_{x1}x}-\omega_{t2}^2H_1^{ST}(j\omega_{t2},j\omega_{x2})e^{j\omega_{t2}t+j\omega_{x2}x}\\ &-\left(\omega_{t1}+\omega_{t2}\right)^2 2H_2^{ST}\left(j\omega_{t1},j\omega_{x1},j\omega_{t2},j\omega_{x2}\right)e^{j(\omega_{t1}+\omega_{t2})t+j(\omega_{x1}+\omega_{x2})x}\\ &-4\omega_{t1}^2H_2^{ST}\left(j\omega_{t1},j\omega_{x1},j\omega_{t1},j\omega_{x1}\right)e^{j2\omega_{t1}t+j2\omega_{x1}x}\\ &-4\omega_{t2}^2H_2^{ST}\left(j\omega_{t2},j\omega_{x2},j\omega_{t2},j\omega_{x2}\right)e^{j2\omega_{t2}t+j2\omega_{x2}x} \end{aligned} \tag{39}$$

and

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2}=&-\omega_{x1}^2H_1^{ST}(j\omega_{t1},j\omega_{x1})e^{j\omega_{t1}t+j\omega_{x1}x}-\omega_{x2}^2H_1^{ST}(j\omega_{t2},j\omega_{x2})e^{j\omega_{t2}t+j\omega_{x2}x}\\ &-\left(\omega_{x1}+\omega_{x2}\right)^2 2H_2^{ST}\left(j\omega_{t1},j\omega_{x1},j\omega_{t2},j\omega_{x2}\right)e^{j(\omega_{t1}+\omega_{t2})t+j(\omega_{x1}+\omega_{x2})x}\\ &-4\omega_{x1}^2H_2^{ST}\left(j\omega_{t1},j\omega_{x1},j\omega_{t1},j\omega_{x1}\right)e^{j2\omega_{t1}t+j2\omega_{x1}x}\\ &-4\omega_{x2}^2H_2^{ST}\left(j\omega_{t2},j\omega_{x2},j\omega_{t2},j\omega_{x2}\right)e^{j2\omega_{t2}t+j2\omega_{x2}x} \end{aligned} \tag{40}$$

Substituting (36) ~ (40) into (28) and the second order spatio-temporal generalised frequency response function is

$$\begin{aligned} &H_2^{ST}\left(j\omega_{t1},j\omega_{x1},j\omega_{t2},j\omega_{x2}\right)\\ &=\frac{\xi_2\omega_{t1}\omega_{t2}H_1(j\omega_{t1},j\omega_{x1})H_1(j\omega_{t2},j\omega_{x2})}{-\left(\omega_{t1}+\omega_{t2}\right)^2+\xi_1 j\left(\omega_{t1}+\omega_{t2}\right)+\omega_0^2+c\left(\omega_{x1}+\omega_{x2}\right)^2} \end{aligned} \tag{41}$$

4.2 Calculation of STGFRF for Discrete Nonlinear Spatio-Temporal Systems

Now the probing method will be developed to calculate the STGFRF for discrete nonlinear spatio-tempoal systems. Discretising system (28) using a forward-time-centred-space finite difference method yields a discrete spatio-temporal system

$$\begin{aligned} y(k+2,h) &= a_1 y(k+1,h) + a_2 y(k,h) \\ &+ a_3 y^2(k+1,h) + a_4 y(k+1,h) y(k,h) + a_5 y^2(k,h) \\ &+ d_1 y(k,h+1) + d_2 y(k,h-1) + b_1 u(k,h) \end{aligned} \tag{42}$$

where

$$\begin{aligned} a_1 &= 2 - \xi_1 \Delta t \\ a_2 &= -1 + \xi_1 \Delta t - \omega_0^2 (\Delta t)^2 - \frac{2c(\Delta t)^2}{(\Delta x)^2} \\ a_3 &= -\xi_2 \\ a_4 &= 2\xi_2 \\ a_5 &= -\xi_2 \\ d_1 &= d_2 = \frac{c(\Delta t)^2}{(\Delta x)^2} \\ b_1 &= b(\Delta t)^2 \end{aligned} \tag{43}$$

Firstly, define the discrete probing input as

$$u(k,h) = e^{j\omega_t k\Delta t + j\omega_x h\Delta x} \tag{44}$$

and the corresponding output as

$$y(k,h) = H_{d1}^{ST}(j\omega_t, j\omega_x) e^{j\omega_t k\Delta t + j\omega_x h\Delta x} \tag{45}$$

The forward time-shifted and space-shifted outputs are

$$\begin{aligned} y(k+1,h) &= e^{j\omega_t \Delta t} H_{d1}^{ST}(j\omega_t, j\omega_x) e^{j\omega_t k\Delta t + j\omega_x h\Delta x} \\ y(k+2,h) &= e^{j\omega_t 2\Delta t} H_{d1}^{ST}(j\omega_t, j\omega_x) e^{j\omega_t k\Delta t + j\omega_x h\Delta x} \\ y(k,h+1) &= e^{j\omega_x \Delta x} H_{d1}^{ST}(j\omega_t, j\omega_x) e^{j\omega_t k\Delta t + j\omega_x h\Delta x} \\ y(k,h-1) &= e^{-j\omega_x \Delta x} H_{d1}^{ST}(j\omega_t, j\omega_x) e^{j\omega_t k\Delta t + j\omega_x h\Delta x} \end{aligned} \tag{46}$$

Substituting equation (44) ~ (46) into (42) and equating the coefficients of $e^{j\omega_t k\Delta t + j\omega_x h\Delta x}$ on both sides yields the first order spatio-temporal generalised frequency response function

$$H_{d1}^{ST}(j\omega_t, j\omega_x) = \frac{b_1}{e^{j\omega_t 2\Delta t} - a_1 e^{j\omega_t \Delta t} - a_2 - d_1 e^{j\omega_x \Delta x} - d_2 e^{-j\omega_x \Delta x}} \tag{47}$$

Probing with inputs consisting of two different frequencies

$$u(k,h) = e^{j(\omega_{t1}+\omega_{t2})k\Delta t + j(\omega_{x1}+\omega_{x2})h\Delta x} \tag{48}$$

The output can be defined as

$$\begin{aligned} y(k,h) = {} & H_{d1}^{ST}(j\omega_{t1}, j\omega_{x1}) e^{j\omega_{t1}k\Delta t + j\omega_{x1}h\Delta x} + H_{d1}^{ST}(j\omega_{t2}, j\omega_{x2}) e^{j\omega_{t2}k\Delta t + j\omega_{x2}h\Delta x} \\ & + 2H_{d2}^{ST}\left(j\omega_{t1}, j\omega_{x1}, j\omega_{t2}, j\omega_{x2}\right) e^{j(\omega_{t1}+\omega_{t2})k\Delta t + j(\omega_{x1}+\omega_{x2})h\Delta x} \\ & + H_{d2}^{ST}\left(j\omega_{t1}, j\omega_{x1}, j\omega_{t1}, j\omega_{x1}\right) e^{j2\omega_{t1}k\Delta t + j2\omega_{x1}h\Delta x} \\ & + H_{d2}^{ST}\left(j\omega_{t2}, j\omega_{x2}, j\omega_{t2}, j\omega_{x2}\right) e^{j2\omega_{t2}k\Delta t + j2\omega_{x2}h\Delta x} \end{aligned} \tag{49}$$

Substituting the input, output and the associated time-shift and space-shift into (42) and equating the coefficients of $e^{j(\omega_{t1}+\omega_{t2})k\Delta t + j(\omega_{x1}+\omega_{x2})h\Delta x}$ yields

$$\begin{aligned} & H_{d2}^{ST}\left(j\omega_{t1}, j\omega_{x1}, j\omega_{t2}, j\omega_{x2}\right) \\ & = \frac{\left(2a_3 e^{j(\omega_{t1}+\omega_{t2})\Delta t} + a_4\left(e^{j\omega_{t1}\Delta t} + e^{j\omega_{t2}\Delta t}\right) + 2a_5\right) H_{d1}^{ST}(j\omega_{t1}, j\omega_{x1}) H_{d1}^{ST}(j\omega_{t2}, j\omega_{x2})}{2e^{2j(\omega_{t1}+\omega_{t2})\Delta t} - 2a_1 e^{j(\omega_{t1}+\omega_{t2})\Delta t} - 2a_2 - 2d_1 e^{j(\omega_{x1}+\omega_{x2})\Delta x} - 2d_2 e^{-j(\omega_{x1}+\omega_{x2})\Delta x}} \end{aligned} \tag{50}$$

Following this idea, all higher-order spatio-temporal generalised frequency response functions can be calculated recursively.

5 Calculation of STGFRF for Boundary-Value Problems

Practical systems will always exist on a finite region so that the physical processes in the boundary region have to be considered. In this section the effects of the boundary conditions on the spatio-temporal generalised frequency response functions will be briefly analysed. The most commonly encountered boundary conditions in the solution of partial differential equations are: Dirichlet boundary conditions, Neumann boundary conditions and Robin Boundary conditions.

Consider a one-dimensional spatio-temporal system $y(t,x)$ which evolves in a region $[0,L]$. A zero boundary condition can be defined as

$$\begin{cases} y(t,0)=0 \\ y(t,L)=0 \end{cases} \tag{51}$$

Probe the system using an input $u(t,x)=\sum_{k=1}^{n} e^{j\omega_{tk}t+j\omega_{xk}x}$ and the corresponding output is $y(t,x)=\sum_{n=1}^{\infty} y_n(t,x)$ where $y_n(t,x)=\sum_{k_1=1}^{n}\cdots\sum_{k_n=1}^{n} H_n^{ST}\left(\omega_{tk_1},\cdots,\omega_{tk_n},\omega_{xk_1},\cdots,\omega_{xk_n}\right)$.

However, the output has to satisfy the boundary conditions (51). The boundary conditions can then be written as

$$\begin{cases} y(t,0)=\sum_{n=1}^{\infty} y_n(t,0)=0 \\ y(t,L)=\sum_{n=1}^{\infty} y_n(t,L)=0 \end{cases} \tag{52}$$

The boundary conditions can further be converted to restrictions for the spatio-temporal generalised frequency response functions $H_n^{ST}\left(\omega_{tk_1},\cdots,\omega_{tk_n},\omega_{xk_1},\cdots,\omega_{xk_n}\right)$. For a linear spatio-temporal system, the conversion can be easily realised.

Consider a boundary-value problem

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}+\xi_1\omega_0\frac{\partial y}{\partial t}+\omega_0^2 y=c\frac{\partial^2 y}{\partial x^2}+bu \\ y(t,0)=0 \\ y(t,L)=0 \end{cases} \tag{53}$$

The linear generalised frequency response function of system (53) has the same form as the first order generalised frequency response function of system (28) in the last section, that is

$$H^{ST}(j\omega_t, j\omega_x)=\frac{b}{c\omega_x^2-\omega_t^2+j\xi_1\omega_t+\omega_0^2} \tag{54}$$

The response of the system for the probing input $u(t,x)=e^{j\omega_t t+j\omega_x x}$ is

$$y(t,x)=H^{ST}\left(j\omega_t,j\omega_x\right)e^{j\omega_t t+j\omega_x x} \tag{55}$$

The zero boundary conditions can then be rewritten as

$$\begin{cases} H^{ST}\left(j\omega_t,j\omega_x\right)e^{j\omega_t t}e^{j\omega_x 0}=0 \\ H^{ST}\left(j\omega_t,j\omega_x\right)e^{j\omega_t t}e^{j\omega_x L}=0 \end{cases} \tag{56}$$

Because $e^{j\omega_t t}$ will not always be zero so the boundary condition restrictions are

$$\begin{cases} H^{ST}\left(j\omega_t,j\omega_x\right)e^{j\omega_x 0}=0 \\ H^{ST}\left(j\omega_t,j\omega_x\right)e^{j\omega_x L}=0 \end{cases} \tag{57}$$

The complex form of the signal $H^{ST}(j\omega_t,j\omega_x)e^{j\omega_x x}$ represents the practical signal $\left|H\left(j\omega_t,j\omega_x\right)\right|\left(A\cos(\omega_x x)+B\sin(\omega_x x)\right)$, where $A^2+B^2=1$. The boundary conditions can now be converted to

$$\begin{cases} \left|H^{ST}\left(j\omega_t,j\omega_x\right)\right|\left(A\cos(\omega_x 0)+B\sin(\omega_x 0)\right)=0 \\ \left|H^{ST}\left(j\omega_t,j\omega_x\right)\right|\left(A\cos(\omega_x L)+B\sin(\omega_x L)\right)=0 \end{cases} \tag{58}$$

According to the first equation of (58), $A=0$ holds. Since $A^2+B^2=1$, $B=1$. The second boundary condition can then be transferred to

$$\left|H^{ST}\left(j\omega_t,j\omega_x\right)\right|\sin(\omega_x L)=0 \tag{59}$$

That is, $\sin\omega_x L=0$ or $\left|H(j\omega_t,j\omega_x)\right|=0$.

The generalised frequency response function of the linear spatio-temporal system can finally be given as

$$H^{ST}(j\omega_t,j\omega_x)=\begin{cases} \dfrac{b}{c\omega_x^2-\omega_t^2+j\xi_1\omega_t+\omega_0^2} & \text{when } \omega_x=\dfrac{k\pi}{L} \\ 0 & \text{otherwise} \end{cases} \tag{60}$$

This means for the boundary-value problem, the spatio-temporal system only has a discrete spatial frequency spectrum at

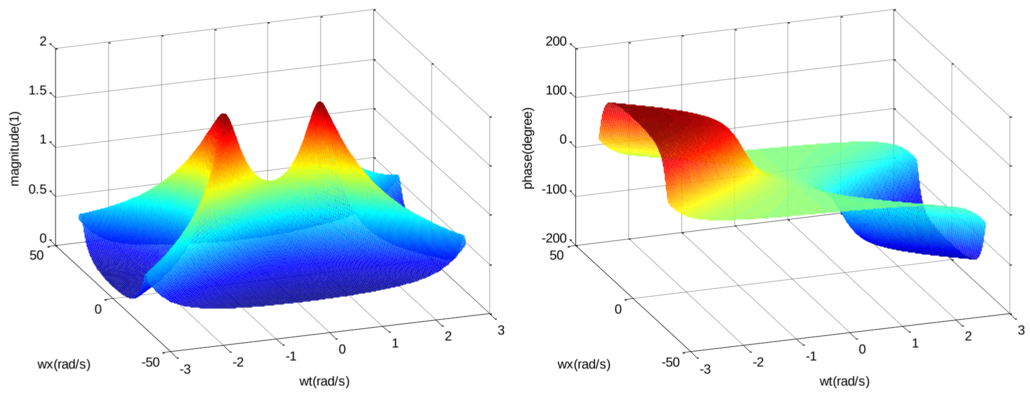
$$\omega_x = \frac{k\pi}{L}, \quad k = 1, 2, \cdots \tag{61}$$

6 Illustrative Examples

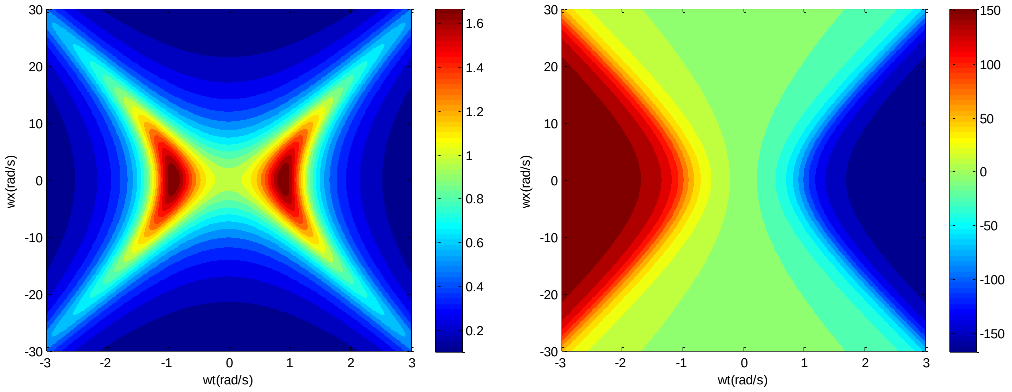
6.1 STGFRF of a Continuous Spatio-Temporal System

In this section some spatio-temporal systems will be analysed using the spatio-temporal generalised frequency response functions obtained in section 4.

Consider the continuous spatio-temporal system (28) in section 4.1. Set the system parameters as $\xi_1 = 0.6, \xi_2 = 1, \omega_0 = 1, b = 1$. The graph of the first order STGFRF $H_1^{ST}(j\omega_t, j\omega_x)$ is given in Fig 5 which graphically describes the magnitude and phase versus the spatial and the temporal frequencies.



(a) (b)



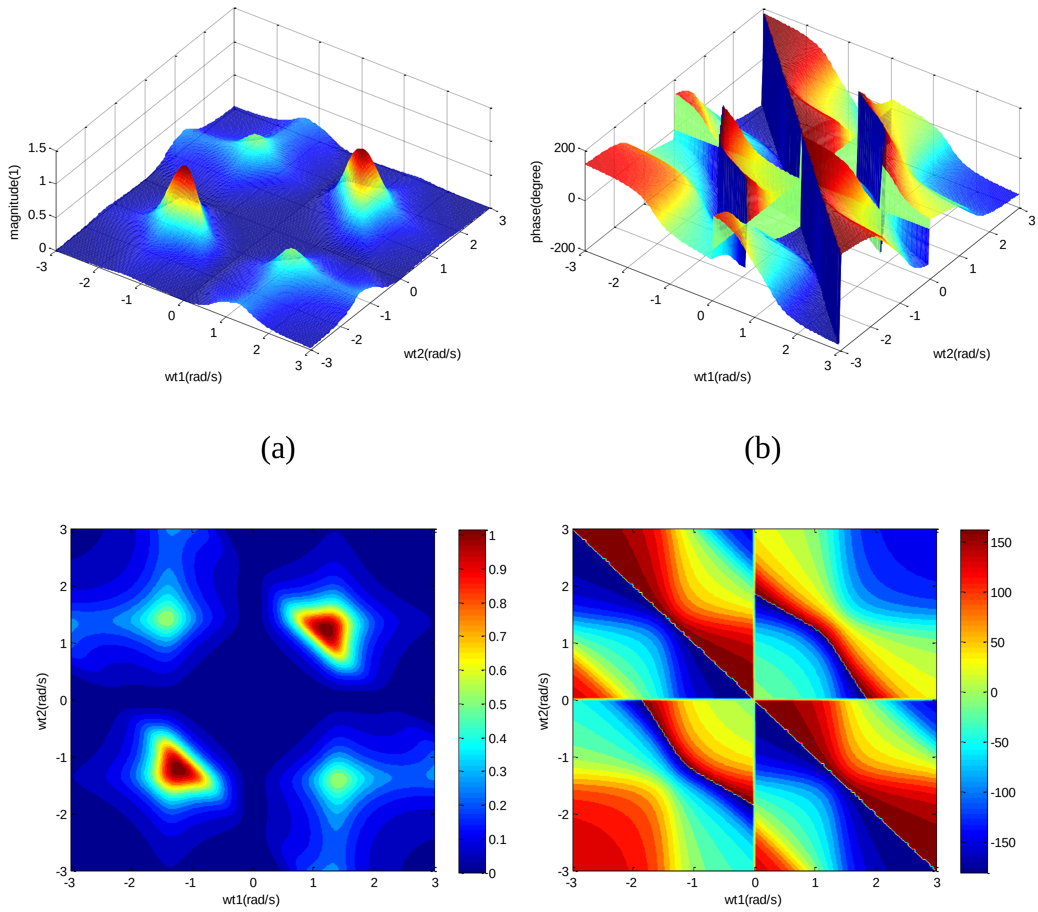
(c) (d)

Fig 5 $H_1^{ST}(j\omega_t, j\omega_x)$

(a) (c) magnitude (b) (d) phase $\xi_1 = 0.6, \xi_2 = 1, \omega_0 = 1, b = 1$

Fig 5 shows that when ω_x is fixed, the system behave as a typical under-damped second order system over the temporal frequency ω_t. However, the STGFRF $H_1^{ST}(j\omega_t, j\omega_x)$ depends on both the spatial frequency ω_x and the temporal frequency ω_t. When the spatial frequency ω_x increases, the resonant frequency of the second order system increases and the peak of the magnitude gets thinner. Simulations show that the larger the diffusion coefficient is, the greater the effect of ω_x becomes.

Given fixed spatial frequencies ω_{x1} and ω_{x2}, the second order STGFRF $H_2^{ST}\left(j\omega_{t1}, j\omega_{x1}, j\omega_{t2}, j\omega_{x2}\right)$ over the temporal frequencies ω_{t1} and ω_{t2} is graphically shown in Fig 6. The STGFRF $H_2^{ST}\left(j\omega_{t1}, j\omega_{x1}, j\omega_{t2}, j\omega_{x2}\right)$ has a similar shape with the temporal second order GFRF $H_2\left(j\omega_{t1}, j\omega_{t2}\right)$ in Fig 2.



(a) (b)

(c) (d)

Fig 6 $H_2^{ST}\left(j\omega_{t1}, j\omega_{x1}, j\omega_{t2}, j\omega_{x2}\right)$

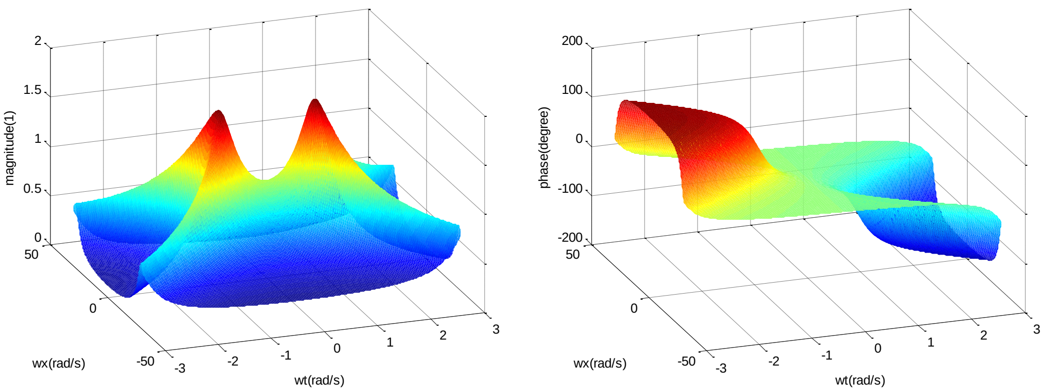
(a) (c) magnitude (b) (d) phase $\xi_1 = 0.6, \xi_2 = 1, \omega_0 = 1, b = 1, \omega_{x1} = \omega_{x2} = 10$

6.2 STGFRF of a Discrete Spatio-Temporal System

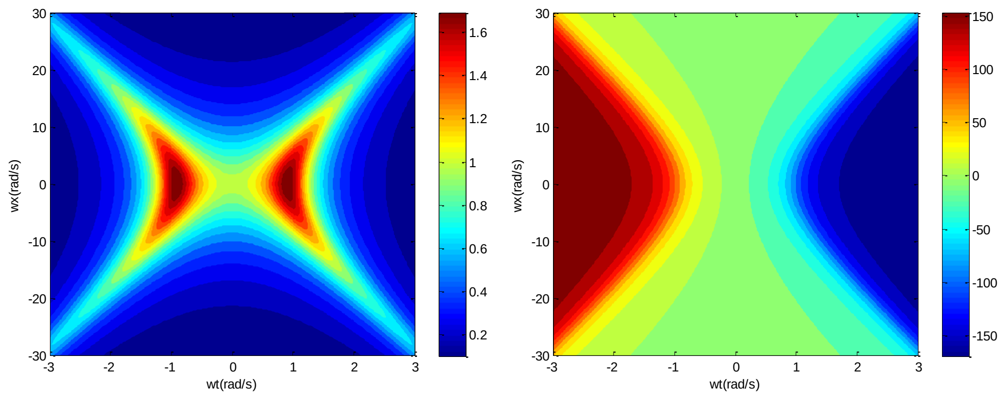
Discretising the continuous system yields the parameters of the discrete system: $a_1 = 1.994, a_2 = -1.0141, a_3 = -1, a_4 = 2, a_5 = -1, d_1 = d_2 = 0.01, b_1 = 0.0001$, where the spatial and the temporal interval are $\Delta x = 0.01$, $\Delta t = 0.01$. The discrete STGFRF in the last section which can be rewritten as

The 3-D graph and the contour graph of the discrete first order STGFRF

$H_{d1}^{ST}(j\omega_t, j\omega_x)$ and the second order STGFRF $H_{d2}^{ST}(j\omega_{t1}, j\omega_{x1}, j\omega_{t2}, j\omega_{x2})$ are given in Fig 7 and Fig 8 respectively. Obviously, we obtain magnitudes and phases which are very close to the continuous case. However in the discrete version, parameters a_1 ~ a_5 represent the effects of the cell states in past time while parameters d_1 and d_2 show the effects of the left and right neighbours. The discrete STGFRFs not only depend on the states of a cell in past time but also the states of its neighbours. The effect of the past states and the neighbourhood can be easily analysed by changing the values of the corresponding coefficients.



(a) (b)



(c) (d)

Fig 7 $H_{d1}^{ST}(j\omega_t, j\omega_x)$ (a) (c) magnitude (b) (d) phase

$$a_1 = 1.994, a_2 = -1.0141, a_3 = -1, a_4 = 2, a_5 = -1, d_1 = d_2 = 0.01, b_1 = 0.0001$$

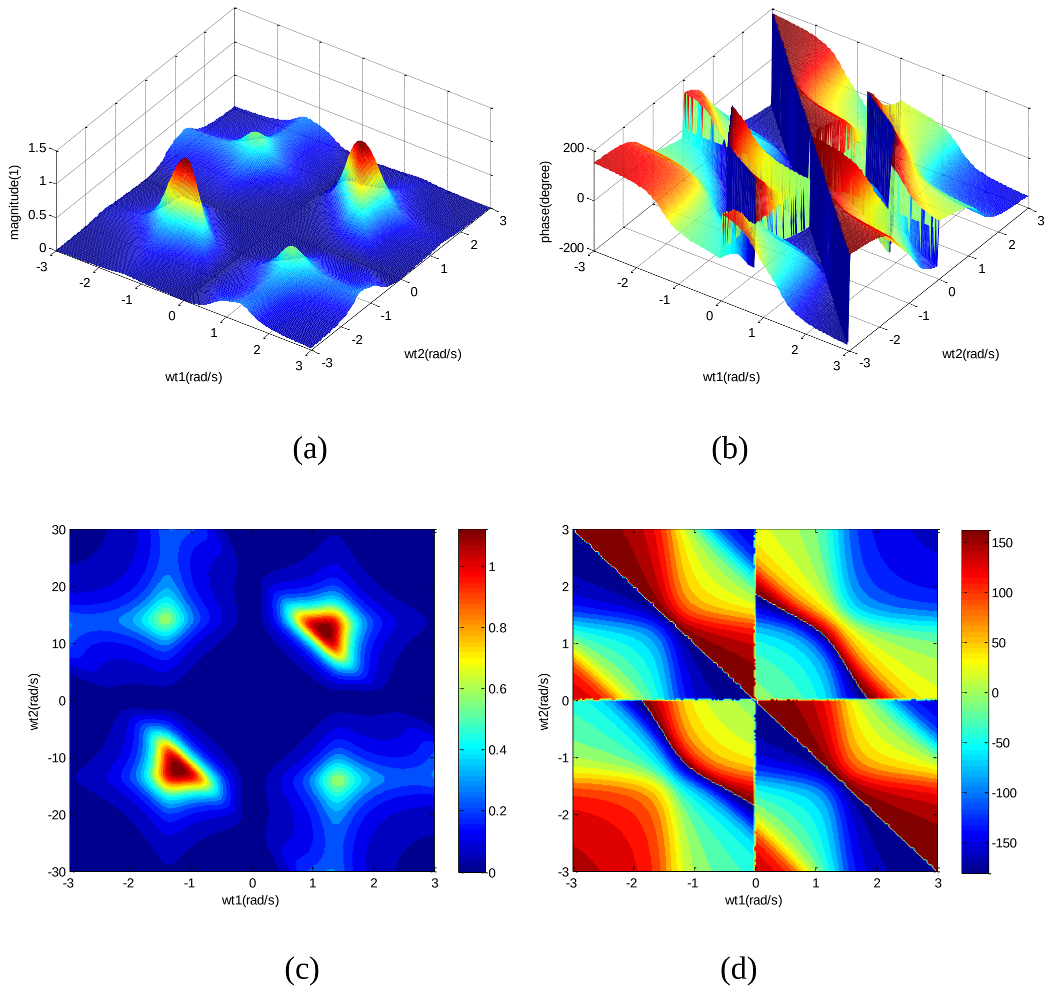


Fig 8 $H_{d2}^{ST}\left(j\omega_{t1}, j\omega_{x1}, j\omega_{t2}, j\omega_{x2}\right)$ (a) (c) magnitude (b) (d) phase

$$a_1 = 1.994, a_2 = -1.0141, a_3 = -1, a_4 = 2, a_5 = -1, d_1 = d_2 = 0.01, b_1 = 0.0001,$$
$$\Delta t = \Delta x = 0.01, \omega_{x1} = \omega_{x2} = 10$$

7 Conclusions

A parametric methodology for the calculation the generalised frequency response functions of spatio-temporal systems has been presented for the first time. The probing method has been developed to determine the generalised frequency response functions from the continuous PDE models and discrete CML models. Several examples were used to demonstrate that the new results are correct.

Although only one-dimensional spatio-temporal systems are considered in this paper, the methods can easily be extended to arbitrary n-dimensional spatio-temporal systems. The spatio-temporal generalised frequency response functions open up a new avenue for the study of spatio-temporal systems. Combined with the identification methods proposed both for continuous spatio-temporal systems and for discrete spatio-temporal systems(Billings and Coca 2002; Billings et al. 2006; Coca and Billings 2001; Guo and Billings 2006; Pan and Billings 2008), the spatio-temporal generalised frequency response functions can be a powerful tool for analysis of spatio-temporal systems.

Acknowledgment

The authors gratefully acknowledge support from the UK Engineering and Physical Sciences Research Council (EPSRC) and the European Research Council (ERC).

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